

Transformations of Ramanujan's Summation Formula and its Applications

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Abstract

In this paper, we obtain some new transformation formulas for Ramanujan's ${}_1\psi_1$ summation formula and also establish some eta-function identities. We also deduce a q -Gamma function identity, a q -integral and some interesting series representations for $\frac{\pi^{3/2}}{2\sqrt{2}\Gamma^2(3/4)}$ and the beta function $B(x, y)$.

Key Words and Phrases: Basic hypergeometric series, Dedekind eta function, q -Gamma function, q -integrals.

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1. Introduction

One of the most celebrated identities of Ramanujan is his ${}_1\psi_1$ -summation formula which can be stated as follows:

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_{\infty} (q/az)_{\infty} (q)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (b/az)_{\infty} (b)_{\infty} (q/a)_{\infty}} \quad (1.1)$$

where $|b/a| < |z| < 1$. This identity was first brought before the mathematical world by G. H. Hardy [7] who described it as ‘‘A remarkable formula with many parameters’’. Hardy did not supply a proof but indicated that a proof could be constructed from the q -binomial theorem. There are several proofs of (1.1) in the literature [cf. 1, 9]. Most proofs of (1.1) found in the literature do depend on the q -binomial theorem. The transformations of the basic hypergeometric series are extremely useful in the theory of partitions. The main purpose of this paper is to obtain some transformations for Ramanujan’s ${}_1\psi_1$ summation formula (1.1).

As customary we employ the following definitions and notations:

$$(a)_\infty := (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1, \quad (1.2)$$

$$(a)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \quad n : \text{any integer}. \quad (1.3)$$

The generalized basic hypergeometric series is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(q)_n (b_1)_n (b_2)_n \dots (b_s)_n} z^n, \quad |z| < 1, |q| < 1. \quad (1.4)$$

The bilateral basic hypergeometric series is defined by

$${}_r\Psi_r \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_r \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(q)_n (b_1)_n (b_2)_n \dots (b_r)_n} z^n, \quad (1.5)$$

where $\left| \frac{b_1 \dots b_r}{a_1 \dots a_r} \right| < |z| < 1, |q| < 1$.

In Section 3 we obtain some special cases of the transformations for Ramanujan’s ${}_1\psi_1$ summation formula obtained in Section 2 and in Section 4 we also deduce some eta-function identities. In the final section we derive some interesting series representations for the beta function $B(x, y)$ and $\frac{\pi^{3/2}}{2\sqrt{2}\Gamma^2(3/4)}$.

2. Main Results

To prove our main result we need the Heine’s transformation [5, Eq.(III.1) p.241, Eq.(III.2), p.241]

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n (z)_n}{(q)_n (az)_n} b^n \quad (2.1)$$

$$= \frac{(c/b)_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c)_n (b)_n}{(q)_n (bz)_n} (c/b)^n. \quad (2.2)$$

Theorem 2.1. If $|b/a| < |z| < 1$ and $|q| < \min\{1, |b|\}$ then,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(b/a)_\infty (az)_\infty}{(b)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (aqz/b)_n}{(q)_n (az)_n} (b/a)^n \\ &\quad + \frac{(q/b)_\infty (bq/az)_\infty}{(q/a)_\infty (b/az)_\infty} \sum_{n=0}^{\infty} \frac{(b/a)_n (b/az)_n}{(q)_n (bq/az)_n} (q/b)^n. \end{aligned} \quad (2.3)$$

Proof: We have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b)_n}{(q/a)_n} (b/az)^n \\ &= -1 + \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n + \sum_{n=0}^{\infty} \frac{(q/b)_n}{(q/a)_n} (b/az)^n. \end{aligned} \quad (2.4)$$

Putting $a = q$, $b = a$ and $c = b$ in (2.2) we obtain

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(b/a)_\infty (az)_\infty}{(b)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (aqz/b)_n}{(q)_n (az)_n} (b/a)^n. \quad (2.5)$$

