

Some Direct Estimates for Szasz-Beta Operators in L_p -Norm

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Abstract

In the year 1995, Vijay Gupta et al [3] introduced an interesting sequence of summation-integral type operators with Beta and Szasz basis functions in summation and integration respectively and studied some direct results for these operators. In this paper, we obtain direct theorems for the linear combination of Szasz-Beta type operators in L_p -approximation. To prove our direct theorems, we use the technique of linear approximating method viz. Steklov mean.

Key Words: Steklov mean, linear combination, linear positive operators, total variation.

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Introduction

For $f \in L_p[0, \infty)$, $p \geq 1$, V. Gupta et al [3] introduced an interesting sequence of linear positive operators to modify the well-known Beta operators by considering the weights of Szasz basis functions. The Beta-Szasz operators, introduced in [3] are defined by

$$B_n(f, x) = \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} b_{n,v}(y) f(y) dy, \quad x \in [0, \infty) \quad (1.1)$$

where

$$p_{n,v}(x) = \frac{e^{-nx} (nx)^v}{v!} \quad \text{and} \quad b_{n,v}(y) = \frac{1}{B(k+1, n)} \cdot \frac{y^k}{(1+y)^{n+k+1}}, \quad (1.2)$$

$B(k+1, n)$ denotes the Beta function given by $\frac{\Gamma(k+1)\Gamma n}{\Gamma(k+1+n)}$.

It is observed that the order of approximation for these operators is $O(n^{-1})$. To improve the order of approximation, we consider the linear combination of these operators $B_n(f, k, x)$ of the operators $B_{d_j n}(f, x)$ as

$$B_n(f, k, x) = \sum_{j=0}^k C(j, k) B_{d_j n}(f, x) \quad (1.3)$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^k \frac{d_j}{d_j - d_i}, \quad k \neq 0 \text{ and } C(0, 0) = 1. \quad (1.4)$$

We may rewrite the operators (1.1) as

$$B_n(f, x) = \int_0^\infty W(n, x, y) f(y) dy \quad (1.5)$$

where $W(n, x, y) = \sum_{v=0}^\infty p_{n,v}(x) b_{n,v}(y)$.

Alternately the k -th linear combinations $B_n(f, k, x)$ of the operators $B_{d_j n}(f, x)$ are defined by

$$B_n(f, k, x) = \begin{vmatrix} 1 & d_0^{-1} \dots d_0^{-k} \\ 1 & d_1^{-1} \dots d_1^{-k} \\ \dots & \dots \\ 1 & d_k^{-1} \dots d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} B_{d_0 n}(f, x) & d_0^{-1} \dots d_0^{-k} \\ B_{d_1 n}(f, x) & d_1^{-1} \dots d_1^{-k} \\ \dots & \dots \\ B_{d_k n}(f, x) & d_k^{-1} \dots d_k^{-k} \end{vmatrix} \quad (1.6)$$

where $d_0, d_1, d_2, \dots, d_k$ are $(k+1)$ arbitrary but fixed distinct positive numbers. Combinations of this type were considered by May [4].

Throughout this paper let $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, $0 < a < b < \infty$ and $I_i = [a_i, b_i]$, $i = 1, 2, 3$. For $f \in L_p[0, \infty)$, $1 \leq p < \infty$, the Steklov mean $f_{\eta, m}$ of m -th order corresponding to f is defined by

$$f_{\eta, m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} \dots \int_{-\eta/2}^{\eta/2} [f(t) + (-1)^{m-1} \Delta_{\sum_{i=1}^m} f(t) dt_1, dt_2, \dots, dt_m], \quad (1.7)$$

where $t \in I$ and $\Delta_h^m f(t)$ is the m -th order forward difference of the function f with step length h . It follows from [2, 5] that :

(i) $f_{\eta, m}$ has derivative up to order m , $f_{\eta, m}^{(m-1)} \in AC(I_1)$, and $f_{\eta, m}^{(m-1)}$ exists a.e. and belong to $L_p(I_1)$;

