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Semigroup Extensions of Isometry Groups of Compactified Spacetimes

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Abstract

We investigate the possibility of semigroup extensions of the isometry group of an identification space, in particular, of a compactified spacetime arising from an identification map $p : \mathbb{R}^n_t \rightarrow \mathbb{R}^n_t/\Gamma$, where $\mathbb{R}^n_t$ is a flat pseudo-Euclidean covering space and Γ is a discrete group of primitive lattice translations on this space. We show that the conditions under which such an extension is possible are related to the index of the metric on the subvector space spanned by the lattice vectors: If this restricted metric is Euclidean, no extensions are possible. Furthermore, we provide an explicit example of a semigroup extension of the isometry group of the identification space obtained by compactifying a Lorentzian spacetime over a lattice which contains a lightlike basis vector. The extension of the isometry group is shown to be isomorphic to the semigroup $(\mathbb{Z}^\times, \cdot)$, i.e. the set of nonzero integers with multiplication as composition and 1 as unit element. A theorem is proven which illustrates that such an extension is obstructed whenever the metric on the covering spacetime is Euclidean.

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1. Introduction

In this work we examine the structure of the isometry group of orbit spaces which are obtained from the action of a discrete group on a flat pseudo-Euclidean covering space. The discrete group will be realized as the set of primitive lattice transformations of a lattice in the covering space which contains a lightlike lattice vector. In this case, the restriction of the pseudo-Euclidean metric to the lattice is no longer (positive or negative) definite. This gives rise to the possibility of having lattice-preserving transformations in the overall isometry group that are injective, but no longer surjective on the lattice; in other words, their inverses do not preserve the lattice. The set of all these transformations therefore will no longer be a group, but only a semigroup. Since it is precisely the lattice–preserving transformations that descend to the quotient of the covering spacetime over the discrete group of primitive lattice translations, these semigroup elements constitute an extension of the isometry group of the compactified space which act non-invertibly on the identification space. We present in detail the case of the compactification of a Lorentzian spacetime over a lightlike lattice, where it is shown that the non-invertible elements wind the lightlike circle k times around itself. We also present a theorem which reveals that the possibility of semigroup extensions arises whenever the restriction of the metric of the covering spacetime to the subspace spanned by the lattice vectors is non-Euclidean. As a consequence, semigroup extensions of isometry groups can be expected in every scenario involving a torus compactification of a higher-dimensional spacetime along a lightlike direction.

The plan of the paper is as follows: In section 2 we provide some background on orbit spaces and the associated fibre-preserving sets. In section 3 we study identifications over lattices and examine a condition which obstructs nontrivial extensions of the isometry group on the identification space. In section 4 we give an explicit example of a non-trivial extension, provided by compactifying a flat Lorentzian spacetime over a lightlike lattice. It will be shown that the extension of the isometry group in this case is isomorphic to the semigroup $(\mathbb{Z}^\times, \cdot)$ of nonzero integers with multiplication as composition and 1 as unit element.

2. Orbit Spaces and Normalizing sets

Assume that a group G has a left action on a topological space X such that the map $G \times X \ni (g, x) \mapsto gx \in X$ is a homeomorphism. When a discrete subgroup $\Gamma \subset G$ acts properly discontinuously and freely on X, then the natural projection $p : X \to X/\Gamma$ of X onto the space of orbits, X/Γ, can be made into a covering map, and X becomes a covering space of X/Γ (e.g. [1, 2, 3, 4]). More specifically, if $X = M$ is a connected pseudo-Riemannian manifold with a metric η, and $G = I(M)$ is the group of isometries of M, so that Γ is a discrete subgroup of isometries acting on M, then there is a unique way to make the quotient M/Γ a pseudo-Riemannian manifold (e.g. [5, 6, 7, 8, 9]); in this construction one stipulates that the projection p be a local isometry, which determines the metric on M/Γ. In such a case, we call $p : M \to M/\Gamma$ a pseudo-Riemannian covering.

In any case the quotient $p : X \to X/\Gamma$ can be regarded as a fibre bundle with bundle space X, base X/Γ, and Γ as structure group, the fibre over $m \in X/\Gamma$ being the orbit of any element $x \in p^{-1}(m)$ under Γ, i.e. $p^{-1}(m) = \Gamma x = \{\gamma x \mid \gamma \in \Gamma\}$. If $g \in G$ induces the homeomorphism $x \mapsto gx$ of X [or an isometry of M], then g gives rise to a well-defined map $g_\# : X/\Gamma \to X/\Gamma$, defined by

$$g_\#(\Gamma x) \equiv \Gamma(gx) \quad , \tag{2.1}$$

on the quotient space **only** when g preserves all fibres, i.e. when $g\,(\Gamma x) \subset \Gamma(gx)$ for all $x \in X$. This is equivalent to saying that $g\Gamma g^{-1} \subset \Gamma$. If this relation is replaced by the stronger condition $g\Gamma g^{-1} = \Gamma$, then g is an element of the *normalizer* [10] $N\,(\Gamma)$ of Γ in G, where

$$N\,(\Gamma) = \left\{ g \in G \mid g\Gamma g^{-1} = \Gamma \right\} \quad . \tag{2.2}$$

The normalizer is a group by construction. It contains all fibre-preserving elements g of G such that g^{-1} is fibre-preserving as well. In particular, it contains the group Γ, which acts trivially on the quotient space; this means, that for any $\gamma \in \Gamma$, the induced map $\gamma_\# : X/\Gamma \to X/\Gamma$ is the identity on X/Γ. This follows, since the action of $\gamma_\#$ on the orbit Γx, say, is defined to be $\gamma_\#\,(\Gamma x) := \Gamma\,(\gamma x) = \Gamma x$, where the last equality holds, since Γ is a group.