Changing a to q , b to q/b , c to q/a and z to b/az in (2.1) we obtain

$$\sum_{n=0}^{\infty} \frac{(q/b)_n}{(q/a)_n} (b/az)^n = \frac{(q/b)_\infty (bq/az)_\infty}{(q/a)_\infty (b/az)_\infty} \sum_{n=0}^{\infty} \frac{(b/a)_n (b/az)_n}{(q)_n (bq/az)_n} (q/b)^n. \quad (2.6)$$

Substituting (2.5) and (2.6) in (2.4) we obtain (2.3).

Theorem 2.2. If $|\frac{b}{a}| < |z| < 1$ and $|a| < 1$, then,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(a)_\infty (qz)_\infty}{(b)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(b/a)_n (z)_n}{(q)_n (qz)_n} (a)^n \\ &\quad + \frac{(b/a)_\infty (q/az)_\infty}{(q/a)_\infty (b/az)_\infty} \sum_{n=0}^{\infty} \frac{(q/z)_n (q/b)_n}{(q)_n (q/az)_n} (b/a)^n. \end{aligned} \quad (2.7)$$

Proof: Changing a to q , b to a and then c to b in (2.1), we deduce

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(a)_\infty (qz)_\infty}{(b)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(b/a)_n (z)_n}{(q)_n (qz)_n} (a)^n. \quad (2.8)$$

Putting $a = q$, $b = q/b$, $c = q/a$ and $z = b/az$ in (2.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(q/b)_n}{(q/a)_n} (b/az)^n = \frac{(b/a)_\infty (q/az)_\infty}{(q/a)_\infty (b/az)_\infty} \sum_{n=0}^{\infty} \frac{(q/z)_n (q/b)_n}{(q)_n (q/az)_n} (b/a)^n. \quad (2.9)$$

On employing (2.8) and (2.9) in (2.4) we obtain Theorem 2.2.

Theorem 2.3. If $|\frac{b}{a}| < |z| < 1$ then,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(b/a)_{\infty} (az)_{\infty}}{(b)_{\infty} (z)_{\infty}} \sum_{n=0}^{\infty} \frac{(aqz/b)_n (a)_n}{(q)_n (az)_n} (b/a)^n \\ &+ \frac{(b/a)_{\infty} (q/az)_{\infty}}{(q/a)_{\infty} (b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(q/z)_n (q/b)_n}{(q)_n (q/az)_n} (b/a)^n. \end{aligned} \quad (2.10)$$

Proof: On employing (2.5) and (2.9) in (2.4) we obtain Theorem 2.3.

3. Special Cases

Putting $a = -1/q$ and $b = -1$ in (2.3), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1+q^{n-1}} = \frac{-q}{1+q} + \frac{q}{1+q} \frac{(q)_{\infty} (-z/q)_{\infty}}{(-1)_{\infty} (z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n (-1/q)_n}{(q)_n (-z/q)_n} q^n + q \sum_{n=0}^{\infty} \frac{1}{1 - \frac{q^{n+1}}{z}} (-q)^n, \quad (3.1)$$

where $|q| < |z| < 1$.

Changing b to $-q^3$, a to $-q$, z to q in (2.7) we get,

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1+q^{n+1})(1+q^{n+2})} = \frac{(1+q^2)(1+q)}{q(1-q)} \quad (3.2)$$

Putting $b = -q^2$, $a = -1/q$ and $z = q^{1/2}$ in (2.10) and then changing q to q^2 , we obtain