- (ii) $\|f_{\mu,m}^{(r)}\|_{L_p(I_2)} \leq K_1 \eta^{-r} \omega_r(f, \eta, p, I_1), r = 1, 2, \dots, m;$
- (iii) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leq K_2 \omega_m(f, \eta, p, I_1);$
- (iv) $\|f_{\eta,m}\|_{L_p(I_2)} \leq K_3 \|f\|_{L_p(I_1)};$
- (v) $\|f_{\mu,m}^{(r)}\|_{L_p(I_2)} \leq K_4 \eta^{-m} \|f\|_{L_p(I_1)}$

where $AC[a, b]$ stands for the class of absolutely continuous function on $[a, b]$ and K_i^s are certain constants defined on I , but are independent of f and n . Let $BV[a, b]$ denotes the set of all functions of bounded variation on $[a, b]$. The semi norm $\|f\|_{BV[a,b]}$ is defined by the total variation of f on $[a, b]$. For $f \in L_p[a, b], 1 < p < \infty$, the Hardy-Little wood majorant of f is defined as:

$$h_f(x) = \sup_{\xi \rightarrow x} \frac{1}{\xi - x} \int_x^\xi f(t) dt, (a \leq \xi \leq b).$$

In the present paper we establish some direct results on L_p -norm for the linear combinations of the Szasz-Beta operators.

Auxiliary Results

This section deals with certain basic Lemmas, which are necessary to prove the inverse theorem.

Lemma 2.1 [3]. Let the m -th order moment be defined by

$$V_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} b_{n,v}(y)(y-x)^m dy, \tag{2.1}$$

then

$$V_{n,0}(x) = 1, V_{n,1}(x) = \frac{x+1}{n-1}, V_{n,2} = \frac{(n+2)x^2 + 2x(2+n) + 2}{(n-1)(n-2)},$$

and there holds a recurrence

relation.

$$(n-m-1)V_{n,m+1}(x) = xV_{n,m}^{(1)}(x) + [(m+1)(2x+1) - x]V_{n,m}(x) + mx(2+x)V_{n,m-1}(x), m-1 \in N. \tag{2.2}$$

Consequently for each $x \geq 0$,

$$V_{n,m}(x) = O_x(n^{-[(m+1)/2]}), \tag{2.3}$$

where $[\alpha]$ denotes the integral part of α . Also by Holder's inequality we conclude from (2.3), for every fixed $x \in [0, \infty)$

$$B_n(|y-x|^r, x) = O(n^{-r/2}), \forall r > 0 \tag{2.4}$$

and for every $x \in [0, \infty)$.

Lemma 2.2. For $p \in N$ and n sufficiently large there holds

$$B_n[(y-x)^p, k, x] = n^{-(k+1)} \{Q(p, k, x) + O(1)\}, \quad y \in [0, \infty)$$

where $Q(p, k, x)$ are certain polynomials in x of degree $p/2$.

Proof: From Lemma 2.1, for sufficiently large n we can write,

$$B_n[(y-x)^p, x] = \frac{P_0(x)}{n^{\lfloor p+1 \rfloor / 2}} + \frac{P_1(x)}{n^{\lfloor p+1 \rfloor / 2}} + \dots + \frac{P_{\lfloor p/2 \rfloor}(y)}{n^p}$$

where P_i 's are certain polynomials in x of degree $\lfloor i/2 \rfloor$. Using the above in (1.3), we get the required result.

Lemma 2.3[1]. Let $1 \leq p \leq \infty$, $f \in L_p[a, b]$, $f^{(k)} \in AC[a, b]$ and $f^{(k+1)} \in L_p[a, b]$, then

$$\|f^{(j)}\|_{L_p[a, b]} \leq K_j^* \left(\|f^{(k+1)}\|_{L_p[a, b]} + \|f\|_{L_p[a, b]} \right)$$

$j=1, 2, \dots, k$ where K_j^* 's are certain constants depending only on j, k, p, a, b .

