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General Projection Systems and Strongly Monotone Nonlinear Variational Inequalities

Ram U. Verma
Department of Mathematics
University of Toledo
Toledo, Ohio 43606, USA
verma99@msn.com

Abstract

Let K_1 and K_2, respectively, be non empty closed convex subsets of real Hilbert spaces H_1 and H_2. The *solvability* of a general system of nonlinear variational inequality $(SNVI)$ problems based on a new projection system is explored. The SNVI problem is stated as follows: find an element $(x^*, y^*) \in K_1 \times K_2$ such that

$$< \rho S(x^*, y^*), x - x^* > \geq 0, \ \forall x \in K_1 \text{ and for } \rho > 0,$$

$$< \eta T(x^*, y^*), y - y^* > \geq 0, \ \forall y \in K_2 \text{ and for } \eta > 0,$$

where $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are nonlinear mappings.

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1. Introduction

Projection-type systems are frequently applied to achieve the convergence analysis for solutions of variational inequality problems arising from several fields, for instance complementarity theory, convex quadratic programming, optimization and control theory, and variational problems. Recently the author [6,7] introduced and studied a two-step system of nonlinear variational inequalities in Hilbert space settings. This study was carried out based new algorithms, which encompass several classes of iterative algorithms. In this paper we explore, based on a general system of projection-type methods, the approximation-solvability of a system of nonlinear strongly monotone variational inequalities in Hilbert spaces. The obtained results extend/improve the results in [1,5,6,7]. For more details, we refer to [1-10].

Let H_1 and H_2 be two real Hilbert spaces with the inner product $< .,. >$ and norm $\|.\|$. Let $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ be any mappings on $K_1 \times K_2$, where K_1 and K_2 are nonempty closed convex subsets of H_1 and H_2, respectively. We consider a system of nonlinear variational inequality (abbreviated as SNVI) problems: find an element $(x^*, y^*) \in K_1 \times K_2$ such that

$$< \rho S(x^*, y^*), x - x^* > \geq 0 \ \forall x \in K_1 \tag{1}$$

$$< \eta T(x^*, y^*), y - y^* > \geq 0 \ \forall y \in K_2, \tag{2}$$

where $\rho, \eta > 0$.
The SNVI $(1)-(2)$ problem is equivalent to the following projection formulas

$$x^* = P_k[x^* - \rho S(x^*, y^*)] \text{ for } \rho > 0$$

$$y^* = Q_k[y^* - \eta T(x^*, y^*)] \text{ for } \eta > 0,$$

where P_k is the projection of H_1 onto K_1 and Q_K is the projection of H_2 onto K_2.
We note that the SNVI $(1)-(2)$ problem extends the NVI problem: determine an element $x^* \in K_1$ such that

$$< S(x^*), x - x^* > \geq 0, \ \forall x \in K_1. \tag{3}$$

Also, we note that the SNVI $(1)-(2)$ problem is equivalent to a system of nonlinear complementarities (abbreviated as SNC): find an element $(x^*, y^*) \in K_1 \times K_2$ such that $S(x^*, y^*) \in K_1^*$, $T(x^*, y^*) \in K_2^*$, and

$$< \rho S(x^*, y^*), x^* > = 0 \text{ for } \rho > 0, \tag{4}$$

$$< \eta T(x^*, y^*), y^* > = 0 \text{ for } \eta > 0, \tag{5}$$

where K_1^* and K_2^*, respectively, are polar cones to K_1 and K_2 defined by

$$K_1^* = \{f \in H_1 : < f, x > \geq 0, \ \forall x \in K_1\}.$$

$$K_2^* = \{g \in H_2 : < g, y > \geq 0, \ \forall g \in K_2\}.$$

Now, we recall some auxiliary results and notions crucial to the problem on hand.

Lemma 1.1. For an element $z \in H$, we have

$$x \in K \text{ and } < x - z, y - x > \geq 0, \ \forall y \in K \text{ if and only if } x = P_k(z).$$

Lemma 1.2. [3] Let $\{\alpha^k\}, \{\beta^k\}$, and $\{\gamma^k\}$ be three nonnegative sequences such that

$$\alpha^{k+1} \leq (1-t^k)\alpha^k + \beta^k + \gamma^k \; for\, k = 0,1,2,...,$$

where $t^k \in [0,1]$, $\Sigma_{k=0}^{\infty} t^k = \infty$, $\beta^k = o(t^k)$, and $\Sigma_{k=0}^{\infty}\gamma^k < \infty$. Then $\alpha^k \to 0$ as $k \to \infty$.

