

## Asymptotic Behavior of Solutions for a Damped Evolution Equation in a Disk

Lin Qun<sup>1</sup>, Shaoyong Lai<sup>1,2</sup> and Yong Hong Wu<sup>2</sup>

<sup>1</sup>*Department of Mathematics,  
Sichuan Normal University  
Chengdu 610066, P.R. China*

<sup>2</sup>*Department of Mathematics and Statistics,  
Curtin University of Technology  
GPO BOX U1987, Perth, WA6845, Australia  
Email: yhwu@maths.curtin.edu.au*

### Abstract

In this paper, we study an initial-boundary value problem for the following damped evolution equation defined in a disk,

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \gamma\Delta(u^2).$$

A strong solution is constructed in the form of a series of the small parameter present in the initial conditions in the space  $C^0([0, \infty), H_r^s(\Omega))$ . For  $s \in (1/2, 5/2)$ , the uniqueness of the solution is proved, and the long-time asymptotics is obtained in explicit form.

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### 1. Introduction

In 1872, Boussinesq [1] derived an equation describing the propagation of small amplitude, long waves on the surface of shallow water. This equation possesses special, travelling-wave solutions called solitary waves discovered by Scott Russell more than thirty years earlier. Boussinesq's theory was the first to give a satisfactory, scientific explanation of the phenomenon of solitary waves. The classical Boussinesq equation is as follows

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad (1.1)$$

where  $u(x, t)$  is the elevation of the free surface of fluid, the subscripts denote partial derivatives, and the constant coefficients  $\alpha$  and  $\beta$  depend on the depth of fluid and the characteristic speed of long waves. Equation (1.1) has been studied from various points of view in references [2, 4, 6, 7].

In [10], Varlamov studied the solution of the following damped Boussinesq equation in a disk region

$$u_{tt} - 2b\Delta u_t = -\beta\Delta^2 u + \Delta u + \gamma\Delta(u^2), \quad (1.2)$$

where  $\Delta = (1/r)\partial_r(r\partial_r) + (1/r^2)\partial_\theta^2$ . The second term on the left-hand side is responsible for strong internal damping. An initial-boundary value problem for equation (1.2) in a disk was studied.

In the present paper, we consider an initial-boundary value problem for the following damped evolution equation

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \gamma\Delta(u^2) \quad (1.3)$$

in a disk, where  $\Delta$  is as defined before. The above equation reduces to (1.2) for  $a = 0$  and  $\alpha = 0$ . The rest of the paper is organized as follows. In section 2, we present the statement of the problem and some auxiliary results. In section 3, we present and prove the existence and uniqueness theorem. In section 4, we derive an asymptotic solution for the problem in question.

## 2. Some Auxiliary Results

The initial-boundary value problem under investigation is defined in a disk region and is as follows

$$\begin{aligned} u_{tt} - a\Delta u_{tt} - 2b\Delta u_t &= \\ &\alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \gamma\Delta(u^2), \quad (r, \theta) \in \Omega, t > 0, \\ u(r, \theta, 0) &= \varepsilon^2\phi(r, \theta), u_t(r, \theta, 0) = \varepsilon^2\psi(r, \theta), \quad (r, \theta) \in \Omega, \\ u|_{\partial\Omega} &= \Delta u|_{\partial\Omega} = \Delta^2 u|_{\partial\Omega} = 0, t > 0, \end{aligned} \quad (2.1)$$

where we have used the cylindrical polar coordinates  $(r, \theta)$  and the  $\Delta$  is as defined before;  $|u(0, \theta, t)| < +\infty$  and  $u(r, \theta, t)$  is periodic in  $\theta$  with a period  $2\pi$ ;  $a, b, \alpha, \beta, \varepsilon$  are positive constants, and  $\gamma$  is a constant in  $R^1$ ;  $\phi(r, \theta)$  and  $\psi(r, \theta)$  are real valued functions; and  $\Omega = \{(r, \theta) | r < 1, \theta \in [-\pi, \pi)\}$ .

Let  $L_{2,r}(\Omega)$  be the space  $L_2(\Omega)$  with the weight  $r$  endowed with the scalar product  $(\cdot, \cdot)_{r,0}$  and the corresponding norm  $\|\cdot\|_{r,0}$ . To study system (2.1), we use the series expansions of eigenfunctions of the Laplace operator in  $\Omega$ . For  $f(r, \theta) \in L_{2,r}(\Omega)$ , the expansion is

$$f(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{f}_{mn} \chi_{mn}(r, \theta), \quad \hat{f}_{mn} = \frac{(f, \chi_{mn})_{r,0}}{\|\chi_{mn}(r, \theta)\|_{r,0}^2}, \quad (2.2)$$

$\chi_{mn}(r, \theta) = J_m(\lambda_{mn}r)e^{im\theta}$ ,  $m \in \mathbf{Z}$ ,  $n \in \mathbf{N}$ , where  $J_m(z)$  is the Bessel function of index  $m$ ,  $\lambda_{mn}$  denote the positive zeroes of  $J_m(z)$  arranged in ascending order of magnitude.