Main Results

Theorem 3.1. Let $f \in L_p[0, \infty)$, $p > 1$. If f has $(2k+2)$ derivatives on I_1 , with $f^{(2k+1)} \in AC(I_1)$ then for n sufficiently large

$$\|B_n(f, k, \cdot) - f\|_{L_p(I_2)} \leq K n^{-(k+1)} \left(\|f^{(2k+2)}\|_{L_p(I_2)} + \|f\|_{L_p[0, \infty)} \right)$$

where the constant K is independent of n and f .

Proof: By the hypothesis, for $x \in I_2$ and $y \in I_1$

$$\begin{aligned} f(y) &= \sum_{j=0}^{2k+1} \frac{(y-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!} \int_x^y (y-w)^{2k+1} f^{(2k+2)}(w) dw. \\ f(y) &= \sum_{j=0}^{2k+1} \frac{(y-x)^j}{j!} f^{(j)}(x) + \frac{1}{(2k+1)!} \int_x^y (y-w)^{2k+1} f^{(2k+2)}(w) dw + \\ &\quad + F(y, x)(1 - \Phi(y)), \end{aligned} \quad (3.1)$$

Where $\Phi(y)$ denotes the characteristic function on I_1 .

$$F(y, x) = f(y) - \sum_{j=0}^{2k+1} \frac{(y-x)^j}{j!} f^{(j)}(x),$$

For all $y \in [0, \infty)$ and $x \in I_2$. Using (3.1) in (1.3), we have

$$\begin{aligned} B_n(f, k, x) - f(x) &= \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!} B_n((y-x)^j, k, x) \\ &\quad + \frac{1}{(2k+1)!} B_n \Phi(y) \int_x^y ((y-w)^{2k+1} f^{(2k+2)}(w) dw, k, x) \end{aligned}$$

$$\begin{aligned}
 &+ B_n(F(y, x)(1 - \Phi(y), k, x) \\
 &= \Delta_1 + \Delta_2 + \Delta_3.
 \end{aligned}$$

In view of Lemma 2.2 and [1]

$$\begin{aligned}
 \|\Delta_1\|_{L_p(I_2)} &\leq K_5 n^{-(k+1)} \left(\sum_{j=1}^{2k+1} \|f^{(j)}\|_{L_p(I_2)} \right) \\
 &\leq K_6 n^{-(k+1)} \left(\|f^{(j)}\|_{L_p(I_2)} + \|f^{(2k+2)}\|_{L_p(I_2)} \right)
 \end{aligned}$$

To estimate I_2 , let h_f be the Hardy- Littlewood majorant [6] of $f^{(2k+2)}$ on I_1 , using Holder's inequality (2.4), we obtain

$$\begin{aligned}
 J_1 &= \left| B_n(\Phi(y)) \int_x^y (y-w)^{2k+1} f^{(2k+2)}(w) dw, x \right| \\
 &\leq B_n(\Phi(y)) \int_y^x (y-w)^{2k+1} \|f^{(2k+2)}(w)\| |dw|, x \\
 &\leq B_n(\Phi(y)) (y-x)^{2k+1} \int_x^y \|f^{(2k+2)}(w)\| |dw|, x \\
 &\leq B_n(\Phi(y)) (y-x)^{2k+2} |h_f(y)|, x \\
 &\leq \left\{ B_n(|y-x|^{(2k+2)q} \Phi(y), x) \right\}^{1/q} \cdot \left\{ B_n(|h_f(y)|^p \Phi(y), x) \right\}^{1/p} \\
 &\leq K_7 n^{-(k+1)} \left\{ B_n(|h_f(y)|^p \Phi(y), x) \right\}^{1/p} \\
 &\leq K_7 n^{-(k+1)} \left(\int_{a_1}^{b_1} W(n, x, y) |h_f(y)|^p dy \right)^{1/p}
 \end{aligned}$$

