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Adjoint operators which are Weak* to weak sequentially continuous

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Abstract

In this paper $W_*(E,F)$ of all bounded operators from E to F which their adjoints are weak* to weak sequentially continuous are considered. We characterize some important classical properties. We will show $W_*(E,F)$ is a subspace of $\mathscr{U}(E,F)$ of all unconditionally converging operators from E to F if F is a Gelfand-Phillips space. We show that the elements of $W_*(E^{***},E^*)$ are adjoints of elements in $L(E,E^*)$ if E^* is an injective Banach space or E is a complemented subspace of E^{**}.

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1. Introduction and Preliminaries

Many authors have studied some class of operators and characterized some important properties; see [1], [2], [4], [9] and [10]. Here, we would study the Banach space $W_*(E,F)$ of all bounded operators from E to F which their adjoints are weak* to weak sequentially continuous. We will compare $W_*(E,F)$ with some class of operators, limited operators, unconditionally converging operators and weakly compact operators. Throughout E and F would be Banach spaces, B_E the unit ball of E, $L(E,F)$ the Banach space of all bounded linear maps from E to F. For (E,τ), a sequence (x_n) is τ null if $x_n \to 0$ in τ. All notations and notions used and not defined can be found in [3].

Proposition 1.1. *$W_*(E,F)$ is a Banach space.*

Proof. We need only show that $W_*(E,F)$ is a closed subspace of $L(E,F)$. Suppose $T \in L(E,F)$ is in the closure of $W_*(E,F)$, so there is a sequence $(T_n)_n$ in $W_*(E,F)$ such that $T_n \to T$. We claim that $T \in W_*(E,F)$. To this aim, assume (y_n^*) is a weak* null sequence in F^*, since $T_n \rightarrow T$ implies that $T_m^* \rightarrow T^*$ in norm topology, therefore

$$\| T^* y_n^* \| \leq \| T^* - T_m^* \| \, (sup_n \| y_n^* \|) + \| T_m^*(y_n^*) \|,$$

for fixed but enough large m which it implies that the right hand of above inequality tend to zero and proof has been completed. □

A Banach space E with the property that weak* null sequences in E^* are weakly null are often called *Grothendieck spaces*. In the following proposition we will see that existing an operator $T \in W_*(E,F)$ can make a situation for which F be a Grothendieck space.

Proposition 1.2. *A sequence (y_n^*) would tend to zero in topology $\sigma(F^*, T^{**}(E^{**}))$ if it converges to zero in weak* topology, where $T \in W_*(E,F)$. If $F \subseteq T^{**}E^{**}$ then weak* topology and $\sigma(F^*, T^{**}(E^{**}))$ topology on sequences in F coincide.*

Proof.

$$\begin{aligned} T \in W_*(E,F) \iff & [y_n^* \xrightarrow{w^*} 0 \Rightarrow x^{**}(T^*(y_n^*)) \rightarrow 0 \ \forall x^{**} \in E^{**}] \\ \iff & [y_n^* \xrightarrow{w^*} 0 \Rightarrow T^{**}x^{**}(y_n^*) \rightarrow 0 \ \forall x^{**} \in E^{**}] \\ \iff & [y_n^* \xrightarrow{w^*} 0 \Rightarrow y_n^* \xrightarrow{\sigma(F^*, T^{**}E^{**})} 0]. \end{aligned}$$

Second assertion would be clear. □

Definition 1.3. A subset L of a Banach space E is *limited* if $Lim_n sup_{x \in L} \mid x_n^* x \mid = 0$, where (y_n^*) is a is any sequence in E^* which converges to zero under weak* topology in E^*. A Banach space E for which all limited sets are relatively compact is said to be a *Gelfand-Phillips space*. An operator $T : E \rightarrow F$ is *limited operator* if $T(B_E)$ is a limited set [2]. The Banach space of all limited operator from E to F is denoted by $\mathscr{L}im(E,F)$.

Proposition 1.4. *Suppose $T \in W_*(E,F)$ and F is a Gelfand-Phillips space, then*

(a) $TS \in \mathscr{L}im(c_0, F)$ if $T \in W_*(E,F)$ and $S \in L(c_0, E)$.