For sufficiently large positive  $\lambda_{mn}$ , it follows (see [9]) that

$$\frac{C_1}{\lambda_{mn}} \leq \|\chi_{mn}(r, \theta)\|_{r,0}^2 = 2\pi \|J_m(\lambda_{mn}r)\|_r^2 \leq \frac{C_2}{\lambda_{mn}}, \tag{2.3}$$

where  $\|\cdot\|_r^2$  denotes the norm of  $L_{2,r}(0, 1)$ ,  $C_{1,2}$  are some positive constants. In the sequel we denote by  $c$  any generic positive constants independent of  $m, n, \varepsilon, r, \theta$  and  $t$ , but may depending on the coefficients of the equation and the initial data.

For large  $m$ , the positive zeros of  $J_m(z)$  have the asymptotic expansion (McMahon's expansion, see [8] P. 247) as follows

$$\lambda_{mn} = \mu_{mn} + O\left(\frac{1}{\mu_{mn}}\right), \quad \mu_{mn} = \left(m + 2n - \frac{1}{2}\right)\frac{2}{\pi}, \quad (n \rightarrow +\infty). \tag{2.4}$$

In the following, we need to use the weighted Sobolev space  $H_r^s(\Omega)$  which is different from  $H^s(\Omega)$  in that the weighted space  $L_{2,r}(\Omega)$  is used instead of  $L_2(\Omega)$ , and we will introduce the norm in  $H_r^s(\Omega)$  as follows [10]

$$\|f\|_{r,s}^2 = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn}^{2s} |\hat{f}_{mn}|^2 \|\chi_{mn}\|_{r,0}^2. \tag{2.5}$$

We will also introduce a Banach space  $C^n([0, \infty), H_r^s(\Omega))$  equipped with the following norm

$$\|u\|_{C^n} = \sum_{k=0}^n \sup_{t \in [0, \infty)} \|\partial_t^k u(t)\|_{r,s}. \tag{2.6}$$

We shall also denote by  $V_0^1(f(r, \theta))$  the total variation of  $f(r, \theta)$  in  $r \in [0, 1]$ , where  $f(r, \theta)$  is defined on  $[0, 1] \times [-\pi, \pi]$ . We are now ready to present some auxiliary results.

**Lemma 2.1.** Assume that  $f(r, \theta), (r, \theta) \in \Omega$ , satisfies the following conditions:  $(H_1)$   $f(r, -\pi) = f(r, \pi)$ ;  $(H_2)$   $V_0^1(\sqrt{r}f(r, \theta)) = V_{0,0}(\theta) \in L_1(-\pi, \pi)$ ,  $\lim_{r \rightarrow 0^+} \sqrt{r}f(r, \theta) = F_{0,0}(\theta) \in L_1(-\pi, \pi)$ ;  $(H_3)$   $V_0^1(\sqrt{r}\partial_\theta f(r, \theta)) = V_{0,1}(\theta) \in L_1(-\pi, \pi)$ ,  $\lim_{r \rightarrow 0^+} \sqrt{r}\partial_\theta f(r, \theta) = F_{0,1}(\theta) \in L_1(-\pi, \pi)$ .

Then the following estimate holds for  $m \geq 0$  and  $n \geq 1$ ,

$$|\hat{f}_{mn}| \leq \frac{c}{\lambda_{mn}^{1/2}(m+1)}. \tag{2.7}$$

*Proof.* See p. 291 in [10]. ■

**Lemma 2.2.** Suppose that  $f(r, \theta), (r, \theta) \in \Omega$ , satisfies the following conditions:  $(H_4)$   $\partial_\theta^k f(r, -\pi) = \partial_\theta^k f(r, \pi)$ , for  $k = 0, 1, 2$ ;  $(H_5)$   $\partial_r f(0, \theta) = f(1, \theta) = \partial_r f(1, \theta) = 0$ ;

