International Journal Pure & of Applied Mathematical Sciences.
ISSN 0972-9828 Vol.3 No.2 (2006), pp. 113-125

http://www.gbspublisher.com/ijpams.htm

Linear Operators Associated with a Subclass of Uniformly Convex Functions

G. Murugusundaramoorthy and N. Magesh
School of Science and Humanities, Vellore Institute of Technology,
Deemed University, Vellore – 632014, India.
E-mail:gmsmoorthy@yahoo.com
Department of Mathematics, Adhiyamaan College of Engineering,
Hosur – 635109, India.
E-mail: nmagi_2000@yahoo.co.in

Abstract

Making use of certain linear operator, we define a new subclass of uniformly convex functions with negative coefficients and obtain coefficient estimates, extreme points, closure and inclusion theorems and the radii of starlikeness and convexity for the new subclass. Furthermore, results on convolution products and partial sums are discussed.

AMS subject classification: 30C45.
Keywords and Phrases: Univalent, starlike, convex, uniformly convex, hadamard product, partial sums.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C}\ |z| < 1\}$. Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.2}$$

Following Gooodman [2, 3], Rønning [4, 5] introduced and studied the following subclasses

(i) A function $f \in A$ is said to be in the class $S_p(\alpha,\beta)$ uniformly $\beta-$starlike functions if it satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)}-1\right|,\ z\in U, \tag{1.3}$$

$-1<\alpha\leq 1$ and $\beta\geq 0.$

(ii) A function $f\in A$ is said to be in the class $UCV(\alpha,\beta)$, uniformly $\beta-$convex functions if it satisfies the condition

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|,\ z\in U, \tag{1.4}$$

and $-1<\alpha\leq 1$ and $\beta\geq 0.$

Indeed it follows from (1.3) and (1.4) that

$$f\in UCV(\alpha,\beta)\Leftrightarrow zf'\in S_p(\alpha,\beta). \tag{1.5}$$

For functions $f\in A$ given by (1.1) and $g(z)\in A$ given by $g(z)=z+\sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad z\in U. \tag{1.6}$$

Let $\phi(a,c;z)$ be the incomplete beta function defined by

$$\phi(a,c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}z^n,\ \ c\neq 0,-1,-2,\dots \tag{1.7}$$

where $(\lambda)_n$ is the Pochhammer symbol defined interms of the Gamma functions, by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\left\{\begin{array}{ll}1 & \text{n = 0}\\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & n\in N\end{array}\right\} \tag{1.8}$$

Further, for $f\in A$

$$L(a,c)f(z)=\phi(a,c;z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^n, \tag{1.9}$$

where $L(a,c)$ is called Carlson - Shaffer operator [1] and the operator $*$ stands for the hadamard product (or convolution product) of two power series is given by (1.6).

We notice that

$$L(a,a)f(z)=f(z),\ \ L(2,1)f(z)=zf'(z).$$

For $-1 \leq \alpha < 1$ and $\beta \geq 0$, we let $S(\alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\text{Re}\left\{\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \alpha\right\} > \beta\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right|, \quad z \in U \tag{1.10}$$

where $L(a,c)f(z)$ is given by (1.9). We also let $TS(\alpha, \beta) = S(\alpha, \beta) \cap T$.

By suitably specializing the values of (a) and (c), the class $S(\alpha, \beta)$ can be reduces to the class studied earlier by Rønning [4, 5]. Also choosing $\alpha = 0$ and $\beta = 1$ the class coincides with the classes studied in [10] and [11] respectively.

The main object of this paper is to study the coefficient bounds, extreme points, radius of starlikeness and convolution results for functions belong to the generalized class $TS(\alpha, \beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S(\alpha, \beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined.

2. Basic Properties

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $S(\alpha, \beta)$ and $TS(\alpha, \beta)$.

Theorem 2.1. A function $f(z)$ of the form (1.1) is in $S(\alpha, \beta)$ if

$$\sum_{n=2}^{\infty}[n(1+\beta) - (\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n| \leq 1 - \alpha, \tag{2.1}$$

$-1 \leq \alpha < 1, \ \ \beta \geq 0.$

Proof. It suffices to show that

$$\beta\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right| - \text{Re}\left\{\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned}\beta\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right| &- \text{Re}\left\{\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right\} \\ &\leq (1+\beta)\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right| \\ &\leq \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}{1 - \sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}.\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n| \leq 1-\alpha,$$

and hence the proof is complete. ■

Theorem 2.2. A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS(\alpha,\beta)$, $-1 \leq \alpha < 1, \beta \geq 0$ is that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n \leq 1-\alpha, \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in TS(\alpha,\beta)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty} n\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}} - \alpha \geq \beta \left| \frac{\sum\limits_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}} \right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n \leq 1-\alpha,\ -1 \leq \alpha < 1,\ \beta \geq 0.$$

Theorem 2.3. Let $f(z)$ defined by (1.2) and $g(z)$ defined by $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n$ be in the class $TS(\alpha,\beta)$. Then the function $h(z)$ defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} q_n z^n,$$

where $q_n = (1-\lambda)a_n + \lambda b_n,\ \ 0 \leq \lambda < 1$ is also in the class $TS(\alpha,\beta)$.

