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On Spectral Theory of Problems on Normal Oscillations of an Ideal Compressible Fluid in Rotating Elastic Containers

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Abstract

In this paper the structure of the spectrum for certain problems involving normal oscillations of an ideal compressible fluid in rotating elastic container is investigated. The equivalent system of equations of the problems involving normal oscillations of an ideal compressible fluid in rotating elastic container is obtained. The case of an elastic container filled with an ideal compressible fluid is studied.

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1. Introduction

Assume that in the elastic container occupying the domain Ω we have an ideal compressible fluid occupying the domain Ω_0 with $\Sigma = \partial\Omega_0$. Let $\Sigma_1 = \partial\Omega\backslash\Sigma$ is outside of domain Ω. We assume that Σ and Σ_1 are of class c^2. Let n is a unit exterior normal to $\partial\Omega$ which is outside to Ω domain. We will consider Cartesian coordinate system $(x_1$, x_2, $x_3)$ with the container rotates uniformly with constant angular velocity εk around the x_3- axis (k is a unit vector along this axis). We denote $u(x,t)$ as the displacement vector of the elastic body at the point $x \in \Omega$ in the moment t. Let $\rho(x)$ is density of elastic body, $\rho_1(x,t) = \rho_0 + k_0 p(x,t)$ is density of compressible fluid, $p(x,t)$ is deviation from the balanced pressure of fluid, ρ_0 and K_0 are positive constants. We also suggest that elastic body is isotropic, so the stress tensor of solid is entered as:

$$\sigma_{jk}(u) = \Lambda\delta_{jk} div\ u + M\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)(k,j = 1,2,3),$$

where δ_{jk} is symbol of Kronecker, Λ and M are constants of Lame. Let

$$(Lu)_j \equiv -\sum_{k=1}^{3} \frac{\partial \sigma_{jk}(u)}{\partial x_k}.$$

In our chosen non-inertial coordinate system, we shall put down the equation of small-scale (linear) motion of this mechanical system as:

$$\begin{cases} Lu + \rho \frac{\partial^2 u}{\partial t^2} = 0 \ (\Omega), \\ \sigma(u)n|_\Sigma = 0, \sigma(u)n|_\Sigma = -pn|_\Sigma \ (\Omega), \\ K_0 \frac{\partial p}{\partial t} + \rho_0 div \frac{\partial \omega}{\partial t} = 0 \, (\Omega_0), \\ \rho_0 \frac{\partial^2 \omega}{\partial t^2} + \bigtriangledown p - 2\varepsilon \rho_0 \frac{\partial \omega}{\partial t} \times k \, (\Omega_0), \\ (\omega, n)|_\Sigma = (u, n)|_\Sigma. \end{cases} \tag{1.1}$$

We consider the normal (or free) oscillations, i.e. the motion of this kind:

$$\rho_0 u(x,t) = u(x)e^{i\lambda t}, \rho_0 \omega(x,t) = \omega(x)e^{i\lambda t}, \rho_0 p(x,t) = p(x)e^{i\lambda t}.$$

Putting them to the system of equation (1.1) we get the following eigenvalue problem with respect to λ

$$\begin{cases} Lu - \lambda^2 \rho u = 0 \ (\Omega), \\ \sigma(u) n|_\Sigma = 0, \sigma(u)n|_\Sigma = -pn|_\Sigma, \end{cases} \tag{1.2}$$

$$\begin{cases} -\frac{1}{k_0} div \, \omega = p \ (\Omega_0), \\ (\omega, n)|_\Sigma = \rho_0 (u, n)|_\Sigma, \end{cases} \tag{1.3}$$

$$\lambda^2 \omega + 2\varepsilon i \lambda \omega x k = \nabla p \ (\Omega_0). \tag{1.4}$$

The strongly theory of problems on small oscillations of a fluid in rotating containers was investigated for the first time by S.L. Sobolev in [1]; he showed that the problem leads to the study of a very complicated spectral theory. A number of papers have been devoted to study of the spectral properties of this problem and certain generalizations of it (see, e.g., [2], [7], [13–15]). In [3–6] the spectral properties of problems on normal oscillations of an ideal incompressible fluid in the rotating elastic containers were investigated. There are also some papers [8-11] dedicated to problems involving small-scale oscillations of an ideal incompressible fluid in nonrotating elastic shells where discovered additional series of points of discrete spectrum appearing with the oscillations of elastic shells or elastic cover. In [12] the asymptotics of the eigenvalues were studied in the problem concerning the oscillations of an elastic shell immersed in an ideal compressible fluid.