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{2(1+1/q^2)(1+q^2)q^n}{(1+q^{2n-2})(1+q^{2n})(1+q^{2n+2})} &= -1 + \frac{(q^6; q^2)_{\infty} (-1/q; q^2)_{\infty}}{(-q^4; q^2)_{\infty} (q; q^2)_{\infty}} \\ \sum_{n=0}^{\infty} \frac{(1/q^3; q^2)_n (-1/q^2; q^2)_n}{(q^2; q^2)_n (-1/q; q^2)_n} q^{6n} &+ \frac{(q^6; q^2)_{\infty} (-q^3; q^2)_{\infty}}{(-q^4; q^2)_{\infty} (q^5; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1/q^2; q^2)_n}{(q^2; q^2)_n (-q^3; q^2)_n} q^{6n} \end{aligned} \quad (3.3)$$

4. Eta-function identities

The Dedekind eta-function is defined by

$$\begin{aligned}\eta(\tau) &:= e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \\ &= q^{1/24} (q; q)_{\infty},\end{aligned}\tag{4.1}$$

where $q = e^{2\pi i \tau}$ and $\text{Im } \tau > 0$.

The eta-function is useful in the study of modular forms. Berndt and L.C.Zhang [3] have obtained a number of eta-function identities found in Ramanujan's notebook employing the theory of modular forms. In [4], Bhargava and Somashekara show how a family of interesting eta-function identities can be obtained from a limiting case of the Ramanujan's ${}_1\psi_1$ -summation formula.

Now we derive some new η -function identities.

Employing Ramanujan's ${}_1\psi_1$ summation formula in (2.3) and then putting $a = 1/q$, $b = q^{1/2}$, $z = q^{1/2}$, and changing q to q^2 , we deduce

$$\frac{\eta(\tau)}{\eta^2(2\tau)} = q^{-1/8} - \frac{q^{7/8}}{1+q} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(q^3; q^2)_n}{(q^4; q^2)_n} q^n.\tag{4.2}$$

This can also be obtained from the q -binomial theorem.

Employing Ramanujan's ${}_1\psi_1$ summation formula in (2.7) and then putting $a = -q$, $b = -q^3$ and $z = q^{1/2}$ and changing q to q^2 , we deduce

$$\begin{aligned}\frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^2(4\tau)} &= \frac{2(1+q)q^{5/6}}{(1+q^2)(1+q^4)} - \frac{2q^{5/6}}{1-q} \sum_{n=0}^{\infty} \frac{1-q^{2n+2}}{1-q^{2n+1}} (-q^2)^n - \\ &\frac{2(1+q)q^{5/6}}{(1+q^2)(1+q^4)} \frac{(q^4; q^2)_{\infty}(-1/q; q^2)_{\infty}}{(-1; q^2)_{\infty}(q^3; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(q; q^2)_n(1-q^4; q^2)_n}{(q^2; q^2)_n(-1/q; q^2)_n} q^{4n}.\end{aligned}\tag{4.3}$$

Employing Ramanujan's ${}_1\psi_1$ summation formula in (2.3) and then putting $a = -q^{-5/3}$, $b = -q^{1/3}$ and $z = q^{4/3}$, then changing q to q^3 , we deduce

$$\begin{aligned}\frac{\eta^3(3\tau)}{\eta(\tau)} &= \frac{q^{4/3}(1+q+q^2)}{(1+q^2)(1+q^5)} \left[-1 + \frac{(q^6; q^3)_{\infty}(-1/q; q^3)_{\infty}}{(-q; q^3)_{\infty}(q^4; q^3)_{\infty}} \sum_{n=0}^{\infty} \frac{(q; q^3)_n(-1/q^5; q^3)_n}{(q^3; q^3)_n(-1/q; q^3)_n} q^{6n} \right] \\ &+ \frac{q^{4/3}(1+q+q^2)}{(1+q^2)(1+q^5)} \frac{(q^6; q^3)_{\infty}(-q^4; q^3)_{\infty}}{(-q^8; q^3)_{\infty}(q^2; q^3)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/q; q^3)_n(-q^2; q^3)_n}{(q^3; q^3)_n(-q^4; q^3)_n} q^{6n}\end{aligned}\tag{4.4}$$

