

## Exact Travelling Wave Solutions for a Class of Nonlinear Partial Differential Equations

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### Abstract

The extended mapping method is developed to obtain the exact solutions to a nonlinear ordinary differential equation. As a simple application, a class of nonlinear partial differential equations are studied, and multiple travelling wave solutions are obtained. Many new results are presented.

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### 1. Introduction

Travelling waves, whether their solution expressions are in explicit or implicit forms, are very interesting from the point of view of applications. These types of waves will not change their shapes during propagation and are thus easy to detect. Of particular interest are three types of travelling waves: the solitary waves, which are localized travelling waves, asymptotically zero at large distances, the periodic waves, and the kink waves, which rise or descend from one asymptotic state to another. Very recently, the extended mapping method is proposed [1] to obtain the combined travelling wave solutions to a new Hamiltonian amplitude equation. The basic idea of the method is as follows. For a given nonlinear partial differential equation, say, in two independent variables,

$$N(u, u_t, u_x, \dots) = 0, \quad (1.1)$$

we seek its travelling wave solution of the form

$$u = u(\xi), \quad \xi = kx - \omega t. \quad (1.2)$$

Substitution of equation (1.2) into equation (1.1) yields an ordinary differential equation for  $u(\xi)$ . Then  $u(\xi)$  is expanded into a polynomial in  $f(\xi)$

$$u(\xi) = A_0 + \sum_{i=1}^n f^{i-1}(A_i f + B_i g), \quad (1.3)$$

where  $A_i$  and  $B_i$  are constants to be determined, and  $n$  is fixed by balancing the linear term of the highest order derivative with nonlinear term, while  $f$  and  $g$  satisfy the system of equations

$$\begin{aligned} f'' &= pf + qf^3, & f'^2 &= pf^2 + \frac{1}{2}qf^4 + r, \\ g'' &= g(c_1 + c_2 f^2), & g^2 &= c_3 + c_4 f^2, \end{aligned} \quad (1.4)$$

with prime meaning derivatives with respect to  $\xi$ . After equation (1.3) with equation (1.4) is substituted into the ordinary differential equation, the coefficients  $A_i$ ,  $B_i$ ,  $k$ ,  $\omega$ ,  $p$ ,  $q$ ,  $r$  and  $c_i$  may be determined. If any of them is left unspecified, it will be regarded as being arbitrary for the solution to equation (1.1). Thus, equation (1.3) establishes a new algebraic mapping relation between the solution for equation (1.1) and that of equation (1.4). When  $B_i = 0$ , it reduces to the mapping method [2]. Due to the presence of the parameters  $p$ ,  $q$ ,  $r$  and  $c_i$ , equation (1.4) has a rich structure of solutions. As  $p = -2$ ,  $q = 2$ ,  $r = 1$  and  $c_1 = -1$ ,  $c_2 = 2$ ,  $c_3 = 1$ ,  $c_4 = -1$ , for example, the solution reads  $f(\xi) = \tanh\xi$ ,  $g(\xi) = \operatorname{sech}\xi$ , and the method is called the two-family truncation method [3–4]. In this paper, we will develop the extended mapping method to study the exact solutions to a nonlinear ordinary differential equation (ODE). Then applying the obtained results, we get the multiple travelling wave solutions to a class of nonlinear partial differential equations (PDEs), and many new results are presented.

## 2. Exact Solutions to a Nonlinear ODE

Many physically important nonlinear PDEs can be reduced to the nonlinear ODE

$$Au'' + Bu + Cu^3 = 0, \quad (2.1)$$

by means of the travelling wave reduction method. So it is significant and very interesting to study the exact solutions to equation (2.1), which is our purpose in this section. According to the idea of the method described above, we assume that equation (2.1) has the solution of the form

$$u = A_0 + A_1 f + B_1 g, \quad (2.2)$$

where  $A_i$  and  $B_i$  are constants to be determined, and  $f$  and  $g$  satisfy equation (1.4). Substituting equation (2.2) with equation (1.4) into equation (2.1) and equating the coefficients of like powers of  $f^i g^j$  ( $j = 0, 1$ ) to zero, we get two sets of solutions

