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Solution of Differential-Algebraic Equations(DAEs) by Adomian Decomposition Method

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Abstract

In this paper, we consider differential-algebraic equations(DAEs) systems . The approximate solutions for the differential-algebraic equations(DAEs) systems are obtained by using the Adomian decomposition method. The method is illustrated by two examples of differential-algebraic equations(DAEs) systems and series solutions are obtained. The solutions have been compared with those obtained by exact solutions.

Keywords: Differential-Algebraic, Equations(DAEs), Adomian Decomposition Method.

1. Introduction

Differential-Algebraic Equations (DAEs) can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical (multibody) systems, power systems, chemical process simulation, and optimal control. Many important mathematical models can be expressed in terms of Differential-Algebraic Equations (DAEs). In resent years, much research has been focused on the numerical solution of systems of differential-algebraic equations (DAEs) [4,7,9]. Some numerical methods have been developed, using both BDF [1,4,5], implicit Runge-Kutta methods [6] and Padé approximation method[13,14]. These methods are only directly suitable for low index problems and often reguire that the problem to have special structure. Although many important applications can be solved by these methods there is a need for more general approaches. Some more general approaches were proposed in [3,8,11,12].

The most general form of a differential-algebraic equations(DAEs) is given by

$$F(x, y, y') = 0 \tag{1}$$

with initial values

$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

where F and y is a vector function for which we assumed sufficient differentiability [4]. $\partial F / \partial y'$ may be singular. The rank and structure of this jacobian matrix may depend, in general, on the solution $y(x)$, and for simplicity we will always assume that it is independent of x. The important special case of semi-explicit differential-algebraic equations(DAEs), or an ODE with constraints,

$$y' = f(x, y, z), \tag{2a}$$

$$0 = g(x, y, z). \tag{2b}$$

This is a special case of (1). The index is 1 if $\partial g / \partial z$ is nonsingular, because then one differentiation of (2b) yields z' in principle. For the semi-explicit index-1 DAE we can distinguish between differential variables $y(x)$ and algebraic variables $z(x)$ [4]. The algebraic variables may be less smooth than the differential variables by one derivative. differential-algebraic equations(DAEs) (1) can be written in the semi-explicit form (2). These types of systems arise, for example, in circuit analysis, chemical process simulation, power systems, and many other applications.

The Adomian decomposition method has been applied to problems in physics, biology and chemical reactions. Resently, there has been a great deal of interest in applying Adomian's decomposition technique for solving a wide class of nonlinear equations, including algebraic, differential, partial-differential, differential-delay and integro-differential equations [2,10]. We applied the Adomian's decomposition method to approximation solution of the system differential-algebraic equations(DAEs).

2. Using Adomian Decomposition Method

A system of differential equations can be considered as:

$$\begin{aligned} y_1' &= f_1(x, y_1, \ldots, y_n) \\ y_2' &= f_2(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, \ldots, y_n) \end{aligned} \tag{3}$$

where each equation represents the first derivative of one of the unknown functions as a mapping depending on the independent variable x and n unknown functions $f_1, \ldots, f_n$ [17].

We can present the system (3), by using the i^{th} equation as:

$$Ly_i = f_i(x, y_1, \ldots y_n) \quad i = 1, 2, \cdots, n \tag{4}$$

where L is the linear operator d/dx with the inverse $L^{-1}=\int_0^x(.)dx$. Applying the inverse operator on (4) we get the following canonical form, which is suitable for applying Adomian decomposition method.

$$y_i = y_i(0)+\int_0^x f_i(x, y_1,\dots, y_n)dx \quad i=1,2,\dots,n \tag{5}$$

As usual in Adomian decomposition method the solution of Eq.(5) is considered to be as the sum of a series:

$$y_i=\sum_{j=0}^{\infty} f_{i,j} \tag{6}$$

And the integrand in the Eq.(5), as the sum of the following series:

$$f_i(x_i, y_1\dots, y_n)=\sum_{j=0}^{\infty} A_{i,j}(f_{i,0}, f_{i,1},\dots, f_{i,j}) \tag{7}$$

where $A_{i,j}(f_{i,0}, f_{i,1}\dots, f_{i,n})$ are called Adomian polynomials [6]. Substituting (6) and (7) into (5). we get

$$\sum_{j=0}^{\infty} f_{i,j} = y_i(0)+\int_0^x \sum_{j=0}^{\infty} A_{i,j}(f_{i,0}, f_{i,1},\dots, f_{i,j})$$

$$= y_i(0)+\sum_{j=0}^{\infty}\int_0^x A_{i,j}(f_{i,0}, f_{i,1},\dots, f_{i,j}) \tag{8}$$

from which we define:

$$f_{i,0}=y_i(0)$$

$$f_{i,n+1}=\int_0^x A_{i,n}(f_{i,0}, f_{i,1},\dots, f_{i,n})dx \quad n=0,1,2,\dots \tag{9}$$