Theorem 2.4. (Extreme points) Let

$$f_1(z) = z \ \text{ and } \ f_n(z) = z - \frac{(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n \ \text{ for } \ n = 2,3,4,\ldots \tag{2.3}$$

Then $f(z) \in TS(\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum\limits_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$ and $\sum\limits_{n=1}^{\infty} \lambda_n = 1$.

The proof of the Theorem 2.4, follows on line similar to the proof of the theorem on extreme points given in Silverman [8].

We prove the following theorem by defining $f_j(z)$ $(j = 1, 2, \dots m)$ of the form

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n \text{ for } a_{n,\, j} \geq 0,\ \ z \in U. \tag{2.4}$$

Theorem 2.5. (Closure theorem) Let the functions $f_j(z)$ $(j = 1, 2, \dots m)$ defined by (2.4) be in the classes $TS(\alpha_j, \beta)$ $(j = 1, 2, \dots m)$ respectively. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} a_{n,j} \right) z^n$$

is in the class $TS(\alpha, \beta)$, where $\alpha = \min\limits_{1 \leq j \leq m} \{\alpha_j\}$ where $-1 \leq \alpha_j < 1$.

Proof. Since $f_j(z) \in TS(\alpha_j, \beta)$ $(j = 1, 2, 3, \dots m)$ by applying Theorem 2.2, to (2.4) we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) \\
&= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_{n,j} \right) \\
&\leq \frac{1}{m} \sum_{j=1}^{m} (1 - \alpha_j) \leq 1 - \alpha
\end{aligned}$$

which in view of Theorem 2.2, again implies that $h(z) \in TS(\alpha, \beta)$ and so the proof is complete. ■

Theorem 2.6. Let $f \in TS(\alpha, \beta)$. Then

(i) f is starlike of order $\delta(0 \leq \delta < 1)$ in the disc $|z| < r_1$; that is, Re $\left\{ \frac{zf'(z)}{f(z)} \right\} > \delta$, $(|z| < r_1\ ;\ 0 \leq \delta < 1)$, where

$$r_1 = \inf_{n \leq 2} \left\{ \frac{(a)_{n-1}}{(c)_{n-1}} \left(\frac{1-\delta}{n-\delta} \right) \frac{[n(1+\beta) - (\alpha+\beta)]}{1-\alpha} \right\}^{\frac{1}{n-1}}.$$

(ii) f is convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_2$, that is Re $\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta$, $(|z| < r_2; 0 \leq \delta < 1)$, where

$$r_2 = \inf_{n \leq 2} \left\{ \frac{(a)_{n-1}}{(c)_{n-1}} \frac{(1-\delta)[n(1+\beta) - (\alpha+\beta)]}{n(n-\delta)(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.3).

Proof. Given $f \in A$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{2.5}$$

For the left hand side of (2.5) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n\,|z|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n\,|z|^{n-1} < 1.$$

Using the fact, that $f \in TS(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\alpha+\beta)]}{1-\alpha}\frac{(a)_{n-1}}{(c)_{n-1}}a_n\; < 1.$$

We can say (2.5) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} < \frac{[n(1+\beta)-(\alpha+\beta)]}{1-\alpha}\frac{(a)_{n-1}}{(c)_{n-1}}.$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1-\delta)[n(1+\beta)-(\alpha+\beta)]}{(n-\delta)(1-\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar the proof of (i). ∎

3. Modified Hadamard Products

Let the functions $f_j(z)(j = 1, 2)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\; z^n \text{ for } a_{n,\,j} \geq 0,\;\; z \in U. \tag{3.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \ = \ z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}\, z^n.$$

Using the techniques of Schild and Silverman [6], we prove the following results.