$$A_0 = 0, \quad A_1 = \pm \sqrt{-\frac{qA}{C}}, \quad B_1 = 0, \quad pA + B = 0, \quad (2.3)$$

and

$$\begin{aligned} A_0 &= 0, \quad A_1 = \pm \sqrt{\frac{(c_4 p - c_3 q)A + c_4 B}{c_3 C}}, \\ B_1 &= \pm \sqrt{-\frac{pA + B}{3c_3 C}}, \quad (3c_1 - p)A + 2B = 0, \\ (3c_4 p - 3c_3 q - c_1 c_4 + c_2 c_3)A + 2c_4 B &= 0. \end{aligned} \quad (2.4)$$

Therefore, we obtain the exact solutions of equation (2.1) as follows

$$u_1 = \pm \sqrt{-\frac{qA}{C}} f(\xi), \quad (2.5)$$

where  $f$  satisfies the first two of equation (1.4) with  $pA + B = 0$ , and

$$u_2 = \pm \sqrt{\frac{(c_4 p - c_3 q)A + c_4 B}{c_3 C}} f(\xi) \pm \sqrt{-\frac{pA + B}{3c_3 C}} g(\xi), \quad (2.6)$$

where  $f$  and  $g$  satisfy equation (1.4) with the constraint among the coefficients  $(3c_1 - p)A + 2B = 0$  and  $(3c_4 p - 3c_3 q - c_1 c_4 + c_2 c_3)A + 2c_4 B = 0$ . It is worth noticing that the choice for the positive and negative signs in equation (2.6) is arbitrary, which is valid throughout the paper. As will be seen in what follows, the constraints enable us to obtain the dispersion relations for the nonlinear PDEs under consideration. In the following, we discuss the specific expressions of  $f(\xi)$  and  $g(\xi)$  according to equation (1.4).

**Case 1.**  $p = -(1 + m^2)$ ,  $q = 2m^2$ ,  $r = 1$

**Subcase 1.1**  $c_1 = -1$ ,  $c_2 = 2m^2$ ,  $c_3 = 1$ ,  $c_4 = -1$

Equation (1.4) has the solution  $f(\xi) = sn\xi$ ,  $g(\xi) = cn\xi$ . Thus we have

$$u_1 = \pm \sqrt{-\frac{2A}{C}} msn(\xi), \quad (2.7)$$

with  $-(1 + m^2)A + B = 0$ , and

$$u_2 = \pm \sqrt{\frac{A}{2C}} m[isn(\xi) + cn(\xi)], \quad (2.8)$$

with  $(-2 + m^2)A + 2B = 0$ .

**Subcase 1.2**  $c_1 = -m^2$ ,  $c_2 = 2m^2$ ,  $c_3 = 1$ ,  $c_4 = -m^2$

The solution of equation (1.4) is  $f(\xi) = sn\xi$ ,  $g(\xi) = dn\xi$ . Hence we get solution (2.7) and

$$u_3 = \pm \sqrt{\frac{A}{2C}} [imsn(\xi) + dn(\xi)], \quad (2.9)$$

with  $(1 - 2m^2)A + 2B = 0$ .

**Case 2.**  $p = 2m^2 - 1$ ,  $q = -2m^2$ ,  $r = 1 - m^2$ ,  $c_1 = m^2$ ,  $c_2 = -2m^2$ ,  
 $c_3 = 1 - m^2$ ,  $c_4 = m^2$

In this case, we have  $f(\xi) = cn\xi$ ,  $g(\xi) = dn\xi$ . So we obtain the exact solutions of equation (2.1) as follows