$V_0^1(\sqrt{r}\partial_r^2 f(r, \theta)) = V_{2,0}(\theta) \in L_1(-\pi, \pi)$ ,  $\lim_{r \rightarrow 0^+} \sqrt{r}\partial_r^2 f(r, \theta) = F_{2,0}(\theta) \in L_1(-\pi, \pi)$ ,  
 $(H_6)$   $\partial_\theta^3 f(0, \theta) = \partial_r \partial_\theta^3 f(0, \theta) = \partial_\theta^3 f(1, \theta) = \partial_r \partial_\theta^3 f(1, \theta) = 0$ ;  $V_0^1(\sqrt{r}\partial_r^2 \partial_\theta^3 f(r, \theta)) =$   
 $V_{2,3}(\theta) \in L_1(-\pi, \pi)$ ,  $\lim_{r \rightarrow 0^+} \sqrt{r}\partial_r^2 \partial_\theta^3 f(r, \theta) = F_{2,3}(\theta) \in L_1(-\pi, \pi)$ .

Then we have that for  $m \geq 0$  and  $n \geq 1$ ,

$$|\hat{f}_{mn}| \leq \frac{c}{\lambda_{mn}^{5/2}(m+1)}. \quad (2.8)$$

*Proof.* See p. 291 in [10].

Set

$$a_{mnpqls} = \frac{\int_0^1 r J_m(\lambda_{mn} r) J_p(\lambda_{pq} r) J_l(\lambda_{ls} r) dr}{\|J_m(\lambda_{mn} r)\|_r^2}.$$

From [10], we have that

$$|a_{mnpqls}| \leq c \sqrt{\frac{\lambda_{mn}}{\lambda_{pq} \lambda_{ls}}}, \quad (2.9)$$

$$|a_{mnpqls}| \leq \frac{c}{\sqrt{\lambda_{mn}}} \left( \sqrt{\frac{\lambda_{pq}}{\lambda_{ls}}} + \sqrt{\frac{\lambda_{ls}}{\lambda_{pq}}} + m \right). \quad (2.10)$$

### 3. Existence and Uniqueness Theorem

**Theorem 3.1.** Let the positive constants  $a, b, \alpha, \beta$  and the initial data in system (2.1) satisfy the following two conditions:

$(H_7)$   $a + \beta - b^2 \geq 0$ ,  $a\beta + \alpha + 2\sqrt{a\alpha(a + \beta - b^2)} > 0$ ,  $0 < a < \frac{\lambda_{11}^2 - 2\lambda_{01}^2}{\lambda_{01}^2 \lambda_{11}^2}$ ;

$(H_8)$   $\psi(r, \theta)$  and  $\phi(r, \theta)$  satisfy the assumptions  $(H_1) - (H_3)$  and  $(H_4) - (H_6)$  respectively.

Then there is an  $\varepsilon_0 > 0$  such that for  $\varepsilon \in [0, \varepsilon_0]$ , problem (2.1) has a strong solution in  $C^2([0, \infty), H_r^{s-4}(\Omega)) \cap C^1([0, \infty), H_r^{s-2}(\Omega)) \cap C^0([0, \infty), H_r^s(\Omega))$  with  $\Delta u \in C^2([0, \infty), H_r^{s-6}(\Omega)) \cap C^1([0, \infty), H_r^{s-4}(\Omega)) \cap C^0([0, \infty), H_r^{s-2}(\Omega))$ ,  $\Delta^2 u, \Delta(u^2) \in C^0([0, \infty), H_r^{s-4}(\Omega))$ ,  $\Delta^3 u \in C^0([0, \infty), H_r^{s-6}(\Omega))$  for any  $s < 5/2$ . If  $1/2 < s < 5/2$ , the solution is unique. Moreover,  $u$  is continuous and bounded in  $\bar{\Omega} \times [0, \infty)$ , and the solution can be represented by

$$u(r, \theta, t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r, \theta, t), \quad (3.1)$$

where the function  $u^{(N)}(r, \theta, t)$  will be defined in the proof (see (3.9) and (3.21)), and the series converges absolutely and uniformly with respect to  $(r, \theta) \in \bar{\Omega}$ ,  $t \geq 0$ ,  $\varepsilon \in [0, \varepsilon_0]$ .

*Proof.* We divide the proof of the theorem into two parts. Firstly, we prove that there exists a solution  $u(r, \theta, t)$  to problem (2.1). Then we prove that the solution is unique. ■

### 3.1. Existence and Construction of a Solution

For satisfying the boundary conditions, we seek a solution of (2.1) in the following form

$$u(r, \theta, t) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{u}_{mn}(t) \chi_{mn}(r, \theta), \quad \hat{u}_{mn}(t) = \frac{(u, \chi_{mn})_{r,0}(t)}{\|\chi_{mn}\|_{r,0}^2}, \quad (3.2)$$

where  $\hat{u}_{-mn}(t) = (-1)^m \overline{\hat{u}_{mn}(t)}$ ,  $\chi_{-mn} = (-1)^m \overline{\chi_{mn}}$ ,  $m \geq 0, n \geq 1$ .