Theorem 3.1. For functions $f_j(z)(j = 1, 2)$ defined by (3.1), let $f_1(z) \in TS(\alpha, \beta)$ and $f_2(z) \in TS(\delta, \beta)$. Then $(f_1 * f_2)(z) \in TS(\gamma, \beta)$ where

$$\gamma = \gamma(\alpha, \beta, \delta) = 1 - \frac{(1-\alpha)(1-\delta)(1+\beta)}{(2+\beta-\alpha)(2+\beta-\delta)\frac{(a)}{(c)} - (1-\alpha)(1-\delta)}, \tag{3.2}$$

and $-1 \leq \delta \leq 1$, $-1 < \gamma \leq 1$; $z \in U$. The result is best possible for

$$\begin{aligned} f_1(z) &= z - \frac{(1-\alpha)}{(2+\beta-\alpha)}\frac{(c)}{(a)}z^2 \\ f_2(z) &= z - \frac{(1-\delta)}{(2+\beta-\delta)}\frac{(c)}{(a)}z^2 \end{aligned}$$

Proof. In view of Theorem 2.2, it suffice to prove that

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta)-(\gamma+\beta)]}{1-\gamma}\frac{(a)_{n-1}}{(c)_{n-1}}a_{n,1}a_{n,2} \ \leq 1, \quad (-1 < \gamma \leq 1)$$

where γ is defined by (3.2). On the other hand, under the hypothesis, it follows from (2.2) and the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta)-(\alpha+\beta)]^{1/2}[n(1+\beta)-(\delta+\beta)]^{1/2}}{\sqrt{(1-\alpha)(1-\delta)}}\frac{(a)_{n-1}}{(c)_{n-1}}\sqrt{a_{n,1}a_{n,2}} \ \leq 1. \tag{3.3}$$

Thus we need to find the largest γ such that

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[n(1+\beta)-(\gamma+\beta)]}{1-\gamma}\frac{(a)_{n-1}}{(c)_{n-1}}a_{n,1}a_{n,2} \\ \leq \ &\sum_{n=2}^{\infty} \frac{[n(1+\beta)-(\alpha+\beta)]^{1/2}[n(1+\beta)-(\delta+\beta)]^{1/2}}{\sqrt{(1-\alpha)(1-\delta)}}\frac{(a)_{n-1}}{(c)_{n-1}}\sqrt{a_{n,1}a_{n,2}} \end{aligned}$$

or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{1-\gamma}{\sqrt{(1-\alpha)(1-\delta)}}\frac{[n(1+\beta)-(\alpha+\beta)]^{1/2}[n(1+\beta)-(\delta+\beta)]^{1/2}}{[n(1+\beta)-(\gamma+\beta)]}, \ \ (n \geq 2).$$

By view of (3.3) it is sufficient to find largest γ such that

$$\begin{aligned}&\frac{\sqrt{(1-\alpha)(1-\delta)}}{[n(1+\beta)-(\alpha+\beta)]^{1/2}[n(1+\beta)-(\delta+\beta)]^{1/2}}\frac{(c)_{n-1}}{(a)_{n-1}}\\ \leq\ &\frac{1-\gamma}{\sqrt{(1-\alpha)(1-\delta)}}\frac{[n(1+\beta)-(\alpha+\beta)]^{1/2}[n(1+\beta)-(\delta+\beta)]^{1/2}}{[n(1+\beta)-(\gamma+\beta)]}\end{aligned}$$

which yields

$$\gamma = 1-\frac{(n-1)(1-\alpha)(1-\delta)(1+\beta)}{[n(1+\beta)-(\alpha+\beta)][n(1+\beta)-(\delta+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}-(1-\alpha)(1-\delta)} \text{ for } n\geq 2.$$

Since

$$\Phi(n) = 1-\frac{(n-1)(1-\alpha)(1-\delta)(1+\beta)}{[n(1+\beta)-(\alpha+\beta)][n(1+\beta)-(\delta+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}-(1-\alpha)(1-\delta)} \text{ for } n\geq 2. \tag{3.4}$$

is an increasing function of n $(n\geq 2)$, for $-1\leq\alpha\leq 1$, $-1\leq\delta\leq 1$ and $\beta\geq 0$, letting $n=2$ in (3.4), we have

$$\gamma\leq\Phi(2) = 1-\frac{(1-\alpha)(1-\delta)(1+\beta)}{[2+\beta-\alpha][2+\beta-\delta]\frac{(a)}{(c)}-(1-\alpha)(1-\delta)}.$$

which completes the proof. ∎

Theorem 3.2. Let the functions $f_j(z)(j=1,2)$ defined by (3.1), be in the class $TS(\alpha,\beta)$ with $-1\leq\alpha\leq 1$, $\beta\geq 0$. Then $(f_1*f_2)(z)\in TS(\eta,\beta)$ where

$$\eta = 1-\frac{(1-\alpha)^2(1+\beta)}{[2+\beta-\alpha]^2\frac{(a)}{(c)}-(1-\alpha)^2}.$$

Proof. By taking $\delta=\alpha$, in the above theorem, the result follows. ∎

Theorem 3.3. Let the functions $f(z)$ defined by (1.2) be in the class $TS(\alpha,\beta)$. Also let $g(z)=z-\sum\limits_{n=2}^{\infty}b_nz^n$ for $|b_n|\leq 1$. Then $(f*g)(z)\in TS(\alpha,\beta)$.

