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Nontrivial Weak Solutions of a Class of Quasilinear Elliptic Equations

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Abstract

In this paper we obtain the existence of a nontrivial weak solution of a class of quasilinear elliptic equations.

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1. Introduction

In this paper, we consider the existence of a nontrivial weak solution of the following Dirichlet problem:

$$\begin{cases} -\sum_{i=1}^{N} D_i(|Du|^{p-2}D_iu) = f(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain, $N > p > 1$, $f(x,0) = 0$, $f(x,u) = o(|u|^{p^*-2}u)$ and $p^* = \frac{NP}{N-p}$.

There has been some literature concerning this problem, for example, [1, 2, 4] and references therein.

In [4], Zhu considers the existence of a nontrivial $W^{1,p}$ solution to the problem (1.1) above. The main purpose of this article is to give a much more general result.

Assume

(f_1) $f(x,u) : \Omega \times R \to R$ is continuous and $f(x,0) = 0$;

(f_2) $\lim_{u\to\infty} \frac{f(x,u)}{|u|^{p^*-2}u} = k(x)$ uniformly for $x \in \Omega$ and there exist k_1 and k_2 such that $0 < k_1 \leq k(x) \leq k_2$;

(f_3) $\lim_{u\to 0} \frac{f(x,u)}{|u|^{p^*-2}\cdot u} = l(x)$ uniformly for $x \in \Omega$ and there exists $\alpha > 0$ such that $l(x) \leq (1-\alpha)c'$, where c' is a constant of Sobolev emersion $W_0^{1,p}(\Omega) \to L^p(\Omega)$.

Define $E = W_0^{l,p}(\Omega)$ and $B_\rho = \{u \in E : \|u\|_E \leq \rho\}$ with $\rho > 0$.

The weak solutions of problem (1.1) are the critical points of the function I defined on E by

$$I(u) = \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega F(x,u),$$

where $F(x,u) = \int_0^u f(x,t)dt$.

Our main result is given in the following theorem.

Theorem 1.1: Assume $f(x,u)$ satisfies conditions (f_1), (f_2) and (f_3). Then

1. there exists $u \in E$ such that

$$-\sum_{i=1}^N D_i\big(|Du|^{p-2}D_iu\big) = f(x,u)$$

in E^*;

2. if for all $u_0 \in E$ and $u_0 \neq 0$, it holds that

$$\sup_{t\geq 0} I(tu_0) < \frac{k_2}{N}\left(\frac{s}{k_2}\right)^{\frac{N}{p}},$$

where s is a constant of Sobolev emersion $E \to L^{p^*}$, then problem (1.1) has a nontrivial solution.

2. Lemmas

Lemma 2.1: There exist ρ_0 and $\alpha_0 > 0$ such that $I(u)|_{\partial B_{\rho_0}(0)} \geq \alpha_0 > 0$.

Proof: By assumptions (f_1) and (f_2), for all $\varepsilon > 0$, $\exists M(\varepsilon) > 0$, if $|u| \leq \delta$, we have

$$\left| \frac{f(x,u)}{|u|^{p-2}u} - L(x) \right| < \varepsilon;$$

if $|u| \geq \delta$, we have $|f(x,u)| \leq c(\varepsilon)|u|^{p^*-1}$. Therefore

$$F(x,u) = \int_0^u f(x,t)dt \leq \frac{L(x)+\varepsilon}{p}|u|^p + \frac{c(\varepsilon)}{p^*}|u|^{p^*}$$

and

$$\begin{aligned} I(u) &\geq \frac{1}{p}\int_\Omega |Du|^p - \frac{1}{p}\int_\Omega L(x)|u|^p - \frac{\varepsilon}{p}\int_\Omega |u|^p - \frac{c(\varepsilon)}{p^*}\int_\Omega |u|^{p^*} \\ &\geq \frac{1}{p}\int_\Omega |Du|^p - \frac{1}{p}\int_\Omega (1-\alpha)c'|u|^p - \frac{\varepsilon}{p}\int_\Omega |u|^p - \frac{c(\varepsilon)}{p^*}\int_\Omega |u|^{p^*} \\ &\geq \frac{1}{p}\int_\Omega |Du|^p - \frac{1-\alpha}{p}\int_\Omega |Du|^p - \frac{\varepsilon}{p}\int_\Omega |u|^p - \frac{c(\varepsilon)}{p^*}\int_\Omega |u|^{p^*}. \end{aligned}$$

Poincaré's inequality and Sobolev's inequality imply that there exist $\delta_1 > 0$ and $c_1 > 0$ such that $I(u) \geq \delta_1\|u\|_E^p - c_1\|u\|_E^{p^*}$. From $p^* > p$, we conclude that $I(u)|_{\partial B_{\rho_0}(0)} \geq \alpha_0 > 0$, where $\|u\|_E = \rho_0$ is small enough. ■

Lemma 2.2: For all $u_0 \in E$ with $u_0 \neq 0$, there exists $t_0 > 0$ such that $I(tu_0) < 0$ for $t \geq t_0$.