Similarly, we have

$$\phi(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{\phi}_{mn} \chi_{mn}(r, \theta), \quad \hat{\phi}_{mn} = \frac{(\phi, \chi_{mn})_{r,0}}{\|\chi_{mn}\|_{r,0}^2} \quad (3.3)$$

$$\psi(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{\psi}_{mn} \chi_{mn}(r, \theta), \quad \hat{\psi}_{mn} = \frac{(\psi, \chi_{mn})_{r,0}}{\|\chi_{mn}\|_{r,0}^2}, \quad (3.4)$$

$$u^2(r, \theta, t) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} (\widehat{u^2})_{mn}(t) \chi_{mn}(r, \theta), \quad (3.5)$$

where

$$(\widehat{u^2})_{mn}(t) = \sum_{p,l \geq 0; q,s \geq 1; m=p+l} \hat{u}_{pq} \hat{u}_{ls} a_{mnpqls} + 2 \sum_{p,q,l,s \geq 1; m=p-l} \hat{u}_{pq} \overline{\hat{u}_{ls}} a_{mnpqls}.$$

Substituting (15)–(18) into equation (2.1), we obtain the following Cauchy problem for  $\hat{u}_{mn}(t)$ ,  $m \geq 0, n \geq 1, t \geq 0$ :

$$\begin{pmatrix} (a\lambda_{mn}^2 + 1)\hat{u}_{mn}''(t) + 2b\lambda_{mn}^2 \hat{u}_{mn}'(t) + (\alpha\lambda_{mn}^6 + \beta\lambda_{mn}^4 + \lambda_{mn}^2)\hat{u}_{mn}(t) \\ = -\gamma\lambda_{mn}^2 (\widehat{u^2})_{mn}(t), \end{pmatrix} \quad (3.6)$$

$$\hat{u}_{mn}(0) = \varepsilon^2 \hat{\phi}_{mn}, \quad \hat{u}_{mn}'(0) = \varepsilon^2 \hat{\psi}_{mn}. \quad (3.7)$$

Setting  $\hat{\Phi}_{mn} = \varepsilon \hat{\phi}_{mn}$ ,  $\hat{\Psi}_{mn} = \varepsilon \hat{\psi}_{mn}$  and

$$\sigma_{mn} = \frac{\lambda_{mn} \sqrt{a\alpha\lambda_{mn}^6 + (\alpha + a\beta)\lambda_{mn}^4 + (a + \beta - b^2)\lambda_{mn}^2 + 1}}{1 + a\lambda_{mn}^2},$$

and integrating (3.6)-(3.7) with respect to  $t$ , we get

$$\begin{pmatrix} \hat{u}_{mn}(t) \\ = \varepsilon \exp\left(-\frac{b\lambda_{mn}^2}{1 + a\lambda_{mn}^2}t\right) \left\{ \left[ \cos(\sigma_{mn}t) + \frac{b\lambda_{mn}^2}{1 + a\lambda_{mn}^2} \cdot \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}} \right] \hat{\Phi}_{mn} + \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}} \hat{\Psi}_{mn} \right\} \\ - \frac{\gamma\lambda_{mn}^2}{(1 + a\lambda_{mn}^2)\sigma_{mn}} \int_0^t \exp\left[-\frac{b\lambda_{mn}^2}{1 + a\lambda_{mn}^2}(t - \tau)\right] \sin[\sigma_{mn}(t - \tau)] (\widehat{u^2})_{mn}(\tau) d\tau. \end{pmatrix} \quad (3.8)$$

To solve the nonlinear integral equation (3.8), we apply the perturbation theory. Representing  $\hat{u}_{mn}(t)$  as a formal series in  $\varepsilon$ ,

$$\hat{u}_{mn}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{v}_{mn}^{(N)}(t), \quad (3.9)$$

then substituting it into equation (3.8) and then comparing the coefficients of equal powers of  $\varepsilon$ , we obtain the following formula for  $m \geq 0, n \geq 1, t > 0$ :

$$\left( \begin{array}{l} \hat{v}_{mn}^{(0)}(t) = \exp\left(-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2}t\right) \left\{ \left[ \cos(\sigma_{mn}t) + \frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} \cdot \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}} \right] \hat{\Phi}_{mn} \right. \\ \left. + \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}} \hat{\Psi}_{mn} \right\}, \end{array} \right) \quad (3.10)$$