In this work we are interested in relaxing the equality in the condition defining $N\,(\Gamma)$; to this end we introduce what we wish to call the *extended normalizer*, denoted by $eN\,(\Gamma)$, through

$$eN\,(\Gamma) := \left\{ g \in G \mid g\Gamma g^{-1} \subset \Gamma \right\}. \tag{2.3}$$

The elements $g \in G$ which give rise to well-defined maps $g_\#$ on X/Γ are therefore precisely the elements of the extended normalizer $eN\,(\Gamma)$, as we have seen in the discussion above. Such elements g are said to *descend to the quotient space* X/G. Hence $eN\,(\Gamma)$ contains all homeomorphisms of X [isometries of M] that descend to the quotient space X/Γ [M/Γ]; the normalizer $N\,(\Gamma)$, on the other hand, contains all those g for which g^{-1} descends to the quotient as well. Thus, $N\,(\Gamma)$ is the group of all g which descend to **invertible** maps $g_\#$ [homeomorphisms; isometries] on the quotient space. In the case of a semi-Riemannian manifold M, for which the group G is the isometry group $I\,(M)$, the normalizer $N\,(\Gamma)$ therefore contains all isometries of the quotient space, the only point being that the action of $N\,(\Gamma)$ is not effective, since $\Gamma \subset N\,(\Gamma)$ acts trivially on M/Γ. However, Γ is a normal subgroup of $N\,(\Gamma)$ by construction, so that the quotient $N\,(\Gamma)\,/\Gamma$ is a group again, which is now seen to act effectively on M/Γ, and the isometries of M/Γ which descend from isometries of M are in a 1–1 relation to elements of this group. Thus, denoting the isometry group of the quotient space M/Γ as $I\,(M/\Gamma)$, we have the well-known result that

$$I(M/\Gamma) = N(\Gamma)/\Gamma \quad . \tag{2.4}$$

Now we turn to the extended normalizer. For an element $g \in eN\,(\Gamma)$, but $g \notin N\,(\Gamma)$, the induced map $g_\#$ is no longer injective on X/Γ: To see this we observe that now the inclusion in definition (2.3) is proper, i.e. $g\Gamma g^{-1} \subsetneqq \Gamma$. It follows that a $\gamma' \in \Gamma$ exists for which

$$g\gamma g^{-1} \neq \gamma' \quad \text{for all } \gamma \in \Gamma. \tag{2.5}$$

Take an arbitrary $x \in M$; we claim that

$$g\Gamma g^{-1} x \subsetneqq \Gamma x. \tag{2.6}$$

To see this, assume to the contrary that the sets $g\Gamma g^{-1}x$ and Γx coincide; then $\gamma_1, \gamma' \in \Gamma$ exist for which $g\gamma_1 g^{-1}x = \gamma' x$; since g is an element of the extended normalizer $eN(\Gamma)$,

$g\gamma_1 g^{-1} \in \Gamma$, i.e., $g\gamma_1 g^{-1} = \gamma_2$, say. It follows that $\gamma_2 x = \gamma' x$ or $\gamma_2^{-1}\gamma' x = x$. The element $\gamma_2^{-1}\gamma'$ belongs to Γ, which, by assumption, acts freely. Free action implies that if a group element has a fixed point then it must be the unit element, implying that $\gamma' = \gamma_2 = g\gamma_1 g^{-1}$, which contradicts (2.5); therefore, (2.6) must hold. Eq. (2.6) can be expressed by saying that the orbit of x is the image of the orbit of $g^{-1}x$ under the action of the induced map $g_\#$, $g_\#(\Gamma g^{-1}x) = \Gamma x$; but that the $g_\#$-image of the orbit $\Gamma g^{-1}x$, regarded as a set, is properly contained in the orbit of x. The latter statement means that a $\gamma' \in \Gamma$ exists, as in (2.5), such that $\gamma' x \neq g\gamma g^{-1}x$ for all γ. It follows that $\gamma g^{-1}x \neq g^{-1}\gamma' x$ for all γ, implying that the point $g^{-1}\gamma' x$ is not contained in the orbit of the point $g^{-1}x$. Its own orbit, $\Gamma g^{-1}\gamma' x$, is therefore distinct from the orbit $\Gamma g^{-1}x$ of the point $g^{-1}x$, since two orbits either coincide or are disjoint otherwise. However, the induced map $g_\#$ maps $\Gamma g^{-1}\gamma' x$ into

$$g_\#(\Gamma g^{-1}\gamma' x) = \Gamma(g g^{-1}\gamma' x) = \Gamma x \quad , \tag{2.7}$$

from which it follows that $g_\#$ maps two distinct orbits into the same orbit Γx, expressing the fact that $g_\#$ is not injective. In particular, if g was an isometry of M, then $g_\#$ can no longer be an isometry on the quotient space, since it is not invertible on the quotient space. From this fact, or directly from its definition (2.3), it also follows that $eN(\Gamma)$ is a semigroup.