A mapping $T : H \to H$ from a Hilbert space H into H is called monotone if $< T(x) - T(y), x-y > \geq 0 \forall x,y \in H$. The mapping T is $(r) - strongly$ monotone if for each $x,y \in H$, we have

$$< T(x) - T(y), x-y > \geq r||x-y||^2 \text{ for a constant } r > 0.$$

This implies that $\|T(x) - T(y)\| \geq r\|x-y\|$, that is, T is (r)-expansive, and when $r = 1$, it is expansive. The mapping T is called (s)-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geq 0$ such that $\|T(x) - T(y)\| \leq s\|x-y\|$, $\forall x,y \in H$. T is called (μ)-cocoercive if for each $x,y \in H$, we have

$$< T(x) - T(y), x-y > \geq \mu||T(x) - T(y)||^2 \text{ for a constant } \mu > 0.$$

Clearly, every (μ)-cocoercive mapping T is $(\frac{1}{\mu})$-Lipschitz continuous. We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{ccc} \text{strong monotonicity} & \Rightarrow & \text{expansiveness} \\ \Downarrow & & \\ \text{monotonicity} & & \end{array}$$

T is called relaxed (γ)-cocoercive if there exists a constant $\gamma > 0$ such that

$$< T(x) - T(y), x-y > \geq (-\gamma)\|T(x) - T(y)\|^2, \; \forall x,y \in H$$

T is said to be (r)-strongly pseudomonotone if there exists a positive constant r such that

$$< T(y), x-y > \geq 0 \Rightarrow < T(x), x-y > \geq r\,\|x-y\|^2, \; \forall x,y \in H.$$

T is said to be relaxed (γ, r)-cocoercive if there exist constants $\gamma, r > 0$ such that

$$< T(x) - T(y), x-y > \geq (-\gamma)\,\|T(x) - T(y)\|^2 + r\,\|x-y\|^2.$$

Clearly, it implies that

$$< T(x) - T(y), x-y > \geq (-\gamma)\,\|T(x) - T(y)\|^2,$$

that is T is relaxed (γ)-cocoercive.
T is said to be relaxed (γ, r)-pseudococoercive if there exist positive constants γ and r such that

$$
\begin{aligned}
& < T(y), x-y > \geq 0 \\
\Rightarrow \quad & < T(x), x-y > \geq (-\gamma)\|T(x)-T(y)\|^2 + r\|x-y\|^2, \forall x,y \in H.
\end{aligned}
$$

2. General Projection Methods

In this section, we discuss the approximation-solvability of the SNVI $(1)-(2)$ problem based on the following algorithms.

Algorithm 2.1. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1-a^k-b^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)] + b^k u^k$$

$$y^{k+1} = (1-\alpha^k-\beta^k)y^k + \alpha^k Q_K[y^k - \eta T(x^k, y^k)] + \beta^k v^k,$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, $\rho,\ \eta > 0$ are constants, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings, and u^k and v^k, respectively, are bounded sequences in K_1 and K_2. The sequences $\{a^k\}, \{b^k\}, \{\alpha^k\}, and \{\beta^k\}$ are in $[0,1]$ with $(k \geq 0)$

$$0 \leq a^k + b^k \leq 1, 0 \leq \alpha^k + \beta^k \leq 1.$$

Algorithm 2.2. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1-a^k-b^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)] + b^k u^k$$

$$y^{k+1} = (1-a^k-b^k)y^k + a^k Q_K[y^k - \eta T(x^k, y^k)] + b^k v^k,$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, $\rho,\ \eta > 0$ are constants, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings, and u^k and v^k, respectively, are bounded sequences in K_1 and K_2. The sequences $\{a^k\}, and \{b^k\}$, are in $[0,1]$ with $(k \geq 0)$

$$0 \leq a^k + b^k \leq 1.$$

Algorithm 2.3. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$
\begin{aligned}
x^{k+1} &= (1-a^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)] \\
y^{k+1} &= (1-a^k)y^k + a^k Q_K[y^k - \eta T(x^k, y^k)],
\end{aligned}
$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, ρ, $\eta > 0$ are constants, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings. The sequence $\{a^k\} \in [0,1]$ for $k \geq 0$.