5. Some q -Gamma Function Identities:

F.H. Jackson [8] defined the q -analogue of the gamma function by

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty} (1-q)^{1-x}, \quad 0 < q < 1. \quad (5.1)$$

Jackson [8] also defined a q -integral by,

$$\int_0^a f(t) d_q(t) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n \quad (5.2)$$

and

$$\int_0^\infty f(t) d_q(t) = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) \cdot q^n. \quad (5.3)$$

In this section, as an application of the transformation formulas of Section 2, we deduce some q -Gamma function identities and an q -integral which gives interesting series representations for the beta function $B(x,y)$ and $\frac{\pi^{3/2}}{2\sqrt{2}\Gamma^2(3/4)}$

Theorem 5.1. If $0 < z < b - a < 1$ and $b < 1$ then,

$$\begin{aligned} \frac{\Gamma_q(b)\Gamma_q(1-a)\Gamma_q(z)\Gamma_q(b-a-z)}{\Gamma_q(b-a)\Gamma_q(a+z)\Gamma_q(1-a-z)} &= -(1-q)^{a+1-b} + \frac{\Gamma_q(1-a)\Gamma_q(b-a-z)}{\Gamma_q(1-b)\Gamma_q(b+1-z-a)} \\ &\quad \sum_{n=0}^{\infty} \frac{(q^{b-a})_n (q^{b-z-a})_n}{(q^{b+1-a-z})_n (q)_n} (q^{1-b})^n + \frac{\Gamma_q(b)\Gamma_q(z)}{\Gamma_q(b-a)\Gamma_q(a+z)} (1-q)^{a+1-b} \\ &\quad \sum_{n=0}^{\infty} \frac{(q^{a+1+z-b})_n (q^a)_n}{(q)_n (q^{a+z})_n} (q^{b-a})^n. \end{aligned} \quad (5.4)$$

Proof: Employing Ramanujan's ${}_1\psi_1$ summation formula in (2.3) and then changing a to q^a , b to q^b and z to q^z we obtain the required result.

Corollary: In Theorem 5.1 if we let $q \rightarrow 1$ and put $a = 0$ we deduce,

$$\frac{\Gamma(1-b)\Gamma(b+1-z)}{\Gamma(1-z)} = \sum_{n=0}^{\infty} \frac{(b)_n}{n!} \frac{b-z}{b-z+n}. \quad (5.5)$$

Remark: Putting $b-z = y$ and $1-b = x$ in (5.5) we obtain a well known [6] representation for the beta function $B(x,y)$ given by,

$$B(x,y) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^n (k-x)}{n!} \frac{1}{n+y}. \quad (5.6)$$

Theorem 5.2. If $0 < z < b - a < 1$ and $a > 0$ then,

$$\frac{\Gamma(a)\Gamma(z)\Gamma(1-a)\Gamma(b-a-z)}{\Gamma(a+z)\Gamma(b-a)\Gamma(1-a-z)} = \sum_{n=0}^{\infty} \frac{(b-a)_n}{n!(n+z)} \quad (5.7)$$

Proof: Employing Ramanujan's ${}_1\psi_1$ summation formula in (2.7) and then changing a to q^a , b to q^b and z to q^z we obtain

$$\begin{aligned} \frac{\Gamma_q(z)\Gamma_q(b-a-z)\Gamma_q(b)\Gamma_q(1-a)}{\Gamma_q(a+z)\Gamma_q(1-a-z)\Gamma_q(b-a)} &= -(1-q)^{a+1-b} + \frac{\Gamma_q(b)\Gamma_q(z)}{\Gamma_q(a)\Gamma_q(z+1)} \\ \sum_{n=0}^{\infty} \frac{(q^{b-a})_n(q^z)_n}{(q)_n(q^{1+z})_n} (q^a)^n + (1-q)^{a+1-b} \frac{\Gamma_q(1-a)\Gamma_q(b-a-z)}{\Gamma_q(1-a-z)\Gamma_q(b-a)} &= \sum_{n=0}^{\infty} \frac{(q^{1-z})_n(q^{1-b})_n}{(q)_n(q^{1-a-z})_n} (q^{b-a})^n. \end{aligned} \quad (5.8)$$

Letting $q \rightarrow 1$ in (5.8) we obtain Theorem 5.2.