In this work we will show that the extended normalizer naturally emerges when we study identification spaces M/Γ, where $M = (\mathbb{R}^n, \eta)$ is flat $\mathbb{R}^n$ endowed with a symmetric bilinear form η with signature $(-t, +s)$, $t + s = n$; to such a space we will also refer to as $M = \mathbb{R}^n_t$. The isometry group $I(\mathbb{R}^n_t)$ of $\mathbb{R}^n_t$ is the semi-direct product

$$I(\mathbb{R}^n_t) = \mathbb{E}(t, n-t) = \mathbb{R}^n \odot O(t, n-t) \quad , \tag{2.8}$$

called *pseudo-Euclidean group* $\mathbb{E}(t, n-t)$, where the translational factor $\mathbb{R}^n$ is normal in $\mathbb{E}(t, n-t)$. Elements of $\mathbb{E}(t, n-t)$ will be denoted by (t, R) with group law $(t, R)(t', R') = (Rt' + t, RR')$. The Lie algebra of the pseudo-Euclidean group $\mathbb{E}(t, n-t)$ will be denoted by $\mathsf{eu}(t, n-t)$ henceforth.

For $t = 1$, $\mathbb{E}(1, n-1)$ is the Poincare group, and $\mathbb{R}^n_1$ is a Lorentzian manifold, with metric $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. In this case we call elements $v \in \mathbb{R}^n_1$ timelike, lightlike, spacelike if $\eta(v, v) < 0, = 0, > 0$, respectively.

The identification group Γ will be realized as a discrete group of translations in M, the elements being in 1–1 correspondence with the points of a lattice $\mathsf{lat} \subset \mathbb{R}^n_t$, which is regarded as a subset of $\mathbb{R}^n_t$. We will show explicitly that in the Lorentzian case, the fact that the identity component $SO_0(1, n-1) \subset O(1, n-1)$ of the Lorentz group is no longer compact gives rise to a natural extension of the isometry group $N(\Gamma)/\Gamma$ of the quotient M/Γ to the set $eN(\Gamma)/\Gamma$, provided that the $\mathbb{R}$-linear envelope of the lattice is a lightlike subvector space. This will be compared to the orthogonal case, and it will be shown in Theorem [3] that the compactness of $SO(n)$ obstructs such an extension. That is probably why such extensions have not been studied in the field of crystallography in the past.

3. Identifications Over a Lightlike Lattice

As mentioned above, we shall study quotient spaces of flat covering spacetimes over a group Γ of primitive lattice transformations. If the $\mathbb{R}$-linear span of the lattice vectors has (real) dimension m, say, the resulting identification space M/Γ is homeomorphic to a product manifold $\mathbb{R}^{n-m} \times T^m$, where T^m denotes an m-dimensional torus. This space inherits the metric from the covering manifold $\mathbb{R}^n_t$, since the metric is a local object, but the identification changes only the global topology. Thus M/Γ is again a semi-Riemannian manifold. Whereas the isometry group of the covering space $\mathbb{R}^n_t$ is $\mathbb{E}(t, n-t)$, the isometry group of the "compactified" space M/Γ is obtained from the normalizer $N(\Gamma)$ of the group Γ in $\mathbb{E}(t, n-t)$ according to formula (2.4).

In this work we restrict attention to lattices that contain the origin $0 \in \mathbb{R}^n_t$ as a lattice point, which suffices for our purposes. Let $1 \leq m \leq n$, let $\underline{u} \equiv (u_1, \ldots, u_m)$ be a set of m linearly independent vectors in $\mathbb{R}^n_t$; then the $\mathbb{Z}$-linear span of $\underline{u}$,

$$\mathsf{lat} \equiv \sum_{i=1}^{m} \mathbb{Z} \cdot u_i = \left\{ \sum_{i=1}^{m} z_i \cdot u_i \mid z_i \in \mathbb{Z} \right\} \quad , \tag{3.1}$$

is called the set of lattice points with respect to $\underline{u}$. Elements of lat are regarded as points in $\mathbb{R}^n_t$ as well as vectors on the tangent space $T\mathbb{R}^n_t$ of $\mathbb{R}^n_t$. Let $[\mathsf{lat}] \equiv [\underline{u}]_{\mathbb{R}}$ denote the $\mathbb{R}$-linear span of lat.

We recall that the *index* $ind(W)$ of a vector subspace $W \subset \mathbb{R}^n_t$ is the maximum in the set of all integers that are the dimensions of $\mathbb{R}$-vector subspaces $W' \subset W$ on which the restriction of the metric $\eta| W'$ is negative definite, see e.g. [8]. Hence $0 \leq ind(W) \leq m$, and $ind(W) = 0$ if and only if $\eta| W$ is positive definite. In the Lorentzian case, i.e. $M = \mathbb{R}^n_1$, we call W timelike $\Leftrightarrow$ $\eta| W$ is nondegenerate, and $ind(W) = 1$; W lightlike $\Leftrightarrow$ $\eta| W$ degenerate, and W contains a 1-dimensional lightlike vector subspace, but no timelike vector; and W spacelike $\Leftrightarrow$ $\eta| W$ is positive definite and hence $ind(W) = 0$.

We define the *index of the lattice* as the index of its $\mathbb{R}$-linear span $[\mathsf{lat}]$, $ind(\mathsf{lat}) \equiv ind([\mathsf{lat}])$.