In the present paper we investigate the structure of the spectrum for certain problems involving normal oscillations of an ideal compressible fluid in rotating elastic container. We prove that there is a continuous spectrum of the internal waves on $[-2\varepsilon, 2\varepsilon]$ for the case a rotating elastic container filled with an ideal compressible fluid.

2. Theorems 2.1 and 2.2

The problems (1.2) and (1.3) can be reduced to the system of operator equations. We introduce the following spaces:

$$H_0 = \left\{\varphi \in W_2^1\left(\Omega_0\right),\ \left(\varphi, 1\right)_{L_2\left(\Omega_0\right)} = 0\right\},\ H_0 \subset W_2^1\left(\Omega_0\right),$$

$$H_1 = \left\{\omega = \nabla\varphi, \varphi \in H_0\right\},\ H_2 = L_2\left(\Omega_0\right) \backslash H_1.$$

We get that [3]

$$W_2^1\left(\Omega_0\right) \cap H_2 = \left\{\omega : dir\ \omega = 0,\ \left(\Omega_0\right),\ \left(\omega, n\right)|_{\Sigma} = 0\right\}. \tag{2.1}$$

Note that in H_0 inner product could be introduced as

$$\left(\nabla\varphi_1, \nabla\varphi_2\right)_{L_2(\Omega_0)}$$

and norm generated by this inner product is equivalent to usual norm in the space $W_2^1(\Omega_0)$. As $L_2(\Omega_0) = H_1 \oplus H_2$, we have that for each element $\omega \epsilon L_2(\Omega_0)$ the element $x \epsilon \Im = H_0 \oplus H_2$ by the rule: $x = (\varphi, \vartheta)$, where

$$\omega = \nabla\varphi + \vartheta,\ \nabla\varphi \in H_1,\ \vartheta \in H_2 \tag{2.2}$$

at that, if x and y are defined by ω and ω_1 in the accordance with rule then

$$\left(\omega, \omega_1\right)_{L_2\left(\Omega_0\right)} = \left(\nabla\varphi, \nabla\varphi_1\right)_{L_2\left(\Omega_0\right)} + \left(v, v_1\right)_{L_2\left(\Omega_0\right)} = \left(x, y\right)_{\Im}$$

that is,

$$\left(x, y\right)_{\Im} = \left(\omega, \omega_1\right)_{L_2\left(\Omega_0\right)}. \tag{2.3}$$

In other words, indicated one-one mapping between spaces $L_2(\Omega_0)$ and $\Im$ keep the inner product. Now we introduce the bounded linear operator $A : \Im \to \Im$, $D(A) = \Im$, defined by the formula

$$\left(Ax, y\right)_{\Im} = -i\left(\omega \mathrm{x} k, \omega_1\right)_{L_2\left(\Omega_0\right)}, \tag{2.4}$$

where ω, ω_1 formed by x, y in the according with the rule (2.3). Now we will transform the equation (1.4). For that, previously we obtain such element $p_0 \in H_0$, that $\nabla p_0 = \nabla p$. From (1.3) it follows that

$$p_0 = p - \frac{1}{mes\Omega_0}\int\limits_{\Omega_0} pdx = p + \frac{1}{mes\Omega_0}\int\limits_{\Omega_0} div\omega dx$$

$$= p - \frac{1}{k_0 mes\ \Omega_0}\int\limits_{\Sigma}\left(\omega, n\right)ds = p - \frac{\rho_0}{k_0 mes\ \Omega_0}\int\left(u, n\right)ds.$$