Remark: Putting $a = 1/2$, $b = 1$ and $z = 1/4$ in Theorem 5.2 we obtain,

$$\frac{\pi^{3/2}}{2\sqrt{2}\Gamma^2(3/4)} = \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!(4n+1)} \quad (5.9)$$

which is very similar to the result obtained by Ramanujan [2, p.24].

Theorem 5.3. If $0 < z < b - a < 1$ then,

$$\begin{aligned} \frac{\Gamma_q(b)\Gamma_q(1-a)\Gamma_q(z)\Gamma_q(b-a-z)}{\Gamma_q(b-a)\Gamma_q(a+z)\Gamma_q(1-a-z)} &= -(1-q)^{a+1-b} + \frac{\Gamma_q(1-a)\Gamma_q(b-a-z)}{\Gamma_q(b-a)\Gamma_q(1-z)\Gamma_q(1-b)} \\ &+ \frac{\Gamma_q(b)\Gamma_q(z)}{\Gamma_q(b-a)\Gamma_q(a+1+z-b)\Gamma_q(a)} \int_0^1 g(x)d_q(x) \end{aligned} \quad (5.10)$$

where,

$$f(x) = \frac{(xq)_\infty(xq^{1-a-z})_\infty}{(xq^{1-z})_\infty(xq^{1-b})_\infty} x^{b-a-1}$$

and

$$g(x) = \frac{(xq)_\infty(xq^{a+z})_\infty}{(xq^{a+1+z-b})_\infty(xq^a)_\infty} x^{b-a-1}.$$

Proof: Employing Ramanujan's ${}_1\psi_1$ summation formula in (2.10) and then changing a to q^a , b to q^b and z to q^z we get,

$$\begin{aligned} \frac{\Gamma_q(b)\Gamma_q(1-a)\Gamma_q(z)\Gamma_q(b-a-z)}{\Gamma_q(b-a)\Gamma_q(a+z)\Gamma_q(1-a-z)}(1-q)^{b-a-1} &= -1 + \frac{\Gamma_q(1-a)\Gamma_q(b-a-z)}{\Gamma_q(b-a)\Gamma_q(1-z)\Gamma_q(1-b)} \\ &+ (1-q)^{b-a} \sum_{n=0}^{\infty} \frac{(q^{n+1})_{\infty}(q^{n+1-a-z})_{\infty}}{(q^{n+1-z})_{\infty}(q^{n+1-b})_{\infty}} (q^{b-a})^n + \frac{\Gamma_q(b)\Gamma_q(z)}{\Gamma_q(b-a)\Gamma_q(a+1+z-b)\Gamma_q(a)} \\ &(1-q)^{b-a} \sum_{n=0}^{\infty} \frac{(q^{n+1})_{\infty}(q^{n+a+z})_{\infty}}{(q^{n+a+1+z-b})_{\infty}(q^{n+a})_{\infty}} (q^{b-a})^n. \end{aligned} \quad (5.11)$$

Applying (5.2) to the right side of (5.10) we obtain Theorem 5.3.

Corollary: In Theorem 5.3 if we let $q \rightarrow 1$ we deduce,

$$\frac{\Gamma(b)\Gamma(1-a)\Gamma(z)\Gamma(b-a-z)}{\Gamma(a+z)\Gamma(1-a-z)} = \left[\frac{\Gamma(1-a)\Gamma(b-a-z)}{\Gamma(1-z)\Gamma(1-b)} + \frac{\Gamma(b)\Gamma(z)}{\Gamma(a)\Gamma(a+1+z-b)} \right] B(b-a, a-b+1). \quad (5.12)$$

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