$$u_4 = \pm \sqrt{\frac{2A}{C}} mcn(\xi), \quad (2.10)$$

with  $(2m^2 - 1)A + B = 0$ , and

$$u_5 = \pm \sqrt{\frac{A}{2C}} [m cn(\xi) + dn(\xi)], \quad (2.11)$$

with  $(1 + m^2)A + 2B = 0$ . By taking the different values of  $p, q, r$  and  $c_i$ , we can obtain 12 pairs of solutions. However, the others have singularities, and we omit them here. In the results above,  $sn\xi = sn(\xi|m)$ ,  $cn\xi = cn(\xi|m)$ , and  $dn\xi = dn(\xi|m)$  are Jacobi elliptic sine, cosine functions and Jacobi elliptic function of the third kind, respectively, where  $m$  ( $0 < m < 1$ ) is the modulus of the elliptic function. They are doubly periodic and possess properties of trigonometric functions, namely,  $sn^2\xi + cn^2\xi = 1$ ,  $dn^2\xi + m^2 sn^2\xi = 1$ ,  $(sn\xi)' = cn\xi dn\xi$ ,  $(cn\xi)' = -sn\xi dn\xi$ ,  $(dn\xi)' = -m^2 sn\xi cn\xi$ . When  $m \rightarrow 0$ , the Jacobi elliptic functions degenerate to the trigonometric functions, i.e.,  $sn\xi \rightarrow \sin\xi$ ,  $cn\xi \rightarrow \cos\xi$ ,  $dn\xi \rightarrow 1$ . When  $m \rightarrow 1$ , the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e.  $sn\xi \rightarrow \tanh\xi$ ,  $cn\xi \rightarrow \operatorname{sech}\xi$ ,  $dn\xi \rightarrow \operatorname{sech}\xi$ . Detailed explanation about Jacobi elliptic functions can be found in Refs. [5–6].

### 3. Applications

As simple applications of the results above, we study the travelling wave solutions for several physically important nonlinear PDEs.

**Example 3.1:** The nonlinear Klein-Gordon equation coupled with a scalar field  $v$  in the form [7]

$$\begin{aligned} u_{xx} - u_{tt} - u + 2u^3 + 2uv &= 0, \\ v_x - v_t - 4uu_t &= 0, \end{aligned} \quad (3.1)$$

has been shown to be integrable through Painleve analysis [8]. However, when  $v = 0$ , it is not integrable. And the travelling wave solutions to equation (3.1) have not been studied earlier. Substituting  $u = u(\xi)$ ,  $v = v(\xi)$ ,  $\xi = kx - \omega t$  into equation (3.1) and integrating the second one, we have

$$\begin{aligned} (k^2 - \omega^2)u'' - u + 2u^3 + 2uv &= 0, \\ v &= -\frac{2\omega}{k + \omega}u^2 + \frac{C_1}{k + \omega}, \end{aligned} \quad (3.2)$$

where  $C_1$  is the constant of integration. The substitution of the second one of Eq.(3.2) into the first yields Eq.(2.1) with  $A = (k - \omega)(k + \omega)^2$ ,  $B = 2C_1 - k - \omega$  and  $C = 2(k - \omega)$ .

According to the results in Sec.2, we obtain the exact travelling wave solutions to equation (3.1) as follows.

$$u_1 = \pm i(k + \omega)msn(kx - \omega t), \quad (3.3)$$

with the dispersion relation  $-(1 + m^2)(k - \omega)(k + \omega)^2 + 2C_1 - k - \omega = 0$ , and

$$u_2 = \pm \frac{1}{2}(k + \omega)m[isn(kx - \omega t) + cn(kx - \omega t)], \quad (3.4)$$

with  $(-2 + m^2)(k - \omega)(k + \omega)^2 + 4C_1 - 2k - 2\omega = 0$ .

$$u_3 = \pm \frac{1}{2}\sqrt{\frac{k - \omega}{1 - 2\omega}}(k + \omega)[imsn(kx - \omega t) + dn(kx - \omega t)], \quad (3.5)$$

with  $(1 - 2m^2)(k - \omega)(k + \omega)^2 + 4C_1 - 2k - 2\omega = 0$ .

$$u_4 = \pm \sqrt{\frac{k - \omega}{1 - 2\omega}}(k + \omega)mcn(kx - \omega t), \quad (3.6)$$

with  $(2m^2 - 1)(k - \omega)(k + \omega)^2 + 2C_1 - k - \omega = 0$ .