Proof. Since

$$\begin{aligned}\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_nb_n| &\leq \sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n|b_n|\\ &\leq \sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\\ &\leq (1-\alpha)\end{aligned}$$

it follows that $(f*g)(z)\in TS(\alpha,\beta)$, by the view of Theorem 2.2. ∎

Theorem 3.4. Let the functions $f_j(z)(j=1,2)$ defined by (3.1) be in the class $TS(\alpha,\beta)$. Then the function $h(z)$ defined by $h(z)=z-\sum\limits_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n$ is in the class $TS(\xi,\beta)$, where

$$\xi = 1-\frac{2(1-\alpha)^2(1+\beta)}{(2+\beta-\alpha)^2\frac{(a)}{(c)}-2(1-\alpha)^2}.$$

Proof. By virtue of Theorem 2.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\xi+\beta)]}{(1-\xi)}\frac{(a)_{n-1}}{(c)_{n-1}}(a_{n,1}^2+a_{n,2}^2)\leq 1 \tag{3.5}$$

where $f_j(z)\in TS(\alpha,\beta)(j=1,2)$ we find from (3.1) and Theorem 2.2, that

$$\sum_{n=2}^{\infty}\left[\frac{[n(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}\right]^2 a_{n,j}^2 \leq \sum_{n=2}^{\infty}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}a_{n,j}\right\}^2 \tag{3.6}$$

which yields

$$\sum_{n=2}^{\infty}\frac{1}{2}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}\right\}^2(a_{n,1}^2+a_{n,2}^2)\leq 1. \tag{3.7}$$

On comparing (3.6) and (3.7), it is easily seen that the inequality (3.5) will be satisfied if

$$\frac{[n(1+\beta)-(\xi+\beta)]}{(1-\xi)}\frac{(a)_{n-1}}{(c)_{n-1}} \leq \frac{1}{2}\left\{\frac{[n(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}\right\}^2 \quad \text{for } n\geq 2.$$

That is, if

$$\xi \leq 1-\frac{2(n-1)(1-\alpha)^2(1+\beta)}{[n(1+\beta)-(\alpha+\beta)]^2\frac{(a)_{n-1}}{(c)_{n-1}}-2(1-\alpha)^2}. \tag{3.8}$$

Since

$$\Psi(n) = 1-\frac{2(n-1)(1-\alpha)^2(1+\beta)}{[n(1+\beta)-(\alpha+\beta)]^2\frac{(a)_{n-1}}{(c)_{n-1}}-2(1-\alpha)^2}.$$

is an increasing function of n $(n\geq 2)$. Taking $n=2$ in (3.8), we have,

$$\xi \leq \Psi(2) = 1-\frac{2(1-\alpha)^2(1+\beta)}{(2+\beta-\alpha)^2\frac{(a)}{(c)}-2(1-\alpha)^2},$$

which completes the proof. ■

4. Partial Sums

Following the earlier works by Silverman [8] and Silvia [9] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $TS(\alpha, \beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$.

Theorem 4.1. Let $f(z) \in TS(\alpha, \beta)$ be given by (1.1) and define the partial sums $f_1(z)$ and $f_k(z)$, by

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (k \in N/1) \tag{4.1}$$

Suppose also that

$$\sum_{n=2}^{\infty} d_n |a_n| \le 1, \tag{4.2}$$

where

$$d_n := \frac{[n(\alpha+\beta) - (\alpha+\beta)]}{(1-\alpha)} \frac{(a)_{n-1}}{(c)_{n-1}}.$$

Then $f \in TS(\alpha, \beta)$. Furthermore,

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_{k+1}} \quad z \in U, k \in N \tag{4.3}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_{k+1}}{1 + d_{k+1}}. \tag{4.4}$$