Fubini's theorem and [7, chapter 2] imply that

$$\begin{aligned}
 \|J_1\|_{L_p(I_2)}^p &\leq K_7 n^{-(k+1)p} \int_{a_2}^{b_2} \int_{a_1}^{b_1} W(n, x, y) |h_f(y)|^p dy dx \\
 &\leq K_7 n^{-(k+1)p} \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} W(n, x, y) dx \right] |h_f(y)|^p dy \\
 &\leq K_7 n^{-(k+1)p} \int_{a_1}^{b_1} |h_f(y)|^p dy \\
 &\leq K_7 n^{-(k+1)p} \|h_f(y)\|_{L_p(I_1)}^p \\
 &\leq K_8 n^{-(k+1)p} \|f^{(2k+2)}\|_{L_p(I_1)}^p.
 \end{aligned}$$

Therefore,

$$\|J_1\|_{L_p(I_2)} \leq K_8 n^{-(k+1)p} \|f^{(2k+2)}\|_{L_p(I_1)}$$

Consequently,

$$1 \quad \|\Delta_2\|_{L_p(I_2)} \leq K_9 n^{-(k+1)p} \|f^{(2k+2)}\|_{L_p(I_1)}$$

For $y \in [0, \infty) \setminus [a_1, b_1]$, $x \in I_2$, there exists a $\delta > 0$ such that $|y - x| \geq \delta$.

Thus

$$\begin{aligned} |B_n(F(y, x)(1 - \Phi(y), x))| &\leq \delta^{-(2k+2)} B_n(|F(y, x)|(y - x)^{2k+2}, x) \\ &\leq \delta^{-(2k+2)} B_n(|f(y)|) + \sum_{j=0}^{2k+1} \frac{|y - x|^j}{j!} |f^{(j)}(x)|(y - x)^{2k+2}, x) \\ &\leq \delta^{-(2k+2)} \left[B_n(|f(y)|(y - x)^{2k+2}, x) + \sum_{j=1}^{2k+1} \frac{|f^{(j)}(x)|}{j!} B_n(|y - x|^{2k+2+j}, x) \right] \\ &= J_2 + J_3. \end{aligned}$$

Using Holder's inequality and (2.4), we get

$$\begin{aligned} |J_2| &\leq \delta^{-(2k+2)} B_n(|f(x)|^p, x)^{1/p} B_n(|y - x|^{(2k+2)q}, x)^{1/q} \\ &\leq K_{10} n^{-(k+1)} B_n(|f(y)|^p, x)^{1/p}. \end{aligned}$$

Again applying Fubini's theorem, we get

$$\begin{aligned} \int_{a_2}^{b_2} |J_2|^p dt &\leq K_{10} n^{-(k+1)p} \int_{a_2}^{b_2} \int_0^\infty W(n, x, y) |f(y)|^p dy dx \\ &\leq K_{11} n^{-(k+1)p} \|f\|_{L_p([0, \infty))}. \end{aligned}$$

Thus

$$\|J_2\|_{L_p(I_2)} \leq K_{11} n^{-(k+1)} \|f\|_{L_p([0, \infty))}.$$

Moreover using (2.4) and [1], we get

$$\begin{aligned} \|J_1\|_{L_p(I_2)} &\leq K_{12} n^{-(k+1)} \sum_{j=0}^{2k+1} \|f^{(j)}\|_{L_p(I_2)} \\ &\leq K_{12} n^{-(k+1)} \left(\|f^{(j)}\|_{L_p(I_2)} + \|f^{(2k+2)}\|_{L_p(I_2)} \right) \end{aligned}$$

Combining the estimates of J_2 and J_1 , we are led to

$$\|\Delta_2\|_{L_p(I_2)} \leq K_{13} n^{-(k+1)} \left(\|f\|_{L_p([0, \infty))} + \|f^{(2k+2)}\|_{L_p(I_2)} \right)$$

Hence we obtain the desired result.

Theorem 3.2. Let $f \in L_1[0, \infty)$. If f has $(2k+1)$ derivatives in I_1 with

$$f^{(2k)} \in AC(I_1)$$

and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large we have

$$\|B_n(f, k, \cdot) - f\|_{L_1(I_2)} \leq Kn^{-(k+1)} \left(\|f^{(2k+1)}\|_{BV(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_p[0, \infty)} \right)$$

where K is a constant independent of f and n .