Let $T_{\mathsf{lat}} \subset \mathbb{E}(t, n-t)$ be the subgroup of all translations in $\mathbb{E}(t, n-t)$ through elements of lat,

$$T_{\mathsf{lat}} = \{(t_z, 1) \in \mathbb{E}(t, n-t) \mid t_z \in \mathsf{lat}\} \, . \tag{3.2}$$

Elements $(t_z, 1)$ of T_{lat} are called *primitive translations*. T_{lat} is taken as the discrete group Γ of identification maps, which give rise to the identification space $p : \mathbb{R}^n_t \to \mathbb{R}^n_t/\Gamma$ under study.

We now examine the normalizer and extended normalizer of the identification group: For an element $(t, R) \in \mathbb{E}(t, n-t)$ to be in the extended normalizer $eN(\Gamma)$ of Γ, the condition

$$(t, R)(t_z, 1)(t, R)^{-1} = (Rt_z, 1) \in T_{\mathsf{lat}} \tag{3.3}$$

must be satisfied for lattice vectors t_z. In other words, $Rt_z \in \mathsf{lat}$, which means that the pseudo-orthogonal transformation R must preserve the lattice lat, $R\mathsf{lat} \subset \mathsf{lat}$. For an element (t, R) to be in the normalizer, $(t, R)^{-1}$ must be in the normalizer as well, implying $R^{-1}\mathsf{lat} \subset \mathsf{lat}$, so altogether $R\mathsf{lat} = \mathsf{lat}$. The elements R occuring in the normalizer therefore naturally form a subgroup G_{lat} of $O(t, n-t)$; on the other hand, the elements R occuring in the extended normalizer form a semigroup $eG_{\mathsf{lat}} \supset G_{\mathsf{lat}}$. Furthermore, no condition on the translations t in (t, R) arises, hence all translations occur in the [extended] normalizer. Thus, the [extended] normalizer has the structure of a semi-direct [semi-]group

$$N(\Gamma) = \mathbb{R}^n \odot G_{\mathsf{lat}} \quad , \tag{3.4}$$

$$eN(\Gamma) = \mathbb{R}^n \odot eG_{\mathsf{lat}} \quad , \tag{3.4'}$$

where $\mathbb{R}^n$ refers to the subgroup of all translations in $\mathbb{E}(t, n-t)$.

We now present a condition under which eG_{lat} coincides with G_{lat}:

Theorem 3.1. [Condition] If $ind(\mathsf{lat}) = 0$ or $ind(\mathsf{lat}) = m$ (i.e. minimal or maximal), then $eG_{\mathsf{lat}} = G_{\mathsf{lat}}$.

Proof. We first assume that $ind(W) = 0$, i.e. $\eta|\,W$ is positive definite. Let $R \in eG_{\mathsf{lat}} \subset O(t, n-t)$. Since R preserves lat, it also preserves its $\mathbb{R}$–linear envelope W, i.e. $RW \subset W$. Let $x, y \in W$ arbitrary, then $Rx, Ry \in W$. Since $\eta(Rx, Ry) = \eta(x, y)$, the restriction $R|\,W$ of R to the subvector space W preserves the bilinear form $\eta|\,W$ on this space. But $\eta|\,W$ is positive definite by assumption of $ind(W) = 0$, hence $R|\,W \in O(W)$, where $O(W)$ denotes the orthogonal group of W.

Now we assume that R has the property

$$R \in eG_{\mathsf{lat}}, \quad \text{but} \quad R \notin G_{\mathsf{lat}} \quad , \tag{3.5}$$

in other words, $R\mathsf{lat} \subsetneq \mathsf{lat}$. This means that the restriction $R|\,\mathsf{lat}$ is not surjective. Hence

$$\exists x \in \mathsf{lat} : Ru \neq x \quad \text{for all } u \in \mathsf{lat} \quad . \tag{3.6}$$

x cannot be zero, since $0 \in \mathsf{lat}$, and R is linear. Hence $r \equiv \|x\| > 0$, where $\|x\| = \sqrt{\eta(x, x)}$ denotes the Euclidean norm on W. Now let S_{m-1} be the $(m-1)$-dimensional sphere <u>in W</u>, centered at 0. Consider the intersection $\mathsf{sct} = \mathsf{lat} \cap r \cdot S_{m-1}$, where $r \cdot S_{m-1}$ is the $(m-1)$-dimensional sphere with radius r in W. Note that $r \cdot S_{m-1}$ coincides with the orbit $O(m-1) \cdot x$ of x under the action of the orthogonal group $O(m-1)$, which is a compact set on account of the fact that $O(m-1)$ is a compact Lie group. From the compactness of $r \cdot S_{m-1}$ it follows that the number of elements $\#\mathsf{sct}$ of sct is **finite**; from $x \in \mathsf{sct}$ it follows that sct is not empty, so $1 \leq \#\mathsf{sct} < \infty$. Then

1. $R|\,W$ orthogonal $\quad \Rightarrow \quad R(\mathsf{sct}) \subset r \cdot S_{n-1}$;

2. R lattice preserving $\quad \Rightarrow \quad R(\mathsf{sct}) \subset \mathsf{lat}$;

3. R injective $\quad \Rightarrow \quad \#R(\mathsf{sct}) = \#(\mathsf{sct})$.

The first two statements imply that $R|W$ preserves sct, $(R|W)(\mathsf{sct}) \subset \mathsf{sct}$; from the third we deduce that $(R|W)(\mathsf{sct}) = \mathsf{sct}$, since the set sct is finite. But this says that all elements of sct are in the image of $(R|W)$, hence $x = (R|W)(x')$ for some $x' \in \mathsf{sct}$, which is a contradiction to (3.6). This implies that our initial assumption (3.5) concerning R was wrong.