3. Numerical examples

Example 1. We consider the following system of differential-algebraic equations(DAEs)

$$u'-xv'+u-(1+x)v=0$$

$$v=\sin x \tag{10}$$

with initial condition

$$u(0)=1,\ v(0)=0$$

The exact solution is

$$u(x)=e^{-x}+x\sin x$$

$$v(x)=\sin x$$

Eq.(10)can be written

$$u'=-u+x\cos x+(1+x)\sin x$$

Using the inverse operator $L^{-1}=\int_0^x(.)dx$ we get:

$$u = 1 + \int_0^x (x\cos x + (1+x)\sin x)dx - \int_0^x u dx$$

Using the alternate algorithm for computing the Adomian polynomials [15,16], the Adomian procedure would be as the following:

$$u_{1,0} = 14x\sin x + \sin x - x\cos x \quad , \quad u_{1,n+1} = -\int_0^x u_{1,n} dx$$

After nine steps we get the solution

$$u^*(x) = 1 - x + 1.500000000x^2 - 0.1666666667x^3 - 0.1250000000x^4 - 0.008333333333x^5 + 0.009722222222x^6 - 0.0001984126984x^7 - 0.0001984126984x^8$$

x	Exact $u(x)$	Adomian $u^*(x)$	$\lvert u(x) - u^*(x) \rvert$
0.1	0.9148207597	0.9148207597	0
0.2	0.85846446193	0.85846446192	$0.1\text{x}10^{-9}$
0.3	0.8294742827	0.8294742811	$0.16\text{x}10^{-8}$
0.4	0.8260873829	0.8260873671	$0.158\text{x}10^{-7}$
0.5	0.8462434290	0.8462433346	$0.944\text{x}10^{-7}$
0.6	0.8875971201	0.8875967131	$0.4070\text{x}10^{-6}$
0.7	0.9475376848	0.9475362815	$0.14033\text{x}10^{-5}$
0.8	1.023213837	1.023209724	$0.4113\text{x}10^{-5}$
0.9	1.111563878	1.111553228	$0.10650\text{x}10^{-4}$
1.0	1.209350426	1.209325396	$0.25030\text{x}10^{-4}$

$u^*(x)$ is approximation solution of $u(x)$

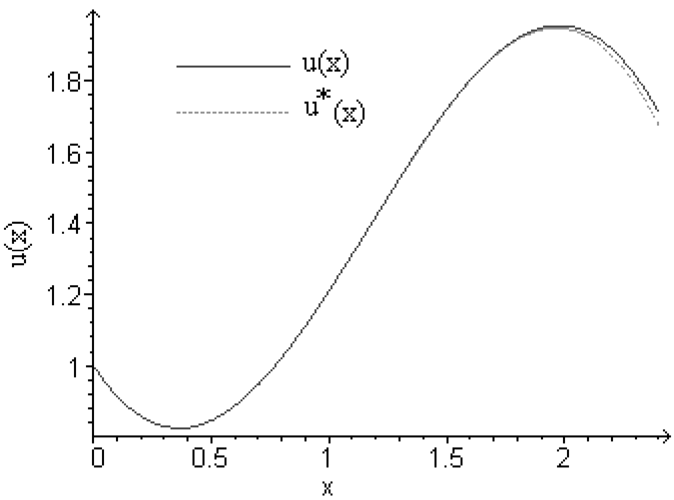

Figure 1: Values of $u(x)$, its Adomian approximation($u^*(x)$).

Example 2. We consider the following system of differential-algebraic equations(DAEs)

$$
\begin{aligned}
y_1' - xy_2' + x^2 y_3' + y_1 - (x+1)y_2 + (x^2+2x)y_3 = 0 \\
y_2' - xy_3' - y_2 + (x-1)y_3 = 0 \\
y_3 = \sin x
\end{aligned}
\tag{11}
$$

with initial condition

$$y_1(0) = 1, y_2(0) = 0, y_3(0) = 0$$

The exact solution is

$$
\begin{aligned}
y_1(x) &= e^{-x} + xe^{x} \\
y_2(x) &= e^{x} + x\sin x \\
y_3(x) &= \sin x
\end{aligned}
$$

Eq.(11) can be written

$$
\begin{aligned}
y_1' &= -y_1 + (2x+1)y_2 - (2x^2+x)\sin x \\
y_2' &= y_2 + x\cos x - (x-1)\sin x
\end{aligned}
$$

Using the inverse operator $L^{-1} = \int_0^x (.)dx$ we get:

$$y_1 = -\int_0^x (2x^2+x)\sin x dx + \int_0^x (-y_1 + (2x+1)y_2)dx$$

$$y_2 = 1 + \int_0^x (x\cos x - (x-1)\sin x)dx + \int_0^x y_2 dx$$

Using the alternate algorithm for computing the Adomian polynomials [15,16], the Adomian procedure would be as the following:

$$y_{1,0} = 1 + 2x^2\cos x - 4\cos x - 4x\sin x - \sin x + x\cos x + 4 \ , \quad y_{1,n+1} = \int_0^x (-y_{1,n} + (2x+1)y_{2,n})dx$$

$$y_{2,0} = -x\sin x + \sin x - x\cos x \quad , \quad y_{2,n+1} = -\int_0^x y_{2,n} dx$$