Proof. By the given assumption on f , for almost all $x \in I_2$ and for all $y \in I_1$, we have

$$f(y) = \sum_{i=0}^{2k+1} \frac{(y-x)^i}{i!} f^{(i)}(x) + \frac{1}{(2k+1)!} \int_x^y ((y-w)^{2k+1} df^{(2k+1)}(w)).$$

We can write

$$f(y) = \sum_{i=0}^{2k+1} \frac{(y-x)^i}{i!} f^{(i)}(x) + \frac{1}{(2k+1)!} \int_x^y ((y-w)^{2k+1} df^{(2k+1)}(w)) \Phi(y) + F(y, x)(1 - \Phi(y)).$$

where $\Phi(y)$ denotes the characteristic function of I_1 and

$$F(y, x) = f(y) - \sum_{i=0}^{2k+1} \frac{(y-x)^i}{i!} f^{(i)}(x).$$

For almost all $x \in I_2$ and for all $y \in [0, \infty)$. Hence we obtain

$$\begin{aligned} B_n(f, k, x) - f(x) &= \sum_{i=0}^{2k+1} \frac{f^{(i)}(x)}{i!} B_n((y-x)^i, k, x) \\ &\quad + \frac{1}{(2k+1)!} B_n \left(\int_y^x (y-w)^{2k+1} df^{(2k+1)}(w) \Phi(y), k, x \right) \\ &\quad + B_n(F(y, x)(1 - \Phi(y)), k, x) = J_1 + J_2 + J_3. \end{aligned}$$

Applying Lemma 2.1 and [1] we get

$$\|J_1\|_{L_1(I_2)} \leq K_{14} n^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right)$$

Further, we have

$$\begin{aligned} K &= \left\| B_n \left(\int_x^y (y-w)^{2k+1} df^{(2k+1)}(w) \Phi(y), x \right) \right\|_{L_1(I_2)} \\ &\leq \int_{a_2}^{b_2} \int_{a_1}^{b_1} W(n, x, y) |y-x|^{2k+1} \left| \int_x^y df^{(2k+1)}(w) \right| dy dx. \end{aligned}$$

For each n there exists a nonnegative integer $r = r(n)$ such that $rn^{-1/2} < \max(b_1 - a_2, b_2 - a_1) \leq (r+1)n^{-1/2}$. Then we have

$$\begin{aligned} K &\leq \sum_{l=0}^r \int_{a_2}^{b_2} \left\{ \int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \Phi(y) W(n, x, y) |y-x|^{2k+1} \right. \\ &\quad \left. \times \left[\int_x^{x+(l+1)n^{-1/2}} \Phi(w) \cdot |df^{(2k+1)}(w)| \right] dy \right\} \end{aligned}$$

$$\begin{aligned}
& + \int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} \Phi(y)W(n, x, y)|y-x|^{2k+1} \\
& \times \left[\int_{x-(l+1)n^{-1/2}}^x \Phi(w) \cdot |df^{(2k+1)}(w)| \right] dy \Big\} dx.
\end{aligned}$$

Let $\Phi_{x,c,d}(w)$ denotes the characteristic function of the interval $[x-cn^{-1/2}, x+dn^{-1/2}]$, where c, d are nonnegative integers. Then we have:

$$\begin{aligned}
K & \leq \sum_{l=1}^r \int_{a_2}^{b_2} \left\{ \int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \Phi(y)W(n, x, y)l^{-4}n^2|y-x|^{2k+5} \right. \\
& \quad \cdot \left[\int_x^{x+(l+1)n^{-1/2}} \Phi(w)\Phi_{x,0,l+1}(w) \cdot |df^{(2k+1)}(w)| \right] dy \\
& \quad + \int_{x-(l+1)n^{-1/2}}^{x-(l)n^{-1/2}} \Phi(y)W(n, x, y)l^{-4}n^2|y-x|^{2k+5} \\
& \quad \cdot \left[\int_{x-(l+1)n^{-1/2}}^x \Phi(w)\Phi_{x,l+1,0}(w) \cdot |df^{(2k+1)}(w)| \right] dy \Big\} dx \\
& \quad + \int_{a_2}^{b_2} \int_{-n^{-1/2}}^{a_1+n^{-1/2}} \Phi(y)W(n, x, y)|y-x|^{2k+1} \\
& \quad \times \left(\int_{x-n^{-1/2}}^{x+n^{-1/2}} \Phi(w)\Phi_{x,l,1}(w) \cdot |df^{(2k+1)}(w)| \right) dy dx \\
& \leq \sum_{l=1}^r \left(l^{-4}n^2 \int_{a_2}^{b_2} \left\{ \int_{x+(l)n^{-1/2}}^{x+(l+1)n^{-1/2}} \Phi(y)W(n, x, y)|y-x|^{2k+5} \right. \right. \\
& \quad \times \left(\int_{a_1}^{b_1} \Phi_{x,0,l+1}(w)|df^{(2k+1)}(w)| \right) dy \Big) \\
& \quad + \int_{x-(l+1)n^{-1/2}}^{x-(l)n^{-1/2}} \Phi(y)W(n, x, y)|y-x|^{2k+5} \\
& \quad \times \left(\int_{a_1}^{b_1} \Phi_{x,l+1,0}(w)|df^{(2k+1)}(w)| \right) dy \Big\} dx \\
& \quad + \int_{a_2}^{b_2} \int_{-n^{-1/2}}^{a_1+n^{-1/2}} \Phi(y)W(n, x, y)|y-x|^{2k+1} \\
& \quad \times \left(\int_{a_1}^{b_1} \Phi_{x,l,1}(w)|df^{(2k+1)}(w)| dy dx \right)
\end{aligned}$$

In the next step, using Lemma 2.1 and Fubini's theorem, we obtain

$$\begin{aligned}
K & \leq K_{15} n^{-(2k+1)/2} \left(\sum_{l=1}^r l^{-4} \left(\int_{a_2}^{b_2} \int_{a_1}^{b_1} \Phi_{x,0,l+1}(w)|df^{(2k+1)}(w)| dx \right. \right. \\
& \quad + \int_{a_2}^{b_2} \int_{a_1}^{b_1} \Phi_{x,l+1,0}(w)|df^{(2k+1)}(w)| dx \\
& \quad \left. \left. + \int_{a_2}^{b_2} \int_{a_1}^{b_1} \Phi_{x,l,1}(w)|df^{(2k+1)}(w)| dx \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= K_{16} n^{-(2k+1)/2} \left(\sum_{l=1}^r l^{-4} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \Phi_{x,0,l+1}(w) dx \left| df^{(2k+1)}(w) \right| \right) \right. \\
&\quad \left. + \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \Phi_{x,0,l+1}(w) dx \right) \left| df^{(2k+1)}(w) \right| \right) \\
&\quad + \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \Phi_{x,l,1}(w) dx \right) \left| df^{(2k+1)}(w) \right| \\
&\leq K_{17} n^{-(2k+1)/2} \left(\sum_{l=1}^r l^{-4} \left(\int_{a_2}^{b_2} \int_{w-(l+1)n^{-1/2}}^w dx \right) \left| df^{(2k+1)}(w) \right| \right. \\
&\quad \left. + \int_{a_1}^{b_1} \left(\int_w^{w+(l+1)n^{-1/2}} dx \right) \left| df^{(2k+1)}(w) \right| \right) \\
&\quad + \int_{a_1}^{b_1} \left(\int_{w-n^{-1/2}}^{w+n^{-1/2}} dx \right) \left| df^{(2k+1)}(w) \right| \right) \\
&\leq K_{18} n^{-(k+1)} \|f^{(2k+1)}\|_{BV(I_1)}.
\end{aligned}$$

Hence, $\|J_2\|_{L_1(I_2)} \leq K_{19} n^{-(k+1)} \|f^{(2k+1)}\|_{BV(I_1)}$, where the constant on the right side depends on k .