Now assume that $ind(\mathsf{lat})$ is maximal. Then $\eta|W$ is negative definite, but the argument given above clearly still applies, since $O(0, m-1) \simeq O(m-1, 0)$, and the only point in the proof was the compactness of the $O(m-1)$-orbits. This completes our proof. ■

We see that the structure of the proof relies on the compactness of orbits $O \cdot x$ of x under the orthogonal group, which, in turn, comes from the fact that the orthogonal groups O are compact. If the metric restricted to $[\mathsf{lat}]$ were pseudo-Euclidean instead, we could have non-compact orbits, related to the non-compactness of the groups $O(t, n-t)$. In this case we expect the possibility that $G_{\mathsf{lat}} \subsetneqq eG_{\mathsf{lat}}$ to arise. An explicit example of this situation will be given now.

4. Compactification of a Lorentzian Spacetime

The covering spacetime is chosen to be $M = \mathbb{R}^n_1$, i.e. an n-dimensional Lorentzian spacetime with metric $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. The lattice $\underline{u}$ is constructed as follows: Let $(\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1})$ denote the canonical basis of $\mathbb{R}^n_1$, where $\mathbf{e}_0$ is the "time direction" with $\eta(\mathbf{e}_0, \mathbf{e}_0) = -1$. We define $\mathbf{e}_\pm \equiv \frac{1}{\sqrt{2}}(\mathbf{e}_0 \pm \mathbf{e}_1)$, so the new basis vectors $\mathbf{e}_\pm$ are both lightlike. We split the new basis according to

$$(\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_2, \ldots, \mathbf{e}_{n-m}, \mathbf{e}_{n-m+1}, \ldots, \mathbf{e}_{n-1}). \tag{4.1}$$

As lattice vectors we choose $\underline{u} \equiv (\mathbf{e}_+, \mathbf{e}_{n-m+1}, \ldots, \mathbf{e}_{n-1})$. Upon compactifying these dimensions, i.e. taking the quotient of $\mathbb{R}^n_1$ over the associated group T_{lat}, we obtain an m-dimensional torus T^m; the remaining $n-m$ dimensions along $(\mathbf{e}_-, \mathbf{e}_2, \ldots, \mathbf{e}_{n-m})$ remain uncompactified, resulting in a product manifold homeomorphic to $\mathbb{R}^{n-m} \times T^m$.

We need to construct the lattice-preserving sets G_{lat} $[eG_{\mathsf{lat}}]$ in order to obtain the [extended] normalizer from formulae (3.4, 3.4'). We proceed in several steps: In step one we identify the set of Lorentz transformations on the covering space $\mathbb{R}^n_1$ which preserve the $\mathbb{R}$-linear span of the lattice vectors $\underline{u}$; thus, we have to identify those generators of Lorentz transformations which map elements of the lattice basis into linear combinations of these basis vectors. As will be seen below, the set of Lorentz transformations so obtained will form a Lie subgroup G_1 of $O(1, n-1)$. In step two we impose the further condition that not only the $\mathbb{R}$-linear span, but the $\mathbb{Z}$-linear span of basis vectors $\underline{u}$, and hence the true lattice points, shall be preserved. This restricts the group G_1 further down to a set G_2 which will be seen to contain a product of a discrete group, a Lie group and a discrete semigroup, which is what we wanted to show.

Let us proceed to step one: We seek the Lorentz transformations Λ which map elements of $\underline{u}$ into real linear combinations of elements of $\underline{u}$. Now, $\underline{u}$ contains one lightlike, and $m-1$ spacelike basis vectors. Assume that the image $\Lambda\,\mathbf{e}_+$ of $\mathbf{e}_+$ is given by a linear combination $\Lambda\,\mathbf{e}_+ = a\,\mathbf{e}_+ + \sum_{i=n-m+1}^{n-1} b_i\,\mathbf{e}_i$. Since $\mathbf{e}_+$ is lightlike and Λ preserves pseudo-Euclidean lengths we must have that $\sum_{i=n-m+1}^{n-1} b_i^2 = 0$, hence $\Lambda\,\mathbf{e}_+ = a\,\mathbf{e}_+$ necessarily. As a consequence, all admissible Λ preserve the one-dimensional lightlike subvector space $\mathbb{R}\cdot\mathbf{e}_+$. The task of step one is therefore performed in two substeps: First we identify the set of Lorentz transformations which preserve this subvector space, and secondly we impose the further restriction that the real linear span of all basis vectors, not only $\mathbf{e}_+$, be preserved.

We now identify the set of Lorentz transformations that preserve the one-dimensional subvector space $\mathbb{R}\cdot\mathrm{e}_+$. We do not wish to compute this set in the full Lorentz group but, for sake of simplicity, focus on the identity component $SO_0(1,n-1)$ of $O(1,n-1)$ instead. The end result will still exhibit the semigroup extension under consideration, and this is what we want to concentrate on.