(b) Let $\sum x_n$ be a w.u.C series in E. There is a subsequence (x_{n_k}) of (x_n) for which $\| T x_{n_k} \| \rightarrow 0$.

Proof. If (a) is in effect, since $(TS)^*$ is weak* to norm sequentially continuous, so TS is a limited operator [2].

To prove (b), in correspondence to a *w.u.C* series $\sum x_n$ there is an operator $S : c_0 \rightarrow E$ which $S(e_n) = x_n$. From (a), $TS(B_{c_0})$ is a limited set in F and from the assumption $TS(B_{c_0})$ is relatively compact, so $(T(x_n)) = (TS(e_n))$ would be relatively compact. Therefore, there is a subsequence (x_{n_k}) such that $(T(x_{n_k}))$ is norm convergent but since it is weakly null so it would be a norm null sequence. □

An operator $T : E \rightarrow F$ is called an *unconditionally converging* if it $\sum T(x_n)$ is unconditionally convergent whenever $\sum x_n$ is *w.u.C*. We denote by $\mathscr{U}(E,F)$ the Banach space of all unconditionally converging operator from E to F.

Proposition 1.5. *Suppose $T \in L(E,F)$, then*

(a) $T \in \mathscr{U}(E,F)$ if and only if $(\| T x_n \|)_n$ be a norm null for any w.u.C series $\sum x_n$ in E.

(b) $T \in \mathscr{U}(E,F)$ if $T \in W_*(E,F)$ and F be a Gelfand-Phillips space.

Proof. (a) follows directly, that is $\| Tx_n \| \to 0$ if T is unconditionally converging operator and $\sum x_n$ is a $w.u.C$ series in E. For the converse suppose T is not unconditionally converging. Then T fixes a copy of c_0 [3], so there is a $w.u.C$ series $\sum x_n$ such that $\sum Tx_n$ is $w.u.C$ but it is not unconditionally convergence. Therefore, there is a block sequence $y_n = \sum_{i=q_n+1}^{q_{n+1}} Tx_i$ such that $\sum y_n$ is $w.u.C$ but $inf \| y_n \| > 0$. Choosing $z_n = \sum_{i=q_n+1}^{q_{n+1}} x_i$, then $\sum z_i$ is $w.u.C$ and $inf_n \| Tz_n \| > 0$. But T is unconditionally converging operator. It shows $\| Tz_n \| \to 0$ which is a contradiction. According to part (b), if T is not unconditionally converging operator there is a bock sequence $z_n = \sum_{i=q_n+1}^{q_{n+1}} x_i$ such that $\sum z_i$ is $w.u.C$ but $inf_n \| Tz_n \| > 0$. From proposition 3 part (b) there is a subsequence (z_{n_k}) such that $\| Tz_{n_k} \| \to 0$ which is a contradiction. □

Definition 1.6. A locally convex space (E,τ) has the *Mazur Property* iff every linear τ-sequential continuous functional is τ-continuous (see [15]). In the Banach space setting, a Banach space X is a *Mazur Space* iff the dual space E^* endowed with the weak* topology has the Mazur property. The Mazur property was introduced by S.Mazur, and for Banach spaces it is investigated in details in [4].

In other result we would like to compare $W_*(E,F)$ with some other classical operator spaces and characterize some classical properties in Banach spaces.

Proposition 1.7. *$W_*(E,F) \subseteq W(E,F)$ (the Banach space of all weakly compact operators from E to F) if one of the following conditions hold:*

(a) F^* contain no copy of ℓ_1,

(b) F is a separable Banach space,

(c) F is a Mazur space.