$$u_5 = \pm \frac{1}{2}\sqrt{\frac{k - \omega}{1 - 2\omega}}(k + \omega)[mcn(kx - \omega t) + dn(kx - \omega t)], \quad (3.7)$$

with  $(1 + m^2)(k - \omega)(k + \omega)^2 + 4C_1 - 2k - 2\omega = 0$ . When  $m \rightarrow 1$ , the corresponding solitary wave solutions read

$$u_1 = \pm i(k + \omega)\tanh(kx - \omega t), \quad (3.8)$$

with the dispersion relation  $-2(k - \omega)(k + \omega)^2 + 2C_1 - k - \omega = 0$ , and

$$u_2 = u_3 = \pm \frac{1}{2}(k + \omega)[i\tanh(kx - \omega t) + \operatorname{sech}(kx - \omega t)], \quad (3.9)$$

with  $-(k - \omega)(k + \omega)^2 + 4C_1 - 2k - 2\omega = 0$ .

$$u_4 = u_5 = \pm (k + \omega)\operatorname{sech}(kx - \omega t), \quad (3.10)$$

with  $(k - \omega)(k + \omega)^2 + 2C_1 - k - \omega = 0$ .

The corresponding  $v$ 's are obtained from the second of equation (3.2).

**Example 3.2:** We consider the nonlinear evolution equation

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0, \quad (3.11)$$

which contains some physically important equations as special cases, such as Duffing equation [9], Klein-Gordon equation [10], Landau-Ginburg-Higgs equation [11] and  $\phi^4$

equation [9]. Bai [12] obtained three singular solitary wave solutions to equation (3.11), i.e., *coth*-type, *csch*-type and the combined *coth-csch*-type. Using the results in Sec.2, we will obtain some new nonsingular solutions to equation (3.11). Substituting equation (1.2) into equation (3.11), we get equation (2.1) with  $A = \omega^2 + \alpha k^2$ ,  $B = \beta$  and  $C = \gamma$ . From equations (2.7)–(2.11) we obtain exact periodic wave solutions to equation (3.11) as follows.

$$u_1 = \pm \sqrt{-\frac{2(\omega^2 + \alpha k^2)}{\gamma}} msn(kx - \omega t), \quad (3.12)$$

with  $-(1 + m^2)(\omega^2 + \alpha k^2) + \beta = 0$ .

$$u_2 = \pm \sqrt{\frac{\omega^2 + \alpha k^2}{2\gamma}} m[isn(kx - \omega t) + cn(kx - \omega t)], \quad (3.13)$$

with  $(-2 + m^2)(\omega^2 + \alpha k^2) + 2\beta = 0$ .

$$u_3 = \pm \sqrt{\frac{\omega^2 + \alpha k^2}{2\gamma}} [imsn(kx - \omega t) + dn(kx - \omega t)], \quad (3.14)$$

with  $(1 - 2m^2)(\omega^2 + \alpha k^2) + 2\beta = 0$ .

$$u_4 = \pm \sqrt{\frac{2(\omega^2 + \alpha k^2)}{\gamma}} mcn(kx - \omega t), \quad (3.15)$$

with  $(2m^2 - 1)(\omega^2 + \alpha k^2) + \beta = 0$ .

$$u_5 = \pm \sqrt{\frac{\omega^2 + \alpha k^2}{2\gamma}} [mcn(kx - \omega t) + dn(kx - \omega t)], \quad (3.16)$$

with  $(1 + m^2)(\omega^2 + \alpha k^2) + 2\beta = 0$ .

When  $m \rightarrow 1$ , from equations (3.12) – (3.16), we obtain three new solitary wave solutions

$$u_1 = \pm \sqrt{-\frac{2(\omega^2 + \alpha k^2)}{\gamma}} \tanh(kx - \omega t), \quad (3.17)$$

with  $-2(\omega^2 + \alpha k^2) + \beta = 0$ .

$$u_2 = u_3 = \pm \sqrt{\frac{\omega^2 + \alpha k^2}{2\gamma}} [\tanh(kx - \omega t) + \operatorname{sech}(kx - \omega t)], \quad (3.18)$$

with  $-(\omega^2 + \alpha k^2) + 2\beta = 0$ .

$$u_4 = u_5 = \pm \sqrt{\frac{2(\omega^2 + \alpha k^2)}{\gamma}} \operatorname{sech}(kx - \omega t), \quad (3.19)$$

with  $\omega^2 + \alpha k^2 + \beta = 0$ .