Thus,

$$\nabla p_0=\nabla p, p_0\in H_0, p_0=p-\frac{\rho_0}{k_0 mes\,\Omega_0}\int(u,n)\,ds. \tag{2.5}$$

We multiply both sides of equation (1.4) by any given element $\omega_1\in L_2(\Omega_0)$ and using (2.3), (2.4), we get the equation in the space

$$\left(\lambda^2E-2\varepsilon\lambda A\right)x=x_{0,}\;x_0=(p_0,0)\,.$$

Writing the operator A as

$$A=\left(\begin{array}{cc}A_{11}&A_{12}\\A_{21}&A_{22}\end{array}\right),$$

where

$$A_{11}:H_0\to H_0,\;A_{12}:H_2\to H_0,$$

$$A_{21}:H_0\to H_2,\;A_{22}:H_2\to H_2.$$

We will rewrite acquired equation in the form of:

$$\lambda^2\varphi-2\varepsilon\lambda\left(A_{11}\varphi+A_{12}\vartheta\right)=p_0, \tag{2.6}$$

$$\lambda^2\varphi-2\varepsilon\lambda\left(A_{21}\varphi+A_{22}\vartheta\right)=0. \tag{2.7}$$

From (2.3) and (2.4) it directly follows that $A^*=A,\;\|A_{ij}\|\le 1$ that is $A^*_{ij}=A_{ij}$, $\|A_{ij}\|\le 1,\;(i,j=1,2)$. In [7] it is proved that for any area Ω_0 with sufficiently smooth boundary, the spectrum of A_{22} operator is just continuous and fills segment [-1,1] entirely. This operator is referred to as gyroscopic. Thus, we reduced the equation (1.4) to the system of equations (2.6) and (2.7). Now we will consider the equation (1.3), we will search the function $\omega\in W_2^1(\Omega_0)$. Putting $\omega=\nabla\varphi+\vartheta\quad(\vartheta\in H_2,\,\varphi\subset H_0)$ to the (1.3) and using (2.1) we obtain that,

$$\left\{\begin{array}{c}-\frac{1}{k_0}\Delta\varphi=p,\;\left(\Omega_0\right),\\ \frac{\partial\varphi}{\partial n}|_\Sigma=\rho_0\left(u,n\right)|_\Sigma,\;\varphi\in H_0.\end{array}\right. \tag{2.8}$$

In other words the equation (1.3) is reduced to the problem (2.8). By multiplying both sides of the equality (2.8) by any function $\alpha\in W_2^1(\Omega_0)$, integrating it by parts and taking into (2.8), we obtain the condition of solvability solution of this problem in the class $W_2^1(\Omega_0)$, and for the solution $\varphi(x)$ the following form holds:

$$\frac{1}{k_0}\left(\nabla\varphi,\nabla\alpha\right)_{L_2(\Omega_0)}+\frac{\rho_0}{k_0}\int\limits_\Sigma(u,n)\,\overline{\alpha}ds=\left(p_0,\alpha\right)_{L_2(\Omega_0)}\text{ for any }\alpha\in H_0. \tag{2.9}$$

Now, from $(\alpha,1)_{L_2(\Omega_0)}=0$ it follows the equality

$$\left(p,\alpha\right)_{L_2(\Omega_0)}=\left(p_0,\alpha\right)_{L_2(\Omega_0)}.$$

In the space H_0 we introduce the inner product by the formula

$$< \varphi, \alpha >_1 = \frac{1}{\rho_0} \left(\nabla \varphi, \nabla \alpha \right)_{L_2(\Omega_0)} . \tag{2.10}$$

Now, we consider the bounded linear operator $B^{-1} : H_0 \to H_0 \to D(B^{-1}) = H_0$ that defined as:

$$\frac{k_0}{\rho_0} \left(\varphi, \alpha \right)_{L_2(\Omega_0)} =< B^{-1} \varphi, \alpha >_1 . \tag{2.11}$$

The operator B^{-1} is self-adjoint, positive definite and completely continuous [6].