Proof. (a): From ℓ_1-Rosenthal theorem [3] any bounded sequence in F^* has a weakly Cauchy subsequence (y_n^*). Therefore, it converges to an element $y^* \in F^*$ in weak* topology. It shows that $T^*y_n^* \xrightarrow{w} T^*y^*$ where, $T \in W_*(E,F)$. Therefore, T is a weakly compact operator.

b): B_{F^*} is weak* sequentially compact so any bounded sequence in F^* has a weak* convergent subsequence (y_n^*) and the rest of proof is similar to a).
c) For a weak* null sequence (y_n^*) in F^* and $T \in W_*(E,F)$ the sequence $(T^*y_n^*)_n$ converges to zero in weak topology of E^*. Therefore, $T^{**}x^{**}(y_n^*) = x^{**}(T^*y_n^*) \to 0$ where, $x^{**} \in E^{**}$. It means that $T^{**}x^{**}$ is weak* sequentially continuous but F is a Mazur space so $T^{**}E^{**} \subseteq F$ and then T is a weakly compact operator. □

We remark that the converse of proposition 5 may be not true. In fact if E is a reflexive Banach space then for any Banach space F, $W_*(E,F) = L(E,F) = W(E,F)$. Next result would partially characterizes Grothendieck spaces.

PROPOSITION 6.
a) $L(E,F) = W_*(E,F)$ if F or F be Grothendieck spaces.

b) Suppose $L(E,F) = W_*(E,F)$ is valid for any Banach space F, then E is a Grothendieck space.

PROOF. Since T^* is weak* to weak* continuous and under assumption $T \in W_*(E,F)$ where, $T \in L(E,F)$ this proves (a). For (b) set $F = E$. Then from $I \in L(E,E)$ we see that I^* must be weak* to weak sequentially continuous so E is a Grothendieck space.

Definition 1.8. Suppose $T : E \to F$ is a bounded operator.
(a) T is called a *completely continuous operator* if T maps weakly convergent sequences in to norm convergent sequences.

(b) T is called a *weakly completely continuous operator* if T maps weakly Cauchy sequences in E to weakly convergent sequences in F.

(c) E is said to be an *injective space* if for any linear sbspace Y of a Banach space F and any bounded operator $T : Y \to E$, T may extended to a bounded operator $S : F \to E$ having the same norm as T.

(d) E is a *Schur space* if any weakly null sequence is norm null sequence.

In the next result we would compare $W_*(E,F)$ and $cc(E,F)$ the Banach space of all completely continuous operators from E to F.

Proposition 1.9. *Suppose E does not have the Schur property. Then $W_*(E,\ell_\infty)$ does not lie in $cc(E,\ell_\infty)$.*

Proof. Under the assumption there is a weakly null (x_n) in E such that $\| x_n \| = 1$ $(\forall n \in \mathcal{N})$. From Bessaga-Pelczynski's theorem ([3] , page 46), (x_n) can be considered as a basic sequence in E. Suppose (x_n^*) is its coefficient functionals on $[x_n]$. From injectivity of ℓ_∞ ([3] , page 71), one can consider (x_n^*) in E^*. Define $S : E \to \ell_\infty$ by $S(x) = (x_n^*(x))_n$ which lies easily in $W_*(E,\ell_\infty)$ because ℓ_∞ is a Grothendieck space ([3] , page 103). Since (x_n) is a weakly null sequence in E and

$$\| S(x_m) \| \geq | x_m^*(x_m) | = 1,$$

$(S(x_n))_n$ is not norm null so S does not lie in $cc(E,\ell_\infty)$. □

Proposition 1.10. *Suppose $T \in W_*(E,E^*)$, then $\tilde{T} = T^*|_E : E \to E^*$ is weakly completely continuous operator.*

Proof. Assume (x_n) is a weakly Cauchy sequence in E. Then it converges to an element x^{**} of E^{**} in weak* topology.

$$\begin{aligned} x^{**} \quad & -x_n \xrightarrow{w^*} 0 \\ T^* \quad & (x^{**} - x_n) \xrightarrow{w} 0 \\ T^* \quad & x^{**} - \tilde{T}(x_n) \xrightarrow{w} 0. \end{aligned}$$

Therefore, $(\tilde{T}(x_n))$ in E^* converges weakly to $T^* x^{**}$ in E^*. □

Definition 1.11. If $i: E \to E^{**}$ is the canonical embedding then E is said to have the *weak Phillips property* (resp. *Phillips property*) if $i \in W_*(E, E^{**})$ (resp. $i^* \in W^*(E^{***}, E^*)$) [13].