**Example 3.3:** The new coupled equation

$$u_{xt} - 2uv = 0, \quad v_t + 2uu_x = 0, \quad (3.20)$$

is given by Konno and Oono [13]. Recently, its Painleve properties were studied by some authors [14–15]. We are interested in the exact solutions to equation (3.20). Substituting  $u = u(\xi)$ ,  $v = v(\xi)$ ,  $\xi = kx - \omega t$  into equation (3.20) and integrating once the second one, we obtain equation (2.1) with  $A = k\omega^2$ ,  $B = -2C_2$  and  $C = 2k$ , and

$$v = \frac{k}{\omega}u^2 - \frac{C_2}{\omega}, \quad (3.21)$$

where  $C_2$  is the constant of integration. By virtue of the results in Sec.2, we can easily obtain exact travelling wave solutions to equation (3.20) as follows:

$$u_1 = \pm i\omega m \operatorname{sn}(kx - \omega t), \quad (3.22)$$

with  $(1 + m^2)k\omega^2 + 2C_2 = 0$ .

$$u_2 = \pm \frac{1}{2}\omega m [i \operatorname{sn}(kx - \omega t) + \operatorname{cn}(kx - \omega t)], \quad (3.23)$$

with  $(-2 + m^2)k\omega^2 - 4C_2 = 0$ .

$$u_3 = \pm \frac{1}{2}\omega [i m \operatorname{sn}(kx - \omega t) + \operatorname{dn}(kx - \omega t)], \quad (3.24)$$

with  $(1 - 2m^2)k\omega^2 - 4C_2 = 0$ .

$$u_4 = \pm \omega m \operatorname{cn}(kx - \omega t), \quad (3.25)$$

with  $(2m^2 - 1)k\omega^2 - 2C_2 = 0$ .

$$u_5 = \pm \frac{1}{2}\omega [m \operatorname{cn}(kx - \omega t) + \operatorname{dn}(kx - \omega t)], \quad (3.26)$$

with  $(1 + m^2)k\omega^2 - 4C_2 = 0$ . When  $m \rightarrow 1$ , the corresponding solitary wave solutions read

$$u_1 = \pm i\omega \tanh\left(\frac{C_2}{\omega^2}x + \omega t\right). \quad (3.27)$$

$$u_2 = u_3 = \pm \frac{1}{2}\omega \left[ i \tanh\left(\frac{4C_2}{\omega^2}x + \omega t\right) + \operatorname{sech}\left(\frac{4C_2}{\omega^2}x + \omega t\right) \right]. \quad (3.28)$$

$$u_4 = u_5 = \pm \omega \operatorname{sech}\left(\frac{2C_2}{\omega^2}x - \omega t\right). \quad (3.29)$$

Equations (3.22) – (3.29) are all new results and the corresponding  $v$ 's are given by equation (3.21).

**Example 3.4:** The governing equation of motion of melt in the earth which neglects phase transition and allows only vertical motions is written as

$$u_t = [u^n \{(u^{-m} u_t)_x - 1\}]_x, \quad (3.30)$$

where  $x$  is the vertical space coordinate and  $t$  the time and  $u(x, t)$  is the mean volume fraction of the melt.  $n$  and  $m$  are related to the permeability  $k$  and viscosity  $\eta$  by the relations  $k = k_0 u^n$  and  $\eta = \eta_0 u^{-m}$ , where  $k_0$  and  $\eta_0$  are constants. Variables  $x$  and  $t$  are normalized by units of length and time,  $\sqrt{k_0 \eta_0 / \eta_l}$  and  $\sqrt{\eta_0 \eta_l / k_0} (g \Delta \rho)^{-1}$  respectively, where  $\eta_l$  is the viscosity of the liquid phase,  $g$  is the gravitational acceleration and  $\Delta \rho$  is the difference in the density between the solid and liquid phase. The equation (3.30) is referred to as the Magma equation. It is suggested that reasonable values of  $n$  and  $m$  are  $2 \sim 5$  and  $0 \sim 1$ , respectively [16].