Let Σ^+ and Σ^- are trace taking operators on the surface Σ from the functions correspondingly regard the spaces $W_2^1(\Omega_0)$ and $W_2^1(\Omega)$. S and S_1 are linear bounded operators defined by the equations

$$S\psi = (\psi, n) , S_1 \varphi = \varphi n.$$

It is easy to see that

$$S_1 = S^*.$$

Since $\alpha \in H_0$ is the arbitrary function, using (2.9) and (2.11), we can rewrite (2.9) as:

$$\varphi + \left(\Sigma^+ \right)^* S \, \Sigma^- u - B^{-1} p_0 = 0, \tag{2.12}$$

where $(\Sigma)^* : L_2(\Sigma) \to W_2^1(\Omega_0)$ is a conjugate operator of the trace operator

$$\Sigma : W_2^1(\Omega_0) \to L_2(\Sigma)$$

to respect the inner product (2.10). So we reduce the equality (1.3) to the equation (2.12). Now we consider the problem (1.2). By multiplying both sides of (1.2) by the arbitrary vector-function τ and integrating it by parts (see [2] p. 72-83), we get [6] the following integral equality:

$$\sum_{p,q=1}^{3} \int_{\Omega} \left(a_{pq} \frac{\partial u}{\partial x_q}, \frac{\partial \tau}{\partial x_p} \right) dx \tag{2.13}$$

$$= \lambda^2 \left(\rho u, \tau \right)_{L_2(\Omega)} - \left(pn, \tau \right)_{L_2(\Sigma)} , \text{ for any } \tau \in W_2^1 \left(\Omega \right) .$$

We will call the generalized solution $u(x)$ of the problem (1.2) of class $W_2^1(\Omega)$ if the vector-function $u(x) \in W_2^1(\Omega)$ satisfied to the integral equality (2.13). In the $W_2^1(\Omega)$ using (2.5) and the last formula, we can introduce the new inner product by the formula:

$$< u, \tau >_2 = \sum_{p,q=1}^{3} \int_{\Omega} \left(a_{pq} \frac{\partial u}{\partial x_q}, \frac{\partial \tau}{\partial x_p} \right) dx + \left(\rho u, \tau \right)_{L_2(\Omega)} \tag{2.14}$$

$$+\frac{\vartheta_0}{k_0 mes\,\Omega_0}\int\limits_{\Sigma}(u,n)\,ds\int\limits_{\Sigma}\overline{(\tau,n)}ds.$$

So, we get

$$< u,\tau >_2 - (\rho u,\tau)_{L_2(\Omega)} + (p_0 n,\tau)_{L_2(\Sigma)} \tag{2.15}$$

$$-\lambda^2(\rho u,\tau)_{L_2(\Omega)} = 0,\ \text{ for any } \tau\in W_2^1(\Omega).$$

Now we consider the linear bounded operator

$$A^{-1}: W_2^1(\Omega)\to W_2^1(\Omega),\quad D(A^{-1}) = W_2^1(\Omega)$$

that defined from the equality

$$(\rho u,\tau)_{L_2(\Omega)} =< A^{-1}u,\tau >_2 \ \text{ for any } \tau,u\in W_2^1(\Omega). \tag{2.16}$$

The operator A^{-1} is self-adjoint, positive definite and completely continuous [6]. Applying (2.15) and (2.16), we obtain the equation

$$\left(E - A^{-1}\right)u + \left(\Sigma^-\right)^* S_1\Sigma^+ p_0 - \lambda^2 A^{-1}u = 0. \tag{2.17}$$

Here $(\Sigma^-)^*: L_2(\Sigma)\to W_2^1(\Omega)$ is the conjugate of the trace operator $\Sigma^-: W_2^1(\Omega)\to L_2(\Sigma)$ to respect the inner product (2.14). Thus we can reduced the problem (1.2) to the equation (2.17). Hence, we have the following result.