Proof. For the coefficients d_n given by (4.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{4.5}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + d_{k+1} \sum_{n=k+1}^{\infty} |a_n| \le \sum_{n=2}^{\infty} d_n |a_n| \le 1 \tag{4.6}$$

by using the hypothesis (4.2). By setting

$$\begin{aligned} g_1(z) &= d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{d_{k+1}}\right)\right\} \\ &= 1 + \frac{d_{k+1} \sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} a_n z^{n-1}} \end{aligned} \tag{4.7}$$

and applying (4.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{n}|a_n|-d_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|} \leq 1, \quad z\in U, \tag{4.8}$$

which readily yields the assertion (4.3) of Theorem 4.1. In order to see that

$$f(z)=z+\frac{z^{k+1}}{d_{k+1}} \tag{4.9}$$

gives sharp result, we observe that for $z=re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)}=1+\frac{z^k}{d_{k+1}}\rightarrow 1-\frac{1}{d_{k+1}}$ as $z\rightarrow 1^-$. Similarly, if we take

$$\begin{aligned} g_2(z) &= (1+d_{k+1})\left\{\frac{f_k(z)}{f(z)}-\frac{d_{k+1}}{1+d_{k+1}}\right\} \\ &= 1-\frac{(1+d_{n+1})\sum\limits_{n=k+1}^{\infty}a_nz^{n-1}}{1+\sum\limits_{n=2}^{\infty}}a_nz^{n-1} \end{aligned} \tag{4.10}$$

and making use of (4.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{k}|a_n|-(1-d_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|} \tag{4.11}$$

which leads us immediately to the assertion (4.4) of Theorem 4.1.

The bound in (4.4) is sharp for each $k\in N$ with the extremal function $f(z)$ given by (4.9). The proof of the Theorem 4.1, is thus complete. ■

Theorem 4.2. If $f(z)$ of the form (1.1) satisfies the condition (2.1). Then

$$Re\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq 1-\frac{k+1}{d_{k+1}}. \tag{4.12}$$

Proof. By setting

$$\begin{aligned} g(z) &= d_{k+1}\left\{\frac{f'(z)}{f_k'(z)} - \left(1-\frac{k+1}{d_{k+1}}\right)\right\} \\ &= \frac{1+\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} na_n z^{n-1} + \sum\limits_{n=2}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}} \\ &= 1+\frac{\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}}. \end{aligned}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}. \tag{4.13}$$

Now

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le 1$$

if

$$\sum_{n=2}^{k} n|a_n| + \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \le 1 \tag{4.14}$$

since the left hand side of (4.14) is bounded above by $\sum\limits_{n=2}^{k} d_n|a_n|$ if

$$\sum_{n=2}^{k}(d_n-n)|a_n| + \sum_{n=k+1}^{\infty} d_n - \frac{d_{k+1}}{k+1}n|a_n| \ge 0, \tag{4.15}$$

and the proof is complete. The result is sharp for the extremal function $f(z) = z + \frac{z^{k+1}}{c_{k+1}}$. ■

Theorem 4.3. If $f(z)$ of the form (1.1) satisfies the condition (2.1) then

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \ge \frac{d_{k+1}}{k+1+d_{k+1}}. \tag{4.16}$$

Proof. By setting

$$\begin{aligned} g(z) &= [(k+1)+d_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{k+1+d_{k+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum_{n=2}^{k} na_n z^{n-1}} \end{aligned}$$

and making use of (4.15), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum_{n=2}^{k} n|a_n| - \left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n|a_n|}. \leq 1,$$

which leads us immediately to the assertion of the Theorem 4.3. ■

References

[1] B.C. Carlson and S.B. Shaffer, Starlike and prestarlike hypergrometric functions, *SIAM J. Math. Anal.*, **15**, pp. 737–745, 2002.

[2] A.W. Goodman, On uniformly convex functions, *Ann. polon. Math.*, **56**, pp. 87–92, 1991.

[3] A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. & Appl.*, **155**, pp. 364–370, 1991.

[4] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**, pp. 189–196, 1993.

[5] F. Rønning, Integral representations for bounded starlike functions, *Annal. Polon. Math.*, **60**, pp. 289–297, 1995.

[6] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, *Ann. Univ. Marie Curie-Sklodowska Sect. A*, **29**, pp. 99–107, 1975.

[7] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51**, pp. 109–116, 1975.

[8] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. & Appl.*, **209**, pp. 221–227, 1997.

[9] E.M. Silvia, Partial sums of convex functions of order α, Houston. *J. Math., Math. Soc.*, **11** (3), pp. 397–404, 1985.

[10] K.G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam and H. Silverman, Subclasses of uniformly convex and uniformly starlike functions. *Math. Japonica*, **42** (3), pp. 517–522, 1995.

[11] K.G. Subramanian, T.V. Sudharsan, P. Balasubrahmanyam and H. Silverman, Classes of uniformly starlike functions, *Publ. Math. Debrecen.*, **53** (3–4), pp. 309–315, 1998.