Here we consider the choice of parameters  $n = 4$  and  $m = 0$  so that equation (3.30) reduces to

$$u_t = \left( u^4 (u_{xt} - 1) \right)_x. \quad (3.31)$$

We seek travelling wave solutions of (3.31) in the form  $u(x, t) = u(z)$ ,  $z = kx - \omega t$  so that (3.31) reduces to

$$\frac{1}{2} \omega k u_z^2 + u - \frac{2}{3} A_1 u^{-3} + \frac{\omega}{2k} u^{-2} - B_1 = 0 \quad (3.32)$$

where,  $A_1$  and  $B_1$  are constants of integration.

Using the independent variable transformation

$$\xi = \int^z u^{-3/2} dz, \quad (3.33)$$

equation (3.32) becomes

$$u_{\xi\xi} = k_1 u^3 + k_2 u^2 + k_3, \quad (3.34)$$

where  $k_1 = -\frac{4}{ck^2}$ ,  $k_2 = \frac{3B_1}{ck^2}$  and  $k_3 = -\frac{1}{2k^2}$  with  $c = \frac{\omega}{k}$ .

Letting  $u = v - \frac{k_2}{3k_1}$ , we can reduce equation (3.34) to the form

$$Av'' + Bv + Cv^3 = 0, \quad (3.35)$$

where  $A = \frac{432}{c^2 k^4}$ ,  $B = -\frac{972}{k^6} \left( \frac{16}{c^7} \right)^{1/3}$  and  $C = \frac{1728}{c^3 k^6}$ .

In view of the results in section 2, we obtain the exact travelling wave solutions to equation (3.35) as follows:

$$v_1 = \pm i \sqrt{2\omega k m s n}(\xi), \quad (3.36)$$

with the dispersion relation  $(1 + m^2) \frac{4}{\omega^2 k^2} + 9 \left( \frac{16}{k^{11} \omega^7} \right)^{1/3} = 0$ ,

$$v_2 = \pm \sqrt{\frac{\omega k}{8}} m [isn(\xi) + cn(\xi)], \quad (3.37)$$

with the dispersion relation  $(-2 + m^2) \frac{2}{\omega^2 k^2} - 9 \left( \frac{16}{k^{11} \omega^7} \right)^{1/3} = 0$ ,

$$v_3 = \pm \sqrt{\frac{\omega k}{8}} [imsn(\xi) + dn(\xi)], \quad (3.38)$$

with the dispersion relation  $(1 - 2m^2) \frac{2}{\omega^2 k^2} - 9 \left( \frac{16}{k^{11} \omega^7} \right)^{1/3} = 0$ ,

$$v_4 = \pm \sqrt{2\omega k m} cn(\xi), \quad (3.39)$$

with the dispersion relation  $(2m^2 - 1) \frac{4}{\omega^2 k^2} - 9 \left( \frac{16}{k^{11} \omega^7} \right)^{1/3} = 0$ ,

$$v_5 = \pm \sqrt{\frac{\omega k}{8}} [m cn(\xi) + dn(\xi)], \quad (3.40)$$

with the dispersion relation  $(1 + m^2) \frac{2}{\omega^2 k^2} - 9 \left( \frac{16}{k^{11} \omega^7} \right)^{1/3} = 0$ .

## 4. Conclusion

A class of nonlinear PDEs can reduce to a nonlinear ODE with different coefficients by means of travelling wave reduction method. The exact solutions to the ODE are obtained in virtue of the extended mapping method. And using the exact solutions of the ODE, we can easily get the exact travelling wave solutions to the PDE in question. As illustrative examples, we consider three physically important nonlinear PDEs and obtain the exact periodic wave solutions of them in terms of Jacobi elliptic functions. Limit cases are studied and corresponding solitary wave solutions are thus obtained. It has been shown that the extended mapping method is applicable to a large variety of nonlinear PDEs, as long as odd- and even-order derivative terms do not coexist in the equation under consideration.

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