We thus seek the subset of orthochronous orientation-preserving Lorentz transformations that preserve the light-like subvector space $\mathbb{R}\cdot\mathbf{e}_+$. Since these matrices are in the identity component of $O(1,n-1)$, they will be (products of) exponentials of Lie algebra elements $L\in so(1,n-1)$ which map the subvector space $\mathbb{R}\cdot\mathbf{e}_+$ into itself, including annihilation of $\mathbf{e}_+$. We use the following conventions for the matrices $L_{\alpha\beta}$ of a standard basis of $so(1,n-1)$:

$$L_{\alpha\beta} = -L_{\beta\alpha} \quad , \quad 0\le\alpha<\beta\le n-1 \quad , \tag{4.2a}$$

$$\left(L_{\alpha\beta}\right)_{\mu\nu} = \eta_{\alpha\mu}\,\delta_{\beta\nu} - \eta_{\beta\mu}\,\delta_{\alpha\nu} \quad , \quad \forall\,\alpha,\beta \quad . \tag{4.2b}$$

The Lie algebra of these matrices, realized by matrix commutators, is

$$[L_{\alpha\beta}, L_{\rho\sigma}] = -\eta_{\alpha\rho}\,L_{\beta\sigma} - \eta_{\beta\sigma}\,L_{\alpha\rho} + \eta_{\alpha\sigma}\,L_{\beta\rho} + \eta_{\beta\rho}\,L_{\alpha\sigma} \quad , \quad \forall\,\alpha,\beta \quad . \tag{4.3}$$

The special Lorentz transformations we seek are generated by those linear combinations of matrices (4.2) which map the vector $\mathbf{e}_+$ onto a multiple of itself, including annihilation. It is straightforward to find that these generators take the form

$$b\cdot L_{01} + \sum_{\alpha=2}^{n-1} v_\alpha\cdot U_\alpha + \sum_{2\le\alpha<\beta} \lambda_{\alpha\beta}\cdot L_{\alpha\beta} \quad , \tag{4.4a}$$

$$U_\alpha = L_{0\alpha} - L_{1\alpha} \quad , \tag{4.4b}$$

where $b\in\mathbb{R}$, $\vec{v}\equiv(v_2,\ldots,v_{n-1})\in\mathbb{R}^{n-2}$ and $\lambda_{\alpha\beta}\in\mathbb{R}$, $2\le\alpha<\beta\le n-1$. The commutator algebra of the set $(L_{01}, U_\alpha, L_{\alpha\beta})$ is given as

$$[L_{01}, U_\alpha] = -U_\alpha \quad , \tag{4.5a}$$

$$[L_{01}, L_{\alpha\beta}] = 0 \quad , \tag{4.5b}$$

$$[U_\alpha, U_\beta] = 0 \quad , \tag{4.5c}$$

$$[L_{\alpha\beta}, U_\gamma] = -\eta_{\alpha\gamma}\,U_\beta + \eta_{\beta\gamma}\,U_\alpha \quad , \tag{4.5d}$$

$$[L_{\alpha\beta}, L_{\rho\sigma}] = -\delta_{\alpha\rho}\,L_{\beta\sigma} - \delta_{\beta\sigma}\,L_{\alpha\rho} + \delta_{\alpha\sigma}\,L_{\beta\rho} + \delta_{\beta\rho}\,L_{\alpha\sigma} \quad . \tag{4.5e}$$

We see that the commutator algebra of the generators (4.4) closes, which is to be expected, since conceptually, the set of Lorentz transformations preserving a one-dimensional subspace must form a group. The set (4.4) therefore forms a Lie algebra $\hat{g}_1$, with associated Lie group G_1. The last three relations, (4.5c, 4.5d, 4.5e), express the fact that the generators $(L_{\alpha\beta}, U_\gamma)$, for $\alpha\beta\gamma \geq 2$, span a subalgebra of $\hat{g}_1$ which is isomorphic to the Euclidean algebra eu$(n-2)$ in $n-2$ dimensions, the U_γ playing the part of translation generators. This subalgebra is actually an ideal in $\hat{g}_1$, as follows from (4.5a, 4.5b). As a consequence, the Lie subgroup $\mathbb{E}(n-2)$ associated with the generators $(L_{\alpha\beta}, U_\gamma)$ is normal in G_1; the latter therefore is homomorphic to

$$G_1 \simeq \mathbb{E}(n-2) \odot \mathbb{R}^+ \quad , \tag{4.6}$$

where " $\odot$ " denotes a semidirect product (accounting for the normality of the $\mathbb{E}$-factor), and $\mathbb{R}^+$ denotes the group of all real numbers b with addition as group composition and zero as the neutral element.

In a certain neighbourhood of the identity, the Lorentz transformations generated by (4.4) may be obtained in the form

$$\Lambda\,(b, \vec{v}, C) = S(\vec{v}) \cdot R(C) \cdot B(b) =$$

$$= \left(\begin{array}{cc|c} 1+\frac{\vec{v}^2}{2} & -\frac{\vec{v}^2}{2} & \vec{v}^T \\ \frac{\vec{v}^2}{2} & 1-\frac{\vec{v}^2}{2} & \vec{v}^T \\ \hline \vec{v} & -\vec{v} & \mathbf{1_{n-2}} \end{array}\right) \cdot \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & C \end{array}\right) \cdot \left(\begin{array}{cc|c} \cosh b & \sinh b & 0 \\ \sinh b & \cosh b & 0 \\ \hline 0 & 0 & \mathbf{1_{n-2}} \end{array}\right) \quad , \tag{4.7}$$

where $\vec{v}^2 = \sum_{\alpha=2}^{n-1} v_\alpha^2$, $C = C(\lambda_{\alpha\beta})$ denotes elements of $SO(n-2)$, whilst $R(C)$ indicates rotations C embedded in $SO_0(1, n-1)$, as in the second matrix in (4.7). In this representation, the parameters $\vec{v}, \lambda_{\alpha\beta}, b$ serve as normal coordinates of the second kind [9].