In the next result we show more connection between $W_*(E,F)$ and weak Phillips property.

Proposition 1.12. *Suppose E^* is an injective Banach space or E is complemented in E^{**}. Then*
(a) $W_*(E,E^*) = L(E,E^*)$ if E has the weak Phillips property.
(b) The adjoints of elements of $L(E,E^*)$ is equal to $W_*(E^{***},E^*)$.

Proof. (a): Consider $T \in L(E,E^*)$ then from the assumption T can be extended to $S: E^{**} \to E^*$. Since $Si = T$, so $T^* = i^*S^*$. i^* is weak* to weak sequentially continuous and then T lies in $W_*(E,E^*)$. (b) is similar. □

Definition 1.13. A subset L of a Banach space E is a V^* *set* if $Lim_n sup_{x\in L} \mid x_n^*(x) \mid \to 0$, where $\sum x_n^*$ is any *w.u.C* series in E^*. The Banach space of all bounded operators $T: E \to F$ for which $T(B_E)$ is a V^* set in F (it is called a V^* operator) is denoted by $V^*(E,F)$. A subset L from E^* is a V*- set* if $Lim_n sup_{x^*\in L} \mid x^*(x_n) \mid \to 0$, where $\sum x_n$ is a *w.u.C* series in E. $T: E \to F$ is called a V*-operator* if $T^*(B_{F^*})$ be a V-set in E^* [10]. The Banach space of all V-operators $T: E \to F$ is denoted by $\mathscr{V}(E,F)$.

Proposition 1.14. *Suppose $T \in L(E,F)$. Then*
(a) $T^* \in \mathscr{U}(F^*,E^*) \Longleftrightarrow T \in \mathscr{V}^*(E,F)$.
(b) $T \in \mathscr{U}(E,F) \Longleftrightarrow T \in \mathscr{V}(E,F)$.

Proof. (a): $(\Rightarrow)$: Suppose $\sum y_n^*$ is *w.u.C* in F^*. Then by proposition 4, $\| T^*y_n^* \| \to 0$ i.e; $sup_{x\in B_E} \mid y_n^*(T(x)) \mid \to 0$. Therefore, $T(B_E)$ is a V^* set.
$(\Leftarrow)$: From the assumption $T(B_E)$ is a V^* set. Hence, given a *w.u.C* series $\sum y_n^*$ in F^*, $\| T^*y_n^* \| = sup_{x\in B_E} \mid y_n^*(Tx) \mid \to 0$. Consequently, T would be an unconditionally converging operator. (b) is similar to (a). □

Proposition 1.15. $cc(E,F) \subseteq \mathscr{U}(E,F)$.

Proof. Suppose $T \in cc(E,F)$ and $\sum x_n$ is any *w.u.C* in E. Since (x_n) is a weakly null sequence so $\| T(x_n) \| \xrightarrow{\|\cdot\|} 0$. Therefore, T is unconditionally converging operator. □

The following example shows that the inclusion in proposition 1.15 can be strictly.

Example 1.16. [8] *R. C. James constructed an example of a Banach space X such that the $2n$-dual of X has a copy of ℓ_1, $X^{(2n-2)}$ has not a copy of ℓ_1 and $X^{(2n-1)}$ has not any copies of ℓ_1 and c_0. Therefore, $I: X^{(2n-2)} \to X^{(2n-2)}$ can not fix a copy of c_0, so it is unconditionally converging operator. On the other hand I being completely continuous is equivalent to say that $X^{(2n-1)}$ has the Schur property. But F is infinite dimensional and $X^{(2n-1)}$ has not a copy of ℓ_1 so $X^{(2n-1)}$ can not have the Schur property.*

For the other result we note the following result for obtaining proposition 1.18.

Proposition 1.17. [6]. *The following are equivalent:*

(a) $K \subseteq E^*$ *is a* V*-set.*

(b) $S^*(K)$ *is relatively compact if* $S \in L(c_0, E)$ *is any bounded operator.*

Proposition 1.18.

$$T \in \mathscr{U}(E,F) \Longleftrightarrow [TS \in K(c_0,F)(\forall S \in L(c_0,E))],$$

where $K(c_0,F)$ *is the Banach space of all compact operators from* c_0 *to* E.