Proof: By assumptions (f_1) and (f_2), for all $\varepsilon > 0$, $\exists M(\varepsilon) > 0$, if $|u| \geq M(\varepsilon)$, we have $|\frac{f(x,u)}{|u|^{p^*-2}u} - k(x)| \leq \varepsilon$. If $|u| \leq M(\varepsilon)$, we have $|f(x,u)| \leq C(\varepsilon)$. Therefore it follows that $F(x,u) \geq \frac{k(x)+\varepsilon}{p^*}|u|^{p^*} - C(\varepsilon)$ and

$$\begin{aligned} I(t,u_0) &= \frac{1}{p}\int_\Omega t^p|Du_0|^p - \int_\Omega F(x,tu_0) \\ &\leq \frac{1}{p}\int_\Omega t^p|Du_0|^p - \frac{1}{p^*}\int_\Omega t^{p^*}(K(x)-\varepsilon)|u|^{p^*} + c(\varepsilon)|\Omega|. \end{aligned}$$

Choosing $\varepsilon = \frac{k_1}{2}$, we obtain

$$I(t,u_0) \leq \frac{t^p}{p}\int_\Omega |Du_0|^p - \frac{k_1 t^{p^*}}{2p^*}\int_\Omega |u|^{p^*} + c(\varepsilon)|\Omega| \to -\infty \quad \text{as } t \to \infty.$$

Hence there exists $t_0 > 0$ such that $I(tu_0) < 0$ for $t \geq t_0$. The proof is complete. ∎

By Lemma 2.2, $\exists e \nsubseteq B_{\rho_0}(0)$, such that $I(e) < 0$.
Define

$$C_0 = \inf_{\gamma \in \tau} \max I(u) \geq \alpha_0 > 0, \tag{2.1}$$

where $\gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = e\}$.

Applying the Mountain Pass Theorem without the Palais-Smale condition (see [2]) for $I(u)$, we obtain a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to C_0, \tag{2.2}$$

$$I'(u_n) \to 0 \quad \text{in } E^*. \tag{2.3}$$

Lemma 2.3: Under conditions (2.2) and (2.3), the sequence $\{u_n\}$ is bounded in E.

Proof: Let $\{u_n\}$ be the sequence defined by (2.2) and (2.3). Then

$$\frac{1}{p}\int_\Omega |Du_n|^p - \int_\Omega F(x, u_n) = C_0 + o(1) \quad \text{as } n \to \infty, \tag{2.4}$$

$$\int_\Omega |Du_n|^p = \int_\Omega u_n f(x, u_n) + \langle \xi_n, u_n \rangle, \tag{2.5}$$

where $\xi_n \to 0$ as $n \to \infty$ in E^*.

Calculating (2.4) $- \frac{1}{p} \times$ (2.5) yields

$$\int_\Omega F(x, u_n) - \frac{1}{p}\int_\Omega u_n f(x, u_n) + C_0 - \frac{1}{p}\langle \xi_n, u_0 \rangle + o(1) \to 0.$$

By (f_1) and (f_2), we have

$$\int_\Omega u_n f(x, u_n) \geq \int_\Omega (k(x) - \varepsilon)|u_n|^{p^*} - c(\varepsilon)|\Omega|,$$

$$\int_\Omega F(x, u_n) \leq \frac{1}{p^*}\int_\Omega (k(x) + \varepsilon)|u_n|^{p^*} + c(\varepsilon)|\Omega|.$$

From the above three equations, it is deduced that

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega k(x)|u_n|^{p^*} \leq c + \left(\frac{1}{p} + \frac{1}{p^*}\right)\varepsilon \int_\Omega |u_n|^{p*} + C\|u\|_E.$$

Hence $\|u\|_E^p \leq C + C\|u_n\|_E$. This means that the sequence $\{u_n\}$ is bounded in E. ∎

By Lemma 2.3 and using the related results in [4], we can obtain a subsequence $\{u_n\}$ and $u \in E$ satisfying

$$\begin{cases} u_n \xrightarrow{w.} U & \text{in } E, \\ D_i u_n \xrightarrow{a.e.} D_i U & \text{in } \Omega, \\ f(x, u_n) \xrightarrow{w.} f(x, u) & \text{in } \left(L^{p^*}(\Omega)\right)^*, \\ |Du_n|^{p-2} D_i u_n \xrightarrow{w.} |Du|^{p-2} D_i u & \text{in } \left(L^{p^*}(\Omega)\right)^*. \end{cases} \tag{2.6}$$

Lemma 2.4 [Strauss Lemma [1]]: Assume $P, Q : \mathbb{R} \to \mathbb{R}$ are two continuous functions such that $\frac{p(s)}{Q(s)} \to 0$ as $|s| \to \infty$ and $\{u_n\} : \mathbb{R}^N \to R$ is a sequence of measurable functions such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| \, \mathrm{d}x < \infty$$

and $P(u_n(x)) \xrightarrow{a.e.} \gamma(x)$ in $\mathbb{R}^N$ as $n \to \infty$. Then for all Borel sets B, we have

$$\int_B |p(u_n(x)) - \gamma(x)| \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Furthermore, assume $\frac{P(s)}{Q(s)} \to 0$ as $s \to 0$ and $u_n(x) \to 0$ as $|x| \to \infty$ uniformly in n. Then we have $p(u_n(x)) \to \gamma(x)$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

