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Semigroup and some Fractional Stochastic Integral Equations

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Abstract

In this paper we study the existence and uniqueness of mild solutions to a fractional stochastic integral equations in the general form

$$u(t)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{F(u(\theta))}{(t-\theta)^{1-\alpha}}d\theta+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{G(u(\theta))}{(t-\theta)^{1-\alpha}}dW(\theta)+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{Au(\theta)}{(t-\theta)^{1-\alpha}}d\theta,$$

where $0\le t\le T,\ \frac{1}{2}\le\alpha\le 1,\ u(0)=u_0,$ *in a real Hilbert space* H . Here $A:D(A)\subset H\to H$ is a linear closed operator generating semigroup,
$F:\gamma([0,T];H)\to L^p([0,T];L^2(\Omega;H))\ \ (1\le p<\infty)$,
$G:\gamma([0,T];H)\to C([0,T];L^2(\Omega;BL(K;H)))$ (where K is a real separable Hilbert space) , W is a K- valued Wiener process with incremental covariance described by the nuclear operator Q and u_0 is an Z_0- measurable H-valued .

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1. Introduction

Let H and K denote real Hilbert space equipped with norm $\|\ .\ \|_H$ and $\|\ .\ \|_K$,respectively , and the space of bounded linear operators from K to H is denoted by BL(K;H). Also, for Banach space X and Y , the space of continuous functions from X into Y (equipped with the usual sup-norm) shall be denoted by $C(X;Y)$, while $L^p(0,T;X)$ shall represent the space of X- valued functions that are p-integrable on [0,T]. Let (Ω,Z,P) be a complete probability space equipped with a normal filtration $\{Z_t:0\le t\le T\}$. An H-valued

random variable is an Z-measurable function $X : \Omega \to H$ and a collection of random variables $\psi = \{X(t;\omega) : \Omega \to H|\ 0 \le t \le T\}$ is called a stochastic process. The collection of all strongly measurable square integrable H-valued random variables, denoted by $L^2(\Omega;H)$, is a Banach space equipped with norm $\| X(.) \|_{L^2(\Omega;H)} = (E \| X(.;\omega) \|_H^2)^{\frac{1}{2}}$.
An important subspace is given by $L_0^2(\Omega;H) = \{f \in L^2(\Omega;H) : f \ is\ Z_0\ measurable\}$.
Next we define the space $\gamma((0,T);H)$ to be the set
$\{v \in C([0,T];L^2(\Omega;H) : v\ is\ Z_t - adapted\}$ with norm $\| v \|_\gamma = \sup_{0 \le t \le T}(E \| v(t) \|_H)^{\frac{1}{2}}$,(see in [2],[4],[5],[6],[12], [13], [14], [15], [16]). In this work we study the existence and uniqueness of the mild solution of the fractional stochastic integral equation in the form

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(u(\theta))}{(t-\theta)^{1-\alpha}} d\theta + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{G(u(\theta))}{(t-\theta)^{1-\alpha}} dW(\theta) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Au(\theta)}{(t-\theta)^{1-\alpha}} d\theta, \tag{1.1}$$

where $0 \le t \le T,\ \frac{1}{2} \le \alpha \le 1,\ u(0) = u_0, A : D(A) \subset H \to H$ is a linear closed operator generating semigroup , $F : \gamma([0,T];H) \to L^p([0,T];L^2(\Omega;H))$
$(1 \le p < \infty),\ G : \gamma([0,T];H) \to C([0,T];L^2(\Omega;BL(K;H)))$ (where K is a real separable Hilbert space) , W is a K- valued Wiener process with incremental covariance described by the nuclear operator Q and u_0 is an Z_0- measurable H-valued random variable independent of W.
The present paper may be regarded as a direct attempt to generalize the work in [1], [3],[10], [11], [13]).
Definition 1.1: An $Z_t - adapted$ stochastic process $u : [0,T] \to H$ is called a mild solution of (1.1) if $u(t)$ is measurable, for all $t \in [0,T]$,

$$\int_0^t \| u(s) \|_H^2 \, ds < \infty,$$

and

$$u(t) = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) u_0 d\theta + \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) F(u(\eta)) d\theta d\eta$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) G(u(\eta)) d\theta dW(\eta), \quad 0 \le t \le T$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$,

$$\int_0^\infty \xi_\alpha(\theta) d\theta = 1,\ for\ all\ 0 \le t \le T\ (see\ [7],\ [8],\ [9]).$$

In the next section, we shall prove the existence and uniqueness of the mild solution of the equation (1.1).

2. Existence and uniqueness

Consider the initial value problem (1.1) in a real separable Hilbert space H under the following assumptions :
(H_1) the linear operator $A : D(A) \subset H \to H$ generates a $C_0 - semigroup$ $\{S(t) : t \geq 0\}$,
(H_2) $F : \gamma([0,T];H) \to L^p(0,T;L^2(\Omega;H))$ is such that there exists $M_F > 0$ for which $\| F(u) - F(v) \|_{L^p} \leq M_F \| u - v \|_\gamma$, for all u , $v \in \gamma([0,T];H)$,
(H_3) $G : \gamma([0,T];H) \to C([0,T];L^2(\Omega;BL(K;H)))$ $(= \gamma_{BL})$ is such that there exists $M_G > 0$ for which $\| G(u) - G(v) \|_{\gamma_{BL}} \leq M_G \| u - v \|_\gamma$ for all $u,\ v \in \gamma([0,T];H)$,
(H_4) $u_0 \in L_0^2(\Omega;H)$.
In this section we shall be concerned with the mild solution of the equation (1.1), which in the form

$$u(t) = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) u_0 d\theta + \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) F(u(\eta)) d\theta d\eta$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) G(u(\eta)) d\theta dW(\eta), \quad 0 \leq t \leq T \quad (2.1)$$