$$\left( \begin{array}{l} \hat{v}_{mn}^{(N)}(t) \\ = -\frac{\gamma\lambda_{mn}^2}{(1+a\lambda_{mn}^2)\sigma_{mn}} \int_0^t \exp\left[-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2}(t-\tau)\right] \sin[\sigma_{mn}(t-\tau)] F_{mn}^{(N)}(\hat{v}(\tau)) d\tau, \end{array} \right) \quad (3.11)$$

where  $N \geq 1$  and

$$\left( \begin{array}{l} F_{mn}^{(N)}(\hat{v}(t)) = \sum_{p,l \geq 0; q,s \geq 1; p+l=m} a_{mnpqls} \sum_{j=1}^N \hat{v}_{pq}^{(j-1)} \hat{v}_{ls}^{(N-j)} \\ + 2 \sum_{p,q,l,s \geq 1; p-l=m} a_{mnpqls} \sum_{j=1}^N \hat{v}_{pq}^{(j-1)} \overline{\hat{v}_{ls}^{(N-j)}}. \end{array} \right) \quad (3.12)$$

Using induction on the number  $N$ , we establish the following estimate for  $m \geq 0, n \geq 1, N \geq 0, t \geq 0$ :

$$|\hat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} \lambda_{mn}^{-5/2} (m+1)^{-1} \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right). \quad (3.13)$$

According to the assumptions ( $H_8$ ) of Theorem 1, for sufficiently small  $\varepsilon$ , inequality (3.13) holds for  $N = 0$ . Suppose that estimate (3.13) holds for all  $\hat{v}_{mn}^{(k)}(t)$  with  $0 \leq k \leq N-1$ , i.e.

$$|\hat{v}_{mn}^{(k)}(t)| \leq c^k (k+1)^{-2} \lambda_{mn}^{-5/2} (m+1)^{-1} \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right). \quad (3.14)$$

From [7], it follows that for  $1 \leq j \leq N$

$$j^{-2}(N+1-j)^{-2} \leq 2^2(N+1)^{-2}[j^{-2} + (N+1-j)^{-2}].$$

Combining (2.4) and (2.10), and applying Fubini-Tonelli's theorem, we obtain

$$|F_{mn}^{(N)}(\hat{v}(t))| \leq c \cdot c^{N-1} (N+1)^{-2} (m+1)^{-1} \lambda_{mn}^{-1/2} \exp\left(-2 \frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t\right). \quad (3.15)$$

Hence it follows

$$|\hat{v}_{mn}^{(N)}(t)| \leq c \cdot c^{N-1} (N+1)^{-2} (m+1)^{-1} \lambda_{mn}^{-5/2} S_{mn}(t), \quad (3.16)$$

where

$$S_{mn}(t) = \exp\left(-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} t\right) \int_0^t \exp\left[\left(\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} - 2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\right) \tau\right] d\tau.$$

Since  $0 < a < \frac{\lambda_{11}^2 - 2\lambda_{01}^2}{\lambda_{01}^2 \lambda_{11}^2}$ ,  $b > 0$  and  $\lambda_{mn} > \lambda_{11}$  for  $m = 0, n \geq 2$  or  $m, n \geq 1$ , it follows

$$\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} \geq \frac{b\lambda_{11}^2}{1+a\lambda_{11}^2} > \frac{2b\lambda_{01}^2}{1+a\lambda_{01}^2}. \quad (3.17)$$

Hence, we have

$$S_{mn}(t) \leq c \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t\right), \quad m \geq 0, n \geq 1; \quad (3.18)$$

$$S_{mn}(t) \leq c \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t\right), \quad m = 0, n \geq 2 \quad \text{or} \quad m, n \geq 1. \quad (3.19)$$

Combining (3.16),(3.18) and (3.19), estimate (3.13) holds for  $k = N$ . Similarly, combining (3.17),(3.18) and (3.19), we get for  $m = 0, n \geq 2$  or  $m, n \geq 1, t \geq 0, N \geq 0$  that

$$|\hat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} (m+1)^{-1} \lambda_{mn}^{-5/2} \exp\left(-2 \frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t\right). \quad (3.20)$$

To derive the representation (3.1), we use (3.2), (3.9) and (3.10), and then preform interchange of summation in the series, namely:

$$u(r, \theta, t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r, \theta, t), \quad (3.21)$$

$$u^{(N)}(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \hat{v}_{mn}^{(N)} \chi_{mn} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overline{\hat{v}_{mn}^{(N)} \chi_{mn}}.$$

Since the series converges absolutely and uniformly for  $(r, \theta) \in \bar{\Omega}, t \geq 0, \varepsilon \in [0, \varepsilon_0], (\varepsilon_0 < \frac{1}{c})$ , this interchange is allowable. To prove that (3.21) is a solution of equation (2.1), we must obtain some estimates of the derivatives of  $\hat{u}_{mn}(t)$ .