The one-dimensional Lie subgroup represented by matrices $b \mapsto B(b)$ can be expressed in a different way: To this end we substitute $a = e^b$ in $\cosh b$ and $\sinh b$, with the result that

$$B(b) = B(\ln a) = \left(\begin{array}{cc|c} \frac{a+\frac{1}{a}}{2} & \frac{a-\frac{1}{a}}{2} & 0 \\ \frac{a-\frac{1}{a}}{2} & \frac{a+\frac{1}{a}}{2} & 0 \\ \hline 0 & 0 & \mathbf{1_{n-2}} \end{array}\right) . \tag{4.8}$$

Since $\ln(aa') = \ln a + \ln a'$, these matrices now represent $\mathbb{R}_+^\times$, i.e. the group of all positive numbers with multiplication as group composition and 1 as the unit element. If the same analysis as before is now repeated with respect to the full Lorentz group $O(1, n-1)$, not only its connected component $SO_+(1, n-1)$, we find that the group $\mathbb{R}_+^\times$ is extended by another component which comprises the negative numbers; as a consequence, the matrices (4.8) then represent the group of nonvanishing real numbers $\mathbb{R}^\times$ with multiplication and unit element as before, according to $\mathbb{R}^\times \ni a \mapsto \pm B(\ln |a|)$.

So far we have obtained the Lie subgroup of the (connected component of the) Lorentz group which preserves the one-dimensional subspace $\mathbb{R} \cdot \mathbf{e}_+$ spanned by the first lattice vector. We now implement the task of substep two of step one and impose the further condition that the whole real subspace spanned by all lattice vectors be preserved. The matrices satisfying this stronger condition comprise a subgroup $G_2 \subset G_1$. We find that the Lie algebra $\hat{g}_2$ of G_2 may be generated by the same set of basis elements that span $\hat{g}_1$, **except** that the generators $L_{\alpha i}$, $2 \leq \alpha \leq n-m$ and $n-m+1 \leq i \leq n-1$, now must be omitted. The generators of $\hat{g}_2$ are therefore

$$L_{01} \quad , \tag{4.9a}$$

$$(L_{\alpha\beta}, U_\gamma \,|\, 2 \leq \alpha < \beta \leq n-m;\, 2 \leq \gamma \leq n-m) \quad , \tag{4.9b}$$

$$(L_{ij}, U_k \,|\, n-m+1 \leq i < j \leq n-1;\, n-m+1 \leq k \leq n-1) \quad . \tag{4.9c}$$

The sets (4.9b) and (4.9c) span Euclidean algebras $\mathsf{eu}(n-m-1)$ and $\mathsf{eu}(m-1)$, respectively; furthermore, both algebras are ideals in the Lie algebra $\hat{g}_2$, as follows from three facts: 1.) the adjoint action of L_{01} maps all U_μ into $-U_\mu$, $\mu = 2, \ldots n-1$, see (4.5a); 2.) the adjoint action of L_{01} annihilates all $L_{\mu\nu}$, $2 \leq \mu < \nu \leq n-1$, see (4.5b); and 3.) both sets (4.9b) and (4.9c) commute, see (4.5c, 4.5d, 4.5e). $\hat{g}_2$ therefore contains a direct sum of two commuting Euclidean algebras $\mathsf{eu}(n-m-1)$ and $\mathsf{eu}(m-1)$ such that this direct sum is an ideal in the full algebra, in which the "conformal" dilation generator L_{01} acts nontrivially only on the Euclidean "translation" generators U_μ. The group G_2 is then obtained upon exponentiation of this Lie algebra as

$$G_2 \simeq [\mathbb{E}(n-m-1) \otimes \mathbb{E}(m-1)] \odot \mathbb{R}_+^\times \quad , \tag{4.10}$$

where the group $\mathbb{R}_+^\times$ is defined in formula (4.8). The first Euclidean group $\mathbb{E}(n-m-1)$ acts only on those dimensions which are linearly independent from the lattice dimensions; these dimensions are not involved in the compactification process and give rise, together with the basis vector $\mathbf{e}_-$, to the $\mathbb{R}^{n-m}$-factor in the compactified spacetime. On the other hand, the second Euclidean group $\mathbb{E}(m-1)$ acts on the remaining $(m-1)$ spacelike lattice dimensions spanned by $\mathbf{e}_{n-m+1}, \ldots, \mathbf{e}_{n-1}$. The notation in eq. (4.10) indicates that the direct product of these groups is normal in the full group G_2, hence the semidirect product with $\mathbb{R}_+^\times$.