Theorem 2.1: Assume that (H_1) - (H_4) hold. Then (1.1) has a unique solution on $[0,T]$, if

$$M_S[C_F T^\alpha + M_G C_G T^\alpha] \leq 1, \quad (2.2)$$

where $1 \leq p,\ q \leq \infty$ *are conjugate indices*.
Proof: Define the solution map $J : \gamma([0,T];H) \to \gamma([0,T];H)$ by

$$(Ju)(t) = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) u_0 d\theta + \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) F(u(\eta)) d\theta d\eta$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) G(u(\eta)) d\theta dW(\eta), \quad 0 \leq t \leq T. \quad (2.3)$$

From Holder's inequality, we get

$$[E \| \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta) F(u(\eta)) d\theta d\eta \|_H^2]^{\frac{1}{2}}$$

$$\leq M_S [\int_0^T \| (T-\eta)^{\alpha-1} F(u(\eta)) \|_{L^2(\Omega;H)}^2 d\eta]^{\frac{1}{2}}$$

$$\leq M_S [\int_0^T (T-\eta)^{2(\alpha-1)} d\eta]^{\frac{1}{2}} [\int_0^T \| F(u(\eta)) \|_{L^2(\Omega;H)}^2 d\eta]^{\frac{1}{2}}$$

$$\leq M_S \frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} [\int_0^T \| F(u(\eta)) \|_{L^2(\Omega;H)}^2 d\eta]^{\frac{1}{2}}$$

$$\leq C_F M_S T^{\alpha-\frac{1}{2}} \parallel F(u) \parallel_{L^p}, \tag{2.4}$$

where C_F is a constant depending on α.
Subsequently, an application of (H_2), together with Minkowski's inequality enables us to continue the string of inequalities in (2.4) to conclude that

$$[E \parallel \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)F(u(\eta))d\theta d\eta \parallel_H^2]^{\frac{1}{2}}$$

$$\leq M_S C_F T^{\alpha-\frac{1}{2}}[M_F \parallel u \parallel_\gamma + \parallel F(0) \parallel_{L^p}]. \tag{2.5}$$

Taking the supremum over $[0,T]$ in (2.5) then implies that

$$\int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)F(u(\eta))d\theta d\eta \ \in \gamma([0,T];H),$$

for any $u \in \gamma([0,T];H)$. Further for such $u,\ G(u(\eta)) \in BL(K;H)$ *and*
$u_0 \in L_0^2(\Omega;H)$ $(by\ (H_4))$. Consequently, one can argue as in ([9], [11]) to conclude that J is will defined .
Next we show that J is a strict contraction .
Observe that for $u,\ v\ \in \gamma([0,T];H)$, we infer from (2.3) that

$$(Ju)(t)-(Jv)(t) = \alpha\int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(F(u(\eta))-F(v(\eta))d\theta d\eta$$

$$+\alpha\int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(G(u(\eta))-G(v(\eta))d\theta dW(\eta), \quad 0\leq t\leq T. \tag{2.6}$$

Squaring both sides and taking the expectation in (2.6) yields, with the help of Young's inequality,

$$E \parallel (Ju)(t)-(Jv)(t) \parallel_H^2$$

$$\leq 2\alpha^2[E \parallel \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(F(u(\eta))-F(v(\eta))d\theta d\eta \parallel_H^2$$

$$+E \parallel \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(G(u(\eta))-G(v(\eta))d\theta dW(\eta) \parallel_H^2]$$

and subsequently,

$$\parallel (Ju)(t)-(Jv)(t) \parallel_\gamma$$

$$\leq 2\alpha^2[\parallel \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(F(u(\eta))-F(v(\eta))d\theta d\eta \parallel_\gamma$$

$$+\parallel \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(G(u(\eta))-G(v(\eta))d\theta dW(\eta) \parallel_\gamma]. \tag{2.7}$$

Using reasoning similar to that which led to (2.4), one can show that

$$\| \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(F(u(\eta))-F(v(\eta))d\theta d\eta \|_\gamma \le C_F M_S T^\alpha \| u-v \|_\gamma . \quad (2.8)$$

Also,

$$[E \| \int_0^t \int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)(G(u(\eta))-G(v(\eta)))d\theta dW(\eta) \|_H^2]^{\frac{1}{2}}$$

$$\le Tr(Q)\frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}M_S[\int_0^T \| G(u(\eta))-G(v(\eta)) \|^2_{L^2(\Omega;H)} d\eta]^{\frac{1}{2}}$$

$$\le C_G T^{\alpha-\frac{1}{2}} M_G \| u-v \|_\gamma, \quad (2.9)$$

where C_G is a constant depending on $(\alpha \ and \ Tr(Q))$. Using (2.8) and (2.9) in (2.7) enables us to conclude that J is a strict contraction, provided that (2.2) is satisfied, has a unique fixed point which coincides with a mild solution of (1.1). This completes the proof.

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