Differentiating (3.10) with respect to  $t$ , we have that for  $k = 1, 2$ ,

$$\begin{aligned} \partial_t^k \hat{v}_{mn}^{(0)}(t) &= \sum_{l=0}^k C_k^l \left( -\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} \right)^l \exp \left( -\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} t \right) \\ &\quad \times \partial_t^{k-l} \left\{ \left[ \cos \sigma_{mn} t + \frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} \cdot \frac{\sin \sigma_{mn} t}{\sigma_{mn}} \right] \hat{\Phi}_{mn} + \frac{\sin \sigma_{mn} t}{\sigma_{mn}} \hat{\Psi}_{mn} \right\}, \\ \partial_t^k \hat{v}_{mn}^{(N)}(t) &= -\frac{\gamma\lambda_{mn}^2}{(1+a\lambda_{mn}^2)\sigma_{mn}} \int_0^t g_k(m, n, t-\tau) F_{mn}^{(N)}(\hat{v}(\tau)) d\tau + R_k(m, n, t), \end{aligned}$$

where  $N \geq 1$  and

$$g_k(m, n, t) = \sum_{l=0}^k C_k^l \left( -\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} \right)^l \exp \left( -\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} t \right) \sigma_{mn}^{k-l} \sin \left[ \sigma_{mn} t + (k-l) \frac{\pi}{2} \right],$$

$$R_1(m, n, t) = 0, \quad R_2(m, n, t) = -\frac{\gamma\lambda_{mn}^2}{(1+a\lambda_{mn}^2)\sigma_{mn}} F_{mn}^{(N)}(\hat{v}(t)).$$

From (3.13), it follows that for  $m \geq 0, n \geq 1, N \geq 0, t \geq 0$  and  $k = 0, 1, 2$ ,

$$|\partial_t^k \hat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} \lambda_{mn}^{2k-5/2} \exp \left( -\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t \right). \quad (3.22)$$

According to (3.9), we have that for  $m \geq 0, n \geq 1, t \geq 0$ , and  $k = 0, 1, 2$ ,

$$|\partial_t^k \hat{u}_{mn}(t)| \leq c \lambda_{mn}^{2k-5/2} \exp \left( -\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t \right). \quad (3.23)$$

Hence, we conclude that the series

$$\sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn}^{2s} \hat{u}_{mn}^2 \| \chi_{mn} \|_{r,0}^2,$$

converges uniformly for all  $t \geq 0$  if  $s < 5/2$ . Therefore  $u \in C^0([0, \infty), H_r^s(\Omega))$ . From (3.23) with  $k = 0$ , the series (3.21) converges absolutely and uniformly with respect to  $(r, \theta) \in \bar{\Omega}, t \geq 0$ , and  $\varepsilon \in [0, \varepsilon_0]$ . Hence  $u(r, \theta, t)$  is continuous and bounded in this domain. Applying (3.23), we conclude that  $u_t, \Delta u \in C^0([0, \infty), H_r^{s-2}(\Omega)); u_{tt}, \Delta u_t, \Delta^2 u, \Delta(u^2) \in C^0([0, \infty), H_r^{s-4}(\Omega)); \Delta u_{tt}, \Delta^3 u \in C^0([0, \infty), H_r^{s-6}(\Omega))$  with  $s < 5/2$ .

### 3.2. Uniqueness of the Solution

To prove the uniqueness we assume that there exist two solutions  $u^{(1)}(r, \theta, t)$  and  $u^{(2)}(r, \theta, t)$  to the problem (2.1). By setting  $w(r, \theta, t) = u^{(1)}(r, \theta, t) - u^{(2)}(r, \theta, t)$ , it follows

$$w(r, \theta, t) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \hat{w}_{mn}(t) \chi_{mn}(r, \theta),$$

where  $\hat{w}_{-mn}(t) = (-1)^m \overline{\hat{w}_{mn}(t)}$  for  $m \geq 0, n \geq 1$ .