Theorem 2.1. Problem (1.2)–(1.4) is equivalent to the following system of equations

$$\begin{cases} (E - A^{-1})\,u + (\Sigma^-)^* S_1\Sigma^+ p_0 - \lambda^2 A^{-1}u = 0,\\ E\varphi + (\Sigma)^* S\,\Sigma^- u - B^{-1}p_0 = 0,\\ \lambda^2\varphi - 2\varepsilon\lambda\,(A_{11}\varphi + A_{12}\vartheta) = p_0,\\ \lambda^2 v - 2\varepsilon\lambda\,(A_{21}\varphi + A_{22}\vartheta) = 0. \end{cases} \tag{2.18}$$

Here $u\in W_2^1(\Omega),\ \varphi,p_0\in H_0, \vartheta\in H_2$.

We note that $Q = (\Sigma^-)^* S_1\Sigma^+, V = QBQ^*$. Since

$$< (E+V)\,u,u >_2 =< u,u >_2 + < QBQ^*u,u >_2$$

$$=< u,u >_2 + < BQ^*u,Q^*u >_1 = \|u\|_2^2 + \left\|B^{\frac{1}{2}}Q^*u\right\|_1^2 \geq \|u\|_2^2,$$

we have that the operator $(E+V)$ is positively definite. Using Friedrichs theorem on the extended of a self-adjoint definite operator, we get existence of bounded, positive self-adjoint operator $(E+V)^{1\backslash 2}$. Making the substitution $\varphi = B^{-1}\zeta,\ u = (E+V)^{-1/2}\xi$, we arrive at the self-adjoint quadratic bundle

$$\pounds\,(\lambda)\,z = 0,\ z = (\zeta,\xi,\nu)$$

in the space $\Im = W_2^1(\Omega) \oplus H_0 \oplus H_2$. Here

$$\pounds(\lambda) = \pounds_0 + 2\varepsilon\lambda\pounds_1 - \lambda^2\pounds_2,$$

$$\pounds_0 = \begin{pmatrix} E - (E+v)^{-\frac{1}{2}} A^{-1} (E+v)^{-\frac{1}{2}} & (E+v)^{-\frac{1}{2}} Q & 0 \\ Q^* (E+v)^{-\frac{1}{2}} & B^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\pounds_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B^{-1}A_{11}B^{-1} & B^{-1}A_{12} \\ 0 & A_{21}B^{-1} & A_{22} \end{pmatrix},$$

$$\pounds_2 = \begin{pmatrix} (E+v)^{-\frac{1}{2}} H^{-1} (E+v)^{-\frac{1}{2}} & 0 & 0 \\ 0 & B^{-2} & 0 \\ 0 & 0 & E \end{pmatrix}.$$

It is easy to show that $\pounds_2 > 0$. Now, we will prove that $\pounds_0 \geq 0$.

Indeed,

$$(\pounds_0 z, z)_{\Im} =< \left(E - (E+v)^{-\frac{1}{2}} A^{-1} (E+v)^{-\frac{1}{2}}\right) \zeta, \zeta >_2$$

$$+ < (E+v)^{-\frac{1}{2}} Q\zeta, \zeta >_2 + < Q^* (E+v)^{-\frac{1}{2}} \zeta, \zeta >_1$$

$$+ < B^{-1}\zeta, \zeta >_1 =< \left(E - (E+v)^{-\frac{1}{2}} H^{-1} (E+v)^{-\frac{1}{2}}\right) \zeta, \zeta >_2$$

$$+ < B^{-1}\zeta, \zeta >_1 + 2\Re < (E+v)^{-\frac{1}{2}} \zeta, Q\zeta >_1 .$$

Making the substitution $u = (E+v)^{-1/2}\xi$, $\varphi = B^{-1}\zeta$, we can write

$$(\pounds_0 z, z)_{\Im} =< (E - H^{-1})u, u >_2$$

$$+ \left\| B^{\frac{1}{2}} Q^* u \right\|_1^2 + \left\| B^{\frac{1}{2}} \varphi \right\|_1^2 + 2\Re < B^{\frac{1}{2}} Q^* u, B^{\frac{1}{2}} \varphi >_1 .$$

From Cauchy-Bunyakowskii-Schwarz inequality and the condition $E - A^{-1} \geq 0$ it follows that $(\pounds_0 z, z) \geq 0$, $z \in J$. This means that $\pounds_0 \geq 0$. Finally, applying the investigation of the quadratic bundle and using the scheme of the papers [3],[6], we obtain the following result.