For all $y \in [0, \infty) \setminus [a_1, b_1]$, $x \in I_2$, we choose a $\delta > 0$ such that $|y - x| \geq \delta$.

Then

$$\begin{aligned}
\|B_n(F(y, x)(1 - \Phi(y)), x)\|_{L_1(I_2)} &\leq \int_{a_2}^{b_2} \int_0^\infty W(n, x, y) |f(y)| (1 - \Phi(y)) dy dx \\
&\quad + \sum_{i=0}^{2k+1} \frac{1}{i!} \int_{a_2}^{b_2} \int_0^\infty W(n, x, y) |f^{(i)}(x)| |y - x|^i (1 - \Phi(y)) dy dx \\
&= J_4 + J_5.
\end{aligned}$$

For sufficiently large y , there exist positive constants M_0 and K_{20} such that $\frac{(y-x)^{2k+2}}{y^{2k+2}+1} > K_{20}$, for all $y \geq M_0$, $x \in I_2$. Also by Fubini's theorem

$$J_4 = \left(\int_0^{M_0} \int_{a_2}^{b_2} + \int_{M_0}^\infty \int_{a_2}^{b_2} \right) W(n, x, y) |f(y)| (1 - \Phi(y)) dx dy = J_6 + J_7.$$

Now, using Lemma 2.1, we have

$$J_6 = \delta^{-(2k+2)} \int_0^{M_0} \int_{a_2}^{b_2} W(n, x, y) |f(y)| (y-x)^{2k+2} dx dy$$

$$\leq K_{21} n^{-(k+1)} \left(\int_0^{M_0} |f(y)| dy \right)$$

And

$$\begin{aligned} J_7 &= \frac{1}{K_{20}} \int_{M_0}^{\infty} \int_{a_2}^{b_2} W(n, x, y) \frac{(y-x)^{2k+2}}{y^{2k+2} + 1} |f(y)| dx dy \\ &\leq K_{22} n^{-(k+1)} \left(\int_{M_0}^{\infty} |f(y)| dy \right) \end{aligned}$$

Combining the estimates of J_6 and J_7 , we get

$$J_4 \leq K_{23} n^{-(k+1)} \|f\|_{L_1[0, \infty)}$$

Further, using (2.4) and [1], we get

$$\begin{aligned} J_5 &\leq \delta^{-(2k+2)} \sum_{i=0}^{2k+1} \frac{1}{i!} \int_{a_2}^{b_2} \int_0^{\infty} W(n, x, y) |f^{(i)}(x)| (y-x)^{2k+i+2} dy dx \\ &\leq K_{24} n^{-(k+1)} \left(\sum_{i=0}^{2k+1} \|f^{(i)}\|_{L_1(I_2)} \right) \\ &\leq K_{24} n^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right) \end{aligned}$$

From above estimates of J_4 and J_5 , we get

$$\|B_n(F(y, x)(1 - \Phi(y)), x)\|_{L_1(I_2)} \leq K_{25} n^{-(k+1)} \left(\|f\|_{L_1[0, \infty)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right)$$

Consequently we obtain

$$\|J_3\|_{L_1(I_2)} \leq K_{26} n^{-(k+1)} \left(\|f\|_{L_1[0, \infty)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right)$$

Finally combining the estimates of J_1, J_2 and J_3 , we obtain the required result.

Theorem 3.3. Let $f \in L_p[0, \infty)$, $p \geq 1$. then for n sufficiently large

$$\|B_n(f, k, \cdot) - f\|_{L_p(I_2)} \leq K_k \left(\omega_{2k+2}(f, n^{-1/2}, p, I_1) + n^{-(k+1)} \|f\|_{L_p[0, \infty)} \right)$$

where K_k is a constant independent of f and n .