Proof. $(\Rightarrow)$ follows from proposition 1.17.
$(\Leftarrow)$: On the contrary suppose T is not unconditionally converging operator. Therefore, it fixes a copy of c_0. There is an isomorphism $S : c_0 \to E$ such that $TS\mid_{c_0}$ is an isomorphism. Therefore, $\{T(S(e_n)) \mid n \in \mathbf{N}\}$ must be relatively compact so $\{e_n \mid n \in \mathbf{N}\}$ must be relatively compact which is impossible. □

Proposition 1.19. *(a)* $W_*(E,F)$ *has a copy of* c_0 *if* E^* *or* F *contains a copy of* c_0.

(b) If $W_*(E,F)$ *has a complemented copy of* c_0 *then* E^* *or* F^{**} *contains a copy of* c_0 .

Proof. To prove (a), it is well known that there is a weak* null but normalized sequence (y_n^*) in F^* ([3], page 219). Suppose (x_n^*) is a c_0-basic sequence in E^*. Choose (y_n) in F such that $1 \leq \| y_n \| \leq 2$; $y_n^* y_n = 1$ $(\forall n \in \mathbf{N})$. From the assumption there are $C_1, C_2 > 0$ such that

$$C_1 sup_n \mid a_n \mid \leq \| \sum a_n x_n^* \| \leq C_2 sup_n \mid a_n \mid .$$

On the other hand we can consider $(x_n^* \otimes y_n)_n$ as a sequence in $W_*(E,F)$. We claim that it is a c_0-basic sequence.

$$\begin{aligned} \| \sum a_n x_n^* \otimes y_n \| &= \sup_{\|y^*\| \leq 1} \| \sum a_n x_n^* y^*(y_n) \| \\ &\geq \frac{C_1}{2} \mid (y_n^* y_n) a_n \mid \ \forall n. \end{aligned}$$

Therefore, $\| \sum a_n x_n^* \otimes y_n \| \geq \frac{C_1}{2} sup_n \mid a_n \mid$. On the other hand

$$\begin{aligned} \| \sum a_n x_n^* \otimes y_n \| &= \sup_{\|y^*\| \leq 1} \| \sum a_n x_n^* y^*(y_n) \| \\ &\leq \sup_{\|y^*\| \leq 1} C_2 \sup_n \mid (y_n^* y_n) a_n \mid \\ &\leq C_2 \sup_n \mid a_n \mid . \end{aligned}$$

Now suppose F contains a c_0-basic sequence (y_n). Consider $T_n = x_0^* \otimes y_n$ where, $x_0^* \in E^*$; $\| x_0^* \| = 1$. Again we show that (T_n) is a c_0-basic sequence in $W_*(E,F)$. Suppose

$C_1, C_2 > 0$ are such that for any $(a_n) \in c_0$,

$$C_1 \sup_n \mid a_n \mid \leq \| \sum a_n y_n \| \leq C_2 \sup_n \mid a_n \mid .$$

Then

$$\begin{aligned} C_1 \sup_n \mid a_n \mid &= \sup_{\|x\| \leq 1} C_1 \sup_n \mid a_n x_n)^*(x) \mid \\ &\leq \| \sum a_n x_0^* \otimes y_n \| \\ &= \sup_{\|x\| \leq 1} \| \sum a_n x_0^*(x) y_n \| \\ &\leq \sup_{\|x\| \leq 1} C_2 sup_n \mid a_n x_n)^*(x) \mid \\ &\leq C_2 sup_n \mid a_n \mid . \end{aligned}$$