Theorem 2.2. The spectrum of this bundle, and hence also of the problem (1.2)-(1.4) is real; moreover, on $[-2\varepsilon; +2\varepsilon]$ it is continuous and fills out the interval and on the intervals $R \backslash [-2\varepsilon; +2\varepsilon]$ it consists of isolated eigenvalues λ_n of finite algebraic multiplicity wit limit points at $\pm\infty$. If eigenvalue λ_n of this problem corresponds to the eigenfunctions (u_n, ω_n) then number $-\lambda_n$ also will be the eigenvalue of this problem corresponds to the eigenfunctions $(\overline{u}_n, \overline{\omega}_n)$ and this system of eigenfunctions is twice complete in the space $W_2^1(\Omega) \oplus H_0$, possibly to within a finite defect.

References

[1] S.L. Sobolev, On one new problem of mathematical phisics, *Izv. AN SSSR, Ser. Mathematics*, **18**, 3–50, 1954 (Russian).

[2] G. Fichera, The theorem of existence in theory of the elacity, Moskow, 1974 (Russian).

[3] A. Garadzhaev, Spectral theory of problems on small oscillations of an ideal fluid in rotating containers, *Differential Equations*, **23**(1), pp. 38–47, 1987 (Russian).

[4] A. Garadzhaev, On spectral theory for problems on normal oscillations of an ideal fluid in rotating containers. *Dokl. AN. SSSR*, **264** (2), pp. 277–281, 1982 (Russian).

[5] A. Garadzhaev, Spectral theory of problems on normal oscillations of an ideal compressible fluid in rotating rigit containers, *Differential Equations*, **23**(12), pp. 2080–2087, 1987 (Russian).

[6] A. Garadzhaev, Spectral theory of problems on small oscillations of an ideal compressible fluid in rotating elastic containers, *Differential Equations*, **24**(1), pp. 21-32, 1988 (Russian).

[7] J.V. Ralston, On stationary modes in inviscid rotating fluids, *J. of Math. Anal. and Appl.*, **44**, pp. 366–383, 1973.

[8] N.N. Moiseev and V.V. Rumyantsev, Dynamic stability of bodies containing fluid, Nauka, Moskow, 1965.

[9] A. Garadzhaev, Some problems on normal oscillations of the compressed fluid in rotating containers, in: Some Problems of Applied Mathematics, Fatih University Publications, Fatih University, Istanbul, 2000, pp. 128–136.

[10] M.B. Orazov, On the problem of normal oscillations of capillary ideal fluid filling partially an elastic vessel, in: Correct boundary value problems for non-classical equations of mathematical physics, *Collect. Artic.*, Novosibirsk 1980, pp. 148–151, 1980 (Russian).

[11] A.L. Goldenveizer, V.B. Lidskii and P.E. Tovstih, Free oscillations of thin elastic shells, Nauka, Moskow, 1979 (Russian).

[12] I.M. Rapoport, Dynamics of elastic containers partially filled with liquid, Berlin-Heidelberg-New York: Springer-Verlag, VIII, 368 p. (1968).

[13] N.D. Kopachevskii, S.G. Krein and Ngo Zui Kan, Operator Approach in Linear Problems of Hydrodynamics, Nauka, Moskow, 1989 (Russian).

[14] S.A. Gabov, Spectrum and basis of eigenvectors of one problem of acoustic oscilations of a fluid in rotating containers. *Dokl. AN. SSSR*, **254**(4), pp. 777–779, 1980 (Russian).

[15] P.K. Pal and V.N. Maslennikova, Spectral properties of operators for the problems on oscillations of the compressible fluid in rotating containers. *Dokl. AN. SSSR*, **281**(3), pp. 529–534, 1985 (Russian).