Since  $u^{(1)}(r, \theta, t)$  and  $u^{(2)}(r, \theta, t)$  are solutions to the problem (2.1), we have that for  $m \geq 0$  and  $n \geq 1$

$$\begin{aligned} & \hat{w}_{mn}(t) \\ &= \frac{-\gamma \lambda_{mn}^2}{(1 + a \lambda_{mn}^2) \sigma_{mn}} \int_0^t \exp \left[ -\frac{b \lambda_{mn}^2}{1 + a \lambda_{mn}^2} (t - \tau) \right] \sin[\sigma_{mn}(t - \tau)] G_{mn}(u^{(1)}(\tau), u^{(2)}(\tau)) d\tau, \end{aligned}$$

where

$$G_{mn}(u^{(1)}(t), u^{(2)}(t)) = \left( \sum_{p,l \geq 0; q,s \geq 1; m=p+l} + 2 \sum_{p,q,l,s \geq 1; m=p-l} \right) a_{mnpqls} (\hat{u}_{pq}^{(1)} \hat{w}_{ls} + \hat{u}_{ls}^{(2)} \hat{w}_{pq}).$$

Fixing  $\delta$  such that  $k > \delta > 0$ , then combining (2.4),(2.9) and (3.22) and then using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left| \sum_{p,l \geq 0; q,s \geq 1; m=p+l} a_{mnpqls} \hat{u}_{pq}^{(1)} \hat{w}_{ls} \right| &\leq c \lambda_{mn}^{1/2} \sum_{l \geq 0; q,s \geq 1} \frac{q^{\frac{1+\delta}{2}}}{\lambda_{(m-l)q}^3 \lambda_{ls}^k} \cdot \frac{|\hat{w}_{ls}| \lambda_{ls}^k}{q^{\frac{1+\delta}{2}} \lambda_{ls}^{1/2}} \\ &\leq c \lambda_{mn}^{1/2} \|w\|_{r,k}. \end{aligned}$$

Here the convergence of the triple series takes place for  $1/2 < k$ . In similar way, we can get

$$|G_{mn}| \leq c \lambda_{mn}^{1/2} \|w\|_{r,k}.$$

It follows that for some positive  $T$ ,

$$|\hat{w}_{mn}(t)| \leq \sup_{t \in [0, T]} \|w\|_{r,k} \frac{|\gamma| \lambda_{mn}^{1/2}}{b \sigma_{mn}} \left[ 1 - \exp \left( -\frac{b}{1+a} T \right) \right], t \in [0, T].$$

Hence, we obtain that for  $t \in [0, T]$ ,

$$\|w\|_{r,k}^2 \leq \left( \sup_{t \in [0, T]} \|w\|_{r,k} \right)^2 \frac{|\gamma|^2 [(1 - \exp(-\frac{b}{1+a} T))]^2}{b^2} \sum_{m,n} \frac{\lambda_{mn}^{2k} \lambda_{mn}}{\sigma_{mn}^2} \|\chi_{mn}\|_{r,0}^2.$$

Since the following inequality holds for  $k < 1$

$$\sum_{m,n} \frac{\lambda_{mn}^{2k} \lambda_{mn}}{\sigma_{mn}^2} \|\chi_{mn}\|_{r,0}^2 \leq C_3,$$

we get

$$\|w\|_{r,k} \leq C(T) \sup_{t \in [0, T]} \|w\|_{r,k}$$

where  $C(T) = \frac{C_3^{1/2} |\gamma| [1 - \exp(-\frac{b}{1+a}T)]}{b}$ . Since  $C(0) = 0$ , we can make appropriate choice of  $T_1$  such that  $C(T_1) < 1$ . Thus we obtain that  $w(r, \theta, t) = 0$  in  $[0, T_1]$ . In similar way we get that  $w(r, \theta, t) = 0$  in  $[T_1, 2T_1], [2T_1, 3T_1], \dots, [nT_1, (n+1)T_1], \dots$  with  $nT_1 \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Hence we establish the uniqueness of solution for all  $t \geq 0$  and  $1/2 < k < 1$ . Noting the fact that  $\|w(t)\|_{k_1, r} \leq \|w(t)\|_{k_2, r}$  for  $k_1 \leq k_2$  and all  $t \geq 0$ , we consequently have  $H_r^{k_2}(\Omega) \subseteq H_r^{k_1}(\Omega)$  for  $t \geq 0$ . Therefore, the uniqueness takes place for  $1/2 < k < 5/2$ . This completes the proof of Theorem 1.  $\blacksquare$