To establish (b), suppose that (T_n) is a c_0-basic sequence in $W_*(E,F)$. Therefore, $\sum \eta_n T_n$ is a $w.u.C$ in $W_*(E,F)$. It is easy to see that for any $y^* \in F^*$ and $x^{**} \in E^{**}$, series $\sum T_n^* y^*$ and $\sum T^{**} x^{**}$ are $w.u.C$. Now on the contrary suppose that E^{**} and F^* do not have any copy of c_o. Therefore, $\sum T_n^* y^*$ and $\sum T^{**} x^{**}$ are unconditionally converging for each $x^{**} \in E^{**}$ and $y^* \in F^*$. The series $\sum \eta_n T_n$ is convergent in strong operator topology. From the closed graph theorem one can easily show that $\varphi(\eta) = \sum \eta_n T_n$ is a bounded operator in $L(E,F)$, where $\eta = (\eta_n)_n \in \ell_\infty$ also again by using the closed graph theorem $\varphi : \ell_\infty \to L(E,F)$ by $\varphi(\eta) = \sum \eta_n T_n$ is a bounded operator. Now suppose $y_n^* \xrightarrow{weak^*} 0$ in F^*. Fixed x^{**} in E^{**}. Then $\sum \eta_n T_n^{**} x^{**}$ is unconditionally converging series since it is $w.u.C$ and F^{**} does not contain a copy of c_0 so, $\varphi(\eta)^* y_n^* \xrightarrow{weak} 0$. It shows that the range of φ lies in $W_*(E,F)$. From the assumption there is a bounded projection $P : W_*(E,F) \to H$, where H is a subspace of $W_*(E,F)$ isomorphic to c_0. It shows that $P\varphi$ is a surjection bounded operator from ℓ_∞ to c_0 which is a contradiction. □

Proposition 1.20. *Let E^* and F^{**} do not contain a copy of c_0. Then $W_*(E,F)$ contain a copy of c_0 if and only if $W_*(E,F)$ contain a copy of ℓ_∞.*

Proof. We prove non-trivial direction. Suppose (T_n) is a copy of c_0 in $W_*(E,F)$. Similar to the proof of proposition 14 there is a bounded operator $\varphi : \ell_\infty \to W_*(E,F)$. But $\varphi(e_n) = T_n$ does not converge in norm to zero. Therefore, there is an infinite subset M of $\mathbf{N}$ such that $\varphi \mid_{\ell_\infty(M)} : \ell_\infty(M) \to W_*(E,F)$ is an isomorphism. But $\ell_\infty(M)$ and ℓ_∞ are isomorphic, this completes the proof. □

References

[1] M. Alimohammady, *Weak*-norm sequentially continuous operators*, Math. Slovaca, **50**(2000). no 30, 357–363.

[2] J. Bourgain and J. Diestel, *Limited operators and strictly cosingularity*, Math. Nachr. **119**(1984), 55–58.

[3] J. Diestel, *Sequences and series in Banach spaces*, Graduate texts in Math. Springer-Verlag, (1984).

[4] G. Edgar, *Measureability in Banach spaces II*, Ind.Math.J. 28, (1979), 559- 579.

[5] G. Emmanuele, *On the containment of c_0 by the space of compact operators*, Bull. Sci. Math. **115**(1991), 177–183.
[6] G. Emmanuele, *A remark on the containment of c_0 in space of compact operators*, Math. Proc. Cambridge. Philos. Soc. **111**(1992), 331–335.
[7] G. Emmanuele, *Some permanence results of properties of Banach Spaces*, to appear on Comm. Math. Univ. Carol.
[8] J. C. Ferrando, *Copies of c_0 in certain vector valued function Banach spaces*, Math. Scand. **77**(1995), 148–152.
[9] R. C. James, *separable conjugate spaces*, Pacific J. Math. **10**(1960), 563–571.
[10] N. J. Kalton, *Spaces of compact operators*, Math. Ann. **208**(1974), 267–278.
[11] A. Pelczynski, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Publ. Sci. **10**(1962), 641–648.
[12] H. P. Rosenthal, *On relatively disjoint families of measures with some applications to Banach space theory*, Studia Math. **32**(1970), 13–36.
[13] T. Schlumprecht, Limited sets in Banach spaces, Ph.D Dissertation, München, 1987.
[14] Ali Ulger *The Weak Phillips Property*, Colloq. Mathematicum, 87 (2001), 147-158.
[15] A. Wilansky, *Modern methods in topological vector spaces*, McGraw Hill, New York (1978).