Proof. Let $f_{\eta, 2k+2}(y)$ be the Steklov mean of $(2k+2)$ -th order corresponding to $f(y)$ where $\eta > 0$ is sufficiently small and $f(y)$ is defined as zero outside $[0, \infty)$. Then we have

$$\begin{aligned} \|B_n(f, k, \cdot) - f\|_{L_p(I_2)} &\leq \|B_n(f - f_{\eta, 2k+2}, k, \cdot)\|_{L_p(I_2)} + \|B_n(f - f_{\eta, 2k+2}, k, \cdot) - f_{\eta, 2k+2}\|_{L_p(I_2)} \\ &\quad + \|f_{\eta, 2k+2} - f\|_{L_p(I_2)} \\ &= \Delta_1 + \Delta_2 + \Delta_3 \text{ (Say)}. \end{aligned}$$

To estimate Δ_1 , let $\Phi(y)$ be the characteristic function of I_3 ; then

$$\begin{aligned} B_n((f - f_{\eta,2k+2})(y), x) &= B_n(\Phi(y)(f - f_{\eta,2k+2})(y), x) \\ &\quad + B_n(1 - \Phi(y)(f - f_{\eta,2k+2})(y), x) \\ &= \Delta_4 + \Delta_5. \end{aligned}$$

The following is true for $p=1$, and it is also same for $p>1$ follows from Holder's inequality

$$\int_{a_2}^{b_2} |\Delta_4|^p dy \leq \int_{a_2}^{b_2} \int_{a_3}^{b_3} W(n, x, y) |(f - f_{\eta,2k+2})(y)|^p dy dx.$$

On applying Fubini's theorem, we get

$$\begin{aligned} \int_{a_2}^{b_2} |\Delta_4|^p dy &\leq \int_{a_2}^{b_2} \int_{a_3}^{b_3} W(n, x, y) |(f - f_{\eta,2k+2})(y)|^p dx dy \\ &\leq \|f - f_{\eta,2k+2}\|_{L_p(I_3)}^p. \end{aligned}$$

Hence

$$\|\Delta_4\|_{L_p(I_2)} \leq \|f - f_{\eta,2k+2}\|_{L_p(I_3)}.$$

Using Holder's inequality, (2.4) and Fubini's theorem, we get the following for $p \geq 1$

$$\|\Delta_5\|_{L_p(I_2)} \leq K_{27} n^{-(k+1)} \|f - f_{\eta,2k+2}\|_{L_p[0,\infty)}.$$

By using Jenson's inequality and Fubini's theorem, we obtain

$$\|f_{\eta,2k+2}\|_{L_p[0,\infty)} \leq K_{28} \|f\|_{L_p[0,\infty)}.$$

Hence

$$\|\Delta_5\|_{L_p[0,\infty)} \leq K_{29} n^{-(k+1)} \|f\|_{L_p[0,\infty)}.$$

Now using third property of Steklov means, we get

$$\Delta_1 \leq K_{30} \left(\omega_{2k+2}(f, n, p, I_1) + n^{-(k+1)} \|f\|_{L_p[0,\infty)} \right)$$

And we know that,

$$\|f_{\eta,2k+2}\|_{BV(I_3)}^{(2k+1)} = \|f_{\eta,2k+2}\|_{L_1(I_3)}^{(2k+2)}$$

Hence by virtue of theorem 3.1 ($p>1$), theorem 3.2 ($p=1$) and Lemma 2.3, we have

$$\begin{aligned} \Delta_2 &\leq K_{31} n^{-(k+1)} \left(\|f_{\eta,2k+2}\|_{L_p(I_3)}^{(2k+2)} + \|f_{\eta,2k+2}\|_{L_p[0,\infty)} \right) \\ &\leq K_{32} \left(\eta^{-(2k+2)} \omega_{2k+2}(f, n, p, I_1) + n^{-(k+1)} \|f\|_{L_p[0,\infty)} \right) \end{aligned}$$

In view of the properties of Steklov means.

To estimate Δ_3 , we use the Steklov means property third, and obtain that

$$\Delta_3 \leq K_{32} \omega_{2k+2}(f, n, p, I_1).$$

Hence the required result follows. This completes the proof of theorem 3.3.

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