3. Proof of Theorem 1.1

By Lemma 2.3 and Equations (2.1) and (2.3), it is easy to see that $\{u_n\} \subset E$ and $u \in E$. As $n \to \infty$, we have

$$\frac{1}{p} \int_\Omega |Du_n|^p - \int_\Omega F(x, u_n) = C_0 + o(1), \tag{3.1}$$

$$-\sum_{i=1}^{N} D_i(|Du_n|^{p-2} D_i u_n) = f(x, u_n) + \xi_n \tag{3.2}$$

in E^*. With the help of (2.6) and taking the limit on both sides of (3.2) as $n \to \infty$, we have

$$-\sum_{i=1}^{N} D_i(|Du|^{p-2} D_i u) = f(x, u)$$

in E^*. This finishes the proof of (1) in Theorem 1.1.

Now we are in a position to prove (2) in Theorem 1.1. Assume $u \equiv 0$, then

$$\begin{cases} u_n \xrightarrow{w.} 0 & \text{in } E, \\ u_n \to 0 & \text{in } L^{p^*}(\Omega), \\ u_n \xrightarrow{s.} 0 & \text{in } L^r(\Omega) \text{ for } p \le r < p^*. \end{cases} \tag{3.3}$$

Equation (3.2) implies

$$\int_\Omega |Du_n|^p = \int_\Omega f(x, u_n) u_n + \langle \xi_n, u_n \rangle. \tag{3.4}$$

Writing

$$l = \lim_{n\to\infty} \left(\int_\Omega |Du_n|^p \right)^{\frac{1}{p}},$$

then from (3.4), we have

$$l^p = \lim_{n\to\infty}\int_\Omega f(x,u_n)u_n.$$

By condition (f_2), we obtain

$$\int_\Omega f(x,u_n)u_n = \int_\Omega k(x)|u_n|^{p^*} + \int_\Omega u_n g(x,u_n),$$

where $\frac{u_n g(x,u_n)}{|u_n|^{p^*}} \to 0$ as $|u_n| \to \infty$.

By Lemma 2 and from (3.3), we have

$$\lim_{n\to\infty}\int_\Omega u_n g(x,u_n) = 0.$$

Moreover,

$$l^p = \lim_{n\to\infty}\int_\Omega k(x)|u_n|^{p^*}. \tag{3.5}$$

The Sobolev inequality implies that

$$s\bigg(\int_\Omega |u_n|^{p^*}\bigg)^{\frac{p}{p^*}} \le \int_\Omega |Du_n|^p. \tag{3.6}$$

By (f_2) and from (3.5) and (3.6), we obtain

$$l^p \le K_2 \lim_{n\to\infty}\int_\Omega |u_n|^{p^*} \le K_2\bigg(\frac{l^p}{s}\bigg)^{\frac{p^*}{r}},$$

$$l^p \ge \bigg(\frac{1}{k_2}\bigg)^{\frac{p}{p^*-p}} S^{\frac{p*}{p^*-p}} = k_2\bigg(\frac{s}{k_2}\bigg)^{\frac{N}{P}}. \tag{3.7}$$

From (3.1), we have

$$C_0 = \frac{l^p}{p} - \lim_{n\to\infty}\int_\Omega F(x,u_n). \tag{3.8}$$

From (3.1) and (f_2), it is deduced that

$$f(x,u_n) = k(x)|u_n|^{p^*-2}u_n + g(x,u_n),$$

where $\frac{g(x,u_n)}{|u_n|^{p^*-2}\cdot u_n} \to 0$ as $|u_n| \to \infty$ and

$$F(x,u_n) = \int_0^{u_n} f(x,t)dt = \frac{k(x)}{p^*}|u_n|^{p*} + \int_0^{u_n} g(x,t)\,\mathrm{d}t.$$

Moreover,

$$\int_\Omega F(x,u_n) = \frac{1}{p^*}\int_\Omega k(x)|u_n|^{p^*} + \int_\Omega\bigg(\int_\Omega^{u_n} g(x,t)\,\mathrm{d}t\bigg). \tag{3.9}$$

By Lemma 2, we have

$$\lim_{n\to\infty}\int_\Omega\left(\int_\Omega^{u_n} g(x,t)\,\mathrm{d}t\right)=0. \tag{3.10}$$

From (3.5) and (3.7)–(3.10), we obtain

$$C_0=\left(\frac{1}{p}-\frac{1}{p^*}\right)l^p\geq\left(\frac{1}{p}-\frac{1}{p^*}\right)k_2\left(\frac{s}{k_2}\right)^{\frac{N}{p}}=\frac{k_2}{N}\left(\frac{s}{k_2}\right)^{\frac{N}{p}}.$$

This leads to a contradiction to (3.2). Thus $u\neq 0$. This means that problem (1.1) has a nontrivial solution.

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