Next, we consider, based on Algorithm 2.2, the approximation solvability of the SNVI $(1)-(2)$ problem involving strongly monotone and Lipschitz continuous mappings in Hilbert space settings.

Theorem 2.1. Let H_1 and H_2 be two real Hilbert spaces and, K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be strongly $(r)-monotone$ and $(\mu)-Lipschitz$ continuous in the first variable and let S be $(\nu)-Lipschitz$ continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be strongly $(s)-monotone$ and $(\beta)-Lipschitz$ continuous in the second variable and let T be $(\tau)-Lipschitz$ continuous in the first variable. Let $\|.\|^*$ denote the norm on $H_1 \times H_2$ defined by

$$\|(x,y)\|^* = (\|x\| + \|y\|)\, \forall (x,y) \in H_1 \times H_2.$$

In addition, let

$$\theta + \eta\tau = \sqrt{1 - 2\rho r + \rho^2\mu^2} + \eta\tau < 1$$

$$\sigma + \rho\nu = \sqrt{1 - 2\eta r + \eta^2\beta^2} + \rho\nu < 1,$$

let $(x^*,y^*) \in K_1 \times K_2$ form a solution to the SNVI $(1)-(2)$ problem, and let sequences $\{x^k\}$, and $\{y^k\}$ be generated by $Algorithm\, 2.2$. Furthermore, let

(i) $0 \leq a^k + b^k \leq 1$

(ii) $\Sigma_{k=0}^{\infty} a^k = \infty$, and $\Sigma_{k=0}^{\infty} b^k < \infty$

(iii) $0 < \rho < \frac{2r}{\mu^2}$ and $0 < \eta < \frac{2s}{\beta^2}$.
Then the sequence $\{x^k, y^k)\}$ converges to (x^*, y^*).

Proof. Since $(x^*,y^*) \in K_1 \times K_2$ forms a solution to the SNVI $(1)-(2)$ problem, it follows that

$$x^* = P_K[x^* - \rho S(x^*,y^*)] \text{ and } y^* = Q_K[x^* - \eta T(x^*,y^*)].$$

Applying Algorithm 2.2, we have

$$\begin{aligned}
&\|x^{k+1}-x^*\| \\
= &\ \|(1-a^k-b^k)x^k+a^kP_K[x^k-\rho S(x^k,y^k)]+b^ku^k \\
- &\ (1-a^k-b^k)x^*-a^kP_K[x^*-\rho S(x^*,y^*)]-b^kx^*\| \\
\leq &\ (1-a^k-b^k)\|x^k-x^*\| \\
+ &\ a^k\|P_K[x^k-\rho S(x^k,y^k)]-P_K[x^*-\rho S(x^*,y^*)]\|+Mb^k \\
\leq &\ (1-a^k)\|x^k-x^*\|+a^k\|x^k-x^*-\rho[S(x^k,y^k)-S(x^*,y^k) \\
+ &\ S(x^*,y^k)-S(x^*,y^*)]\|+Mb^k \\
\leq &\ (1-a^k)\|x^k-x^*\|+a^k\|x^k-x^*-\rho[S(x^k,y^k)-S(x^*,y^k)]\| \\
+ &\ a^k\rho\|[S(x^*,y^k)-S(x^*,y^*)]\|+Mb^k,
\end{aligned} \tag{6}$$

where

$$M=max\{sup\|u^k-x^*\|,sup\|v^k-y^*\|\}<\infty.$$

Since S is strongly $(r)-$ monotone and $(\mu)-$Lipschitz continuous in the first variable, and S is $(\nu)-Lipschitz$ continuous in the second variable, we have

$$\begin{aligned}
&\|x^k-x^*-\rho[S(x^k,y^k)-S(x^*,y^k)]\|^2 \\
= &\ \|x-x^*\|^2-2\rho\langle S(x^k,y^k)-S(x^*,y^k),x^k-x^*\rangle \\
+ &\ \rho^2\|S(x^k,y^k)-S(x^*,y^k)\|^2 \\
= &\ \|x-x^*\|^2-2\rho\langle S(x^k,y^k)-S(x^k,y^k,x^k-x^*\rangle \\
+ &\ \rho^2\|S(x^k,y^k)-S(x^*,y^k)\|^2 \\
\leq &\ \|x^k-x^*\|^2-2\rho r\|x^k-x^*\|^2 \\
+ &\ \rho^2\mu^2\|x^k-x^*\|^2 \\
\leq &\ \|x^k-x^*\|^2-2\rho r\|x^k-x^*\|^2 \\
+ &\ \rho^2\mu^2\|x^k-x^*\|^2 \\
= &\ [1-2\rho r+\rho^2\mu^2]\|x^k-x^*\|^2.
\end{aligned} \tag{7}$$