After nine steps we get the solution

$$y^*{}_1(x) = 1 + 1.500000000x^2 + 0.3333333333x^3 + 0.20833333333x^4 + 0.03333333333x^5 + 0.008333333333x^6 + 0.0009920634921x^7 - 0.0001984126984x^8$$

$$y^*{}_2(x) = 1 + x + 1.500000000x^2 + 0.1666666667x^3 - 0.1250000000x^4 + 0.008333333333x^5 + 0.008333333333x^6 - 0.0002480158730x^8$$

x	Exact $y_1(x)$	Adomian $y^*{}_1(x)$	$\lvert y_1(x)-y^*{}_1(x)\rvert$
0.1	1.015354510	1.015354521	$0.11\text{x}10^{-7}$
0.2	1.063011305	1.063011200	$0.105\text{x}10^{-6}$
0.3	1.145775863	1.145774763	$0.1100\text{x}10^{-5}$
0.4	1.267049925	1.267043625	$0.6300\text{x}10^{-5}$
0.5	1.430891295	1.430866344	$0.24951\text{x}10^{-4}$
0.6	1.642082916	1.642005198	$0.77718\text{x}10^{-4}$
0.7	1.906212199	1.906006967	$0.205232\text{x}10^{-3}$
0.8	2.229761706	2.229281110	$0.480596\text{x}10^{-3}$
0.9	2.620212460	2.619185426	$0.1027034\text{x}10^{-2}$
1.0	3.086161269	3.084118661	$0.2042608\text{x}10^{-2}$

$y^*{}_1(x)$ is approximation solution of $y_1(x)$

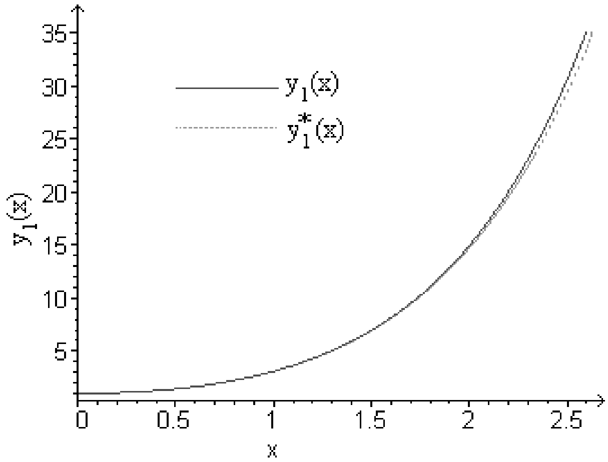

Figure 2. Values of $y_1(x)$, its Adomian approximation ($y^*{}_1(x)$).

x	Exact $y_2(x)$	Adomian $y^*{}_2(x)$	$\lvert y_2(x)-y^*{}_2(x)\rvert$
0.1	1.115154260	1.115154258	$0.2\text{x}10^{-8}$
0.2	1.261136624	1.261136532	$0.92\text{x}10^{-7}$
0.3	1.438514870	1.438513809	$0.1061\text{x}10^{-5}$
0.4	1.647592035	1.647585970	$0.6065\text{x}10^{-5}$
0.5	1.888434040	1.888410489	$0.23551\text{x}10^{-4}$
0.6	2.160904284	2.160832634	$0.71650\text{x}10^{-4}$
0.7	2.464705088	2.464520860	$0.184228\text{x}10^{-3}$
0.8	2.799425801	2.799006923	$0.418878\text{x}10^{-3}$
0.9	3.164597330	3.163730162	$0.867168\text{x}10^{-3}$
1.0	3.559752813	3.558085317	$0.1667496\text{x}10^{-2}$

$y^*{}_2(x)$ is approximation solution of $y_2(x)$

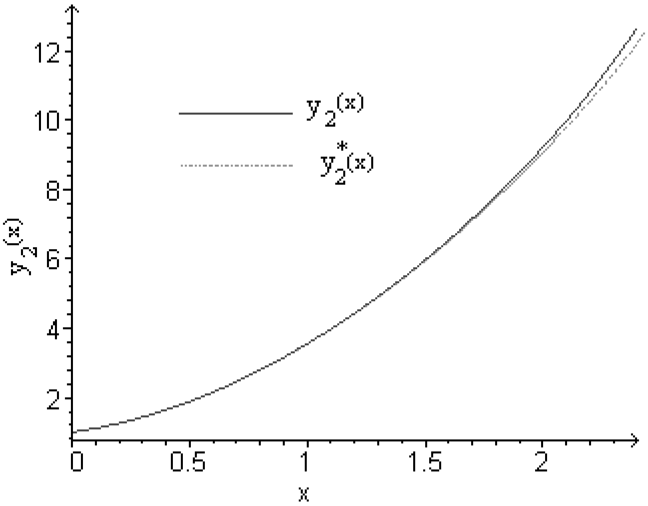

Figure 3: Values of $y_2(x)$, its Adomian approximation ($y^*{}_2(x)$).

Conclusion

The purpose of this paper is to implement the Adomian's decomposition method to system of differential-algebraic equations(DAEs). The Adomian decomposition method is that the solution expressed as an infinite series converges very fast to exact solutions. Results have been found very accurate when they are compared with analytical solutions.

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