#### 4. Long time Asymptotics

**Theorem 3.2.** Under the assumptions of Theorem 3.1, the asymptotic expansion can be expressed as follows

$$\left( \begin{array}{l} u(r, \theta, t) = \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right) \{ [A \cos \sigma_{01}t + B \sin \sigma_{01}t] J_0(\lambda_{01}r) \\ + O\left(\exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right)\right) \} \end{array} \right) \quad (4.1)$$

where  $t$  is sufficiently large and

$$\sigma_{01} = \frac{\lambda_{01} \sqrt{a\alpha\lambda_{01}^6 + (\alpha + a\beta)\lambda_{01}^4 + (a + \beta - b^2)\lambda_{01}^2 + 1}}{1 + a\lambda_{01}^2},$$

in which the coefficients  $A$  and  $B$  are as defined by (4.2) and (4.3) in the proof below, and the estimate of the residual term in (4.1) is uniform with respect to  $(r, \theta) \in \bar{\Omega}$ ,  $\varepsilon \in [0, \varepsilon_0]$ .

*Proof.* In order to obtain the asymptotic expansion (4.1), we single out the term  $\hat{u}_{01}(t)J_0(\lambda_{01}r)$  in (15) and estimate the remaining series  $Q_1(r, t)$  and  $Q_2(r, \theta, t)$ .

From (3.9) and (3.10), we have

$$\hat{u}_{01}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \hat{v}_{01}^{(N)}(t), \quad (4.2)$$

$$\hat{v}_{01}^{(0)}(t) = \varepsilon \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right) [A^{(0)} \cos(\sigma_{01}t) + B^{(0)} \sin(\sigma_{01}t)],$$

$$\hat{v}_{01}^{(N)}(t) = \exp\left[-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right] \{ [A^{(N)} + R_A^{(N)}(t)] \cos(\sigma_{01}t) + [B^{(N)} + R_B^{(N)}(t)] \sin(\sigma_{01}t) \},$$

where  $N \geq 1$

$$\begin{aligned} A^{(0)} &= \varepsilon \hat{\phi}_{01}, & B^{(0)} &= \frac{\varepsilon}{\sigma_{01}} \left( \frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} \hat{\phi}_{01} + \hat{\psi}_{01} \right), \\ A^{(N)} &= \frac{\gamma\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}} \int_0^\infty \exp\left(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau\right) \sin(\sigma_{01}\tau) F_{01}^{(N)}(\hat{v}(\tau)) d\tau, \\ R_A^{(N)}(t) &= -\frac{\gamma\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}} \int_t^\infty \exp\left(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau\right) \sin(\sigma_{01}\tau) F_{01}^{(N)}(\hat{v}(\tau)) d\tau, \\ B^{(N)} &= -\frac{\gamma\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}} \int_0^\infty \exp\left(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau\right) \cos(\sigma_{01}\tau) F_{01}^{(N)}(\hat{v}(\tau)) d\tau, \\ R_B^{(N)}(t) &= \frac{\gamma\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}} \int_t^\infty \exp\left(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau\right) \cos(\sigma_{01}\tau) F_{01}^{(N)}(\hat{v}(\tau)) d\tau. \end{aligned}$$

According to (3.15) with  $m = 0$  and  $n = 1$ , we obtain that for  $N \geq 1$  and  $t > 0$

$$|R_{A,B}^{(N)}(t)| \leq c^N \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right).$$

Hence, it follows

$$\hat{u}_{01}(t) = \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right) \left\{ [A \cos \sigma_{01}t + B \sin \sigma_{01}t] + O\left(\exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right)\right) \right\}, \quad (4.3)$$

where  $A = \sum_{N=0}^{\infty} \varepsilon^{N+1} A^{(N)}$ ,  $B = \sum_{N=0}^{\infty} \varepsilon^{N+1} B^{(N)}$  and the series converges absolutely and uniformly with respect to  $\varepsilon \in [0, \varepsilon_0]$ . Now, we represent the solution as follows

$$u(r, \theta, t) = \hat{u}_{01}(t) J_0(\lambda_{01}r) + Q_1(r, t) + Q_2(r, \theta, t), \quad (4.4)$$

where

$$\begin{aligned} Q_1(r, t) &= \sum_{n=2}^{\infty} u_{0n}(t) J_0(\lambda_{0n}r), \\ Q_2(r, \theta, t) &= \sum_{m,n \geq 1} (\hat{u}_{mn}(t) \chi_{mn}(r, \theta) + \overline{\hat{u}_{mn}(t) \chi_{mn}(r, \theta)}) \end{aligned}$$

Applying (3.9) and (3.10), it follows

$$|Q_1(r, t)| \leq c \exp\left(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right), \quad |Q_2(r, \theta, t)| \leq c \exp\left(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t\right). \quad (4.5)$$

Combining formulas (4.3)-(4.5), we deduce (4.1). The proof of Theorem 2 is thus complete.  $\blacksquare$

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