It follows that

$$\begin{aligned}
&\|x^k-x^*-\rho[S(x^k,y^k)-S(x^*,y^k)]\|^2 \\
\leq &\ [1-2\rho r+(\rho\mu)^2]\|x^k-x^*\|^2.
\end{aligned}$$

As a result, we have

$$\begin{aligned}
&\|x^{k+1}-x^*\| \\
\leq &\ (1-a^k)\|x^k-x^*\|+a^k\theta\|x^k-x^*\| \\
+ &\ a^k\rho\nu\|y^k-y^*\|+Mb^k,
\end{aligned} \tag{8}$$

where $\theta = \sqrt{1-2\rho r+\rho^2\mu^2}$.
Similarly, we have

$$\begin{aligned} \|y^{k+1}-y^*\| \leq (1-a^k)\|y^k-y^*\| + a^k\sigma\|y^k-y^*\| \\ +a^k\eta\tau\|x^k-x^*\| + Mb^k, \end{aligned} \tag{9}$$

where $\sigma = \sqrt{1-2\eta r+\eta^2\beta^2}$.
It follows from (8) and (9) that

$$\begin{aligned} & \|x^{k+1}-x^*\| + \|y^{k+1}-y^*\| \\ \leq\ & (1-a^k)\|x^k-x^*\| + a^k\theta\|x^k-x^*\| \\ +\ & a^k\eta\tau\|x^k-x^*\| + Mb^k \\ +\ & (1-a^k)\|y^k-y^*\| + a^k\sigma\|y^k-y^*\| \\ +\ & a^k\rho\nu\|y^k-y^*\| + Mb^k \\ =\ & [1-(1-\delta)a^k](\|x^k-x^*\| + \|y^k-y^*\|) + 2Mb^k, \end{aligned} \tag{10}$$

where $\delta = max\{\theta+\eta\tau, \sigma+\rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|.\|^*$.
If we set

$$\alpha^k = \|x^k-x^*\| + \|y^k-y^*\|, t^k = (1-\delta)a^k,$$
$$\beta^k = 2Mb^k \ for k = 0,1,2,...,$$

in $Lemma 1.2$, and apply $(i) and (ii)$, we conclude that

$$\|x^k-x^*\| + \|y^k-y^*\| \to 0$$

as $k \to \infty$.

Hence,

$$\|x^{k+1}-x^*\| + \|y^{k+1}-y^*\| \to 0.$$

Consequently, the sequence $\{(x^k,y^k)\}$ converges strongly to (x^*,y^*), a solution to the $SNVI\,(1)-(2)$ problem. This completes the proof.

Note that the proof of the following theorem follows rather directly without using Lemma 1.2.

Theorem 2.2. Let H_1 and H_2 be two real Hilbert spaces and, K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be strongly $(r)-monotone$ and $(\mu)-Lipschitz$ continuous in the first variable and let S be $(\nu)-Lipschitz$ continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be strongly $(s)-monotone$ and $(\beta)-Lipschitz$ continuous in the second variable and let T be $(\tau)-Lipschitz$ continuous in the first variable. Let $\|.\|^*$ denote the norm on $H_1 \times H_2$ defined by

$$\|(x,y)\|^* = (\|x\| + \|y\|)\,\forall (x,y) \in H_1 \times H_2.$$

In addition, let

$$\theta+\eta\tau=\sqrt{1-2\rho r+\rho^2\mu^2}+\eta\tau<1$$

$$\sigma+\rho\nu=\sqrt{1-2\eta r+\eta^2\beta^2}+\rho\nu<1,$$

let $(x^*,y^*)\in K_1\times K_2$ form a solution to the SNVI $(1)-(2)$ problem, and let sequences $\{x^k\}$, and $\{y^k\}$ be generated by $Algorithm$ 2.3. Furthermore, let

(i) $0\le a^k\le 1$

(ii) $\Sigma_{k=0}^{\infty}a^k=\infty$

(iii) $0<\rho<\frac{2r}{\mu^2}$ and $0<\eta<\frac{2s}{\beta^2}$.
Then the sequence $\{(x^k,y^k)\}$ converges strongly to (x^*,y^*).