So far we have identified the group G_2 that preserves the $\mathbb{R}$-linear envelope of the lattice vectors. We now implement step two and replace this $\mathbb{R}$-linear span by the original $\mathbb{Z}$-linear span, which gives the original lattice lat. The subgroup G_{lat} of the Lorentz group we seek is therefore a proper subgroup of G_2 which preserves the lattice points in the sense of $G_{\mathsf{lat}}\,\mathsf{lat} \subset \mathsf{lat}$. It is clear that in the process of this reduction the second of the Euclidean groups, $\mathbb{E}(m-1)$, now must be replaced by a *discrete* group D (which is called *maximal point group* in crystallography [11, 12]). It is also clear that the first Euclidean group $\mathbb{E}(n-m-1)$ remains unchanged, since it is not affected by the replacement $\mathbb{R} \to \mathbb{Z}$. Finally, the group $\mathbb{R}_+^\times$, which acts on the lightlike lattice vector $\mathbf{e}_+$ according to matrices (4.8), must be replaced by a discrete set of transformations which we now examine:

The group elements $B(\ln a)$, $a \in \mathbb{R}$, act on $\mathbf{e}_+$ according to

$$B(\ln a)\,\mathbf{e}_+ = a \cdot \mathbf{e}_+ \quad . \tag{4.11}$$

The main point comes now: Since the lattice must be preserved, the values of a must be restricted to integers, $B(\ln k)$, $k \in \mathbb{Z}$, in order to satisfy $B(\ln k)\,\mathbf{e}_+ = k \cdot \mathbf{e}_+ \in \mathsf{lat}$; we see that this transformation winds the lightlike circle associated with $\mathbf{e}_+$ k times around itself. Although the original set $B(\ln a)$ was a group, the set $B(\ln k)$ involving *integers* clearly is no longer a group, but only a *semi*group, since the operation $B(\ln k)$, $k \neq 1$, has no inverse in this set. This semigroup is isomorphic to the set $(\mathbb{N}_+, \cdot)$ of positive integers with multiplicative composition $(z, z') \mapsto z \cdot z'$, and 1 as unit. Clearly, $B(\ln k)$ is still an invertible element of the original Lorentz group $SO_0(1, n-1)$; however, it has no inverse in G_{lat}, since $B(\ln k)^{-1} = B(\ln \frac{1}{k})$ is not lattice-preserving, as it maps $\mathbf{e}_+ \mapsto \frac{1}{k} \cdot \mathbf{e}_+ \notin \mathsf{lat}$, for $k > 1$.

This analysis can be extended to include the full Lorentz group $O(1, n-1)$; in this case we find, following the line of arguments after eq. (4.8), that the semigroup $(\mathbb{N}_+, \cdot)$ is extended to the semigroup $(\mathbb{Z}^\times, \cdot)$ of nonzero integers with multiplication as composition.

In contrast to the semigroup structure that arises in the transition $\mathbb{R}^\times \to \mathbb{Z}^\times$, the discrete subgroup D which acts on the spacelike lattice vectors only is indeed a group, as follows from theorem [3], since the sub-lattice spanned by the spacelike lattice vectors satisfies the conditions of this theorem.

The [semi-]group $[eG_{\mathsf{lat}}]$ G_{lat} then has the structure

$$G_{\mathsf{lat}} \simeq [\mathbb{E}(n-m-1) \otimes D], \tag{4.12a}$$

$$eG_{\mathsf{lat}} \simeq [\mathbb{E}(n-m-1) \otimes D] \odot (\mathbb{Z}^\times, \cdot). \tag{4.12b}$$

Comparison with (3.4, 3.4') shows that second line (4.12b) contains the extension of the "ordinary" normalizer $N(\Gamma)$ by a semigroup $(\mathbb{Z}^\times, \cdot)$, which is what we wanted to show.

We finish our arguments by giving the expressions for the isometry group $I(M/\Gamma)$ and its semigroup extension $eI(M/\Gamma)$; they can be obtained from formulas (2.4, 3.4, 3.4', 4.12):

$$\begin{aligned} I(M/\Gamma) &= \Big[\mathbb{R}^n/T_{\mathsf{lat}}\Big] \odot G_{\mathsf{lat}}, \\ eI(M/\Gamma) &= \Big[\mathbb{R}^n/T_{\mathsf{lat}}\Big] \odot \Big[G_{\mathsf{lat}} \odot (\mathbb{Z}^\times, \cdot)\Big]. \end{aligned} \tag{4.13}$$

5. Summary

We have shown that the isometry group of identification spaces which arise as a result of compactifying a certain number of dimensions in a flat pseudo-Euclidean spacetime $\mathbb{R}^n_t$ can admit a semigroup extension, related to the concept of the extended normalizer of the identification group Γ in the group of isometries $I(M)$ of the covering space $\mathbb{R}^n_t$. The possibility of such an extension hinges upon the fact whether the restriction of the metric on the covering space to the $\mathbb{R}$-linear envelope of the lattice vectors is Euclidean or not. In the first case, the compactness of the restricted isometry group on the envelope obstructs such an extension. On the other hand, if this restricted metric is pseudo-Euclidean, then nontrivial extensions of the isometry group of the identification space M/Γ exist. We have explicitly provided an example of such an extension: In the case of a Lorentzian spacetime compactified over a lightlike lattice, the extension of the isometry group of the compactified space $\mathbb{R}^{n-m} \times T^m$ is provided by a semigroup which is isomorphic to $(\mathbb{Z}^\times, \cdot)$, i.e., the set of all nonzero integers with multiplication as composition, and 1 as unit element.

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