Proof. Since $(x^*,y^*)\in K_1\times K_2$ forms a solution to the SNVI $(1)-(2)$ problem, it follows that

$$x^*=P_K[x^*-\rho S(x^*,y^*)] \text{ and } y^*=Q_K[x^*-\eta T(x^*,y^*)].$$

Applying Algorithm 2.3, we have

$$\begin{aligned}
&\|x^{k+1}-x^*\|\\
&\quad= \|(1-a^k)x^k+a^kP_K[x^k-\rho S(x^k,y^k)]\\
&\quad- (1-a^k)x^*-a^kP_K[x^*-\rho S(x^*,y^*)]\|\\
&\quad\le (1-a^k)\|x^k-x^*\|\\
&\quad+ a^k\|P_K[x^k-\rho S(x^k,y^k)]-P_K[x^*-\rho S(x^*,y^*)]\|\\
&\quad\le (1-a^k)\|x^k-x^*\|+a^k\|x^k-x^*-\rho[S(x^k,y^k)-S(x^*,y^k)\\
&\quad+ S(x^*,y^k)-S(x^*,y^*)]\|\\
&\quad\le (1-a^k)\|x^k-x^*\|+a^k\|x^k-x^*-\rho[S(x^k,y^k)-S(x^*,y^k)]\|\\
&\quad+ a^k\rho\|[S(x^*,y^k)-S(x^*,y^*)]\|. \qquad (11)
\end{aligned}$$

Since S is strongly $(r)-$ monotone and $(\mu)-$Lipschitz continuous in the first variable, and S is $(\nu)-Lipschitz$ continuous in the second variable, we have

$$\begin{aligned}
& \|x^k - x^* - \rho[S(x^k,y^k) - S(x^*,y^k)]\|^2 \\
= \;& \|x - x^*\|^2 - 2\rho\langle S(x^k,y^k) - S(x^*,y^k), x^k - x^*\rangle \\
+ \;& \rho^2\|S(x^k,y^k) - S(x^*,y^k)\|^2 \\
= \;& \|x - x^*\|^2 - 2\rho\langle S(x^k,y^k) - S(x^*,y^k,x^k - x^*\rangle \\
+ \;& \rho^2\|S(x^k,y^k) - S(x^*,y^k)\|^2 \\
\leq \;& \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 \\
+ \;& \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\langle S(x^*,y^k), x^k - x^*\rangle \\
\leq \;& \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 \\
+ \;& \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\langle S(x^*,y^k), x^k - x^*\rangle \\
= \;& [1 - 2\rho r + \rho^2\mu^2]\|x^k - x^*\|^2
\end{aligned} \tag{12}$$

It follows that

$$\begin{aligned}
& \|x^{k+1} - x^*\| \\
\leq \;& (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| \\
+ \;& a^k\rho\nu\|y^k - y^*\|,
\end{aligned} \tag{13}$$

where $\theta = \sqrt{1 - 2\rho r + \rho^2\mu^2}$.
Similarly, we have

$$\begin{aligned}
\|y^{k+1} - y^*\| \leq (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| \\
+ a^k\eta\tau\|x^k - x^*\|,
\end{aligned} \tag{14}$$

where $\sigma = \sqrt{1 - 2\eta r + \eta^2\beta^2}$.
It follows from (13) and (14) that

$$\begin{aligned}
& \|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \\
\leq \;& (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| \\
+ \;& a^k\eta\tau\|x^k - x^*\| \\
+ \;& (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| \\
+ \;& a^k\rho\nu\|y^k - y^*\| \\
= \;& [1 - (1 - \delta)a^k](\|x^k - x^*\| + \|y^k - y^*\|) \\
\leq \;& \prod_{j=0}^{k}[1 - (1 - \delta)a^j](\|x^0 - x^*\| + \|y^0 - y^*\|),
\end{aligned} \tag{15}$$

where $\delta = max\{\theta + \eta\tau, \sigma + \rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|.\|^*$. Since $\delta < 1$ and $\sum_{k=0}^{\infty} a^k$ is divergent, it follows that

$$lim_{k\to\infty} \prod_{j=0}^{k} [1-(1-\delta)a^j] = 0 \, as \, k \to \infty.$$

Therefore,

$$\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \to 0,$$

and consequently, the sequence $\{(x^k, y^k)\}$ converges strongly to (x^*, y^*), a solution to the $SNVI\,(1)-(2)$ problem. This concludes the proof.

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