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A Global Condition for the Triviality of an Almost Quaternionic Structure on Complex Manifolds

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Abstract

Let M be an almost quaternionic manifold on which its almost quaternionic structure is defined by a three dimensional subbundle V of $T(M) \otimes T^*(M)$ and $\{F, G, H\}$ be a local basis for V. Suppose that the (global) (1,2) tensor field $[V, V]$ is defined by $[V, V] = [F, F] + [G, G] + [H, H]$, where [,] denotes the Nijenhuis bracket. In Ref.[1], for the almost hypercomplex structure $H{=}J_\alpha$, $\alpha = 1, 2, 3$ and the Obata connection ∇^H, it is proved that the torsion tensor of ∇^H vanishes if and only if H is hypercomplex.

In this work, we give an alternative short proof, in particular, prove that if either M is a quaternionic Kahler manifold, or if M is a complex manifold with almost complex structure F, then the vanishing of $[V, V]$ is equivalent to that of all the Nijenhuis brackets of $\{F, G, H\}$. It follows that the bundle V is trivial if and only if $[V, V] = 0$.

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1. Introduction

An almost complex structure on a manifold M is a tensor field $J\colon TM \to TM$ satisfying the identity $J^2 = -id$. *An almost hypercomplex structure* on a $4n$-dimensional manifold M is a triple $S = (F, G, H)$ of almost complex structures F, G and H satisfying the conditions

$$F^2 = G^2 = H^2 = -I, \quad H = FG, \quad FG + GF = FH + HF = GH + HG = 0, \tag{1.1}$$

where I denotes the identity transformation of $T_x(M)$. When each of the tensor fields F, G and H is a complex structure, then S is called hypercomplex structure on M.

Almost quaternionic structures have intensively been studied in the literature [1–10]. An extensive review of the subject can be found in Kirichenko and Arseneva [5]. In the early literature, almost quaternionic structures are defined as global (1,1) tensor fields F, G and H satisfying the conditions (1.1). Later on, it has been realized that such a definition is too restrictive and the following definition is now generally accepted.

Definition 1.1: Let M be a $4n$-dimensional Riemannian manifold which admits a three dimensional vector bundle V of (1,1) tensors such that on a neighborhood U of each $x \in M$, V has a local base $\{F, G, H\}$. If on each such neigborhood, the tensors F, G and H satisfy the conditions (1.1), then the bundle V is called an **almost quaternionic structure** on M.

The Nijenhuis bracket of two tensor fields A and B of type $(1, 1)$ is a tensor field of type $(1, 2)$ defined by [11]

$$[A, B](X, Y) = [AX, BY] - A[BX, Y] - B[X, AY] + [BX, AY] - B[AX, Y] - A[X, BY] + (AB + BA)[X, Y]. \quad (1.2a)$$

In particular, if $A = B$ we have

$$[A, A](X, Y) = 2\big([AX, AY] + A^2[X, Y] - A[AX, Y] - A[X, AY]\big). \quad (1.2b)$$

In [10], it has been shown that the vanishing of any two of the Nijenhuis brackets of F, G and H is a necessary and sufficient condition for the existence of a torsion free connection ∇' such that $\nabla' F = \nabla' G = \nabla' H = 0$.

Since $[F, F]$, $[G, G]$ and $[H, H]$ are locally defined objects, to obtain a global condition for the triviality of V, we globally define the (1,2) tensor field $[V, V]$ by

$$[V, V] = [F, F] + [G, G] + [H, H]. \quad (1.3)$$

By the Newlander-Nirenberg Theorem an almost complex structure is a complex structure if and only if it is integrable i.e., it has no torsion. Thus, if the tensor fields F, G and H are integrable, then their brackets vanish. In [10], it has been shown that any two of the equations

$$[F, F] = 0, \quad [G, G] = 0, \quad [H, H] = 0, \quad [F, G] = 0, \quad [F, H] = 0, \quad [G, H] = 0, \quad (1.4)$$

hold, so do the others. It is then shown that there is a torsion-free connection with respect to which F, G and H are covariantly constant, and it follows that V is a trivial bundle.

V is locally spanned by almost hypercomplex structures $S = (F, G, H)$ and the structure tensor of S is defined as in [1]

$$T^S = (1/12)([F, F] + [G, G] + [H, H]). \quad (1.5)$$

For the almost hypercomplex structure $S = (F, G, H)$, there exists a unique linear connection ∇^S, which preserves S, that is, $\nabla^S F = 0$, $\nabla^S G = 0$, $\nabla^S H = 0$, where ∇^S is the Obata connection of S [6] and its torsion tensor is T^S. The torsion T^S of ∇^S vanishes if and only if $S = (F, G, H)$ is hypercomplex [1].

In Sections 2 and 3, we show that the vanishing of $[V, V]$ is equivalent to the triviality of V, in the cases where M is quaternionic Kahler manifold and where M is a complex manifold with almost complex structure F, respectively, and we give an alternative short proof of this fact.

2. Triviality of almost Quaternionic Structures on Quaternionic Kahler Manifolds

We start by recalling basic results and definitions. If M admits an almost quaternionic structure then at each point x in M there is an orthonormal basis of $T(M)$,of the form

$$\{X_1, \ldots, X_n, FX_1, \ldots, FX_n, GX_1, \ldots, GX_n, HX_1, \ldots, HX_n\}, \tag{2.1}$$

and the set of all such frames at all points $x \in M$ constitute a subbundle of the bundle $\mathcal{O}(M)$ of the orthonormal frames denoted by $\mathcal{H}(M)$. Such a reduction of the frame bundle is possible if and only if the structure group of the tangent bundle is reducible to $Sp(n)Sp(1)$ [9]. The torsion tensor of a connection ∇ is defined by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, and the connection is called "torsion-free" if $T = 0$. A torsion free connection ∇ is compatible with the metric $\langle\ ,\ \rangle$, hence, $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$, and this is equivalent to the reducibility of ∇ to $\mathcal{O}(M)$. Furthermore, if ∇ is reducible to $\mathcal{H}(M)$, the manifold is called "quaternionic Kahler manifold" [11].

Our aim is to obtain global conditions for the triviality of V, here, we study the case where M is a quaternionic Kahler manifold and use connections, which are reducible to $\mathcal{H}(M)$, to obtain our main results.

We quote the following theorem [11]:

Theorem 2.1: Let M be an almost quaternionic manifold. A connection ∇ on $\mathcal{O}(M)$ is reducible to a connection on $\mathcal{H}(\mathcal{M})$ if and only if the covariant derivatives of the tensor fields of F, G and H satisfy the following conditions:

$$\nabla F = aG - bH, \quad \nabla G = -aF + cH, \quad \nabla H = bF - cG, \tag{2.2}$$

where a, b and c are 1-forms.

We first prove the following lemma.

Lemma 2.1: Let M be a $4n$ dimensional manifold, ∇ be a torsion-free connection reducible to $\mathcal{H}(M)$ and a, b and c be as in Theorem 2.1. Then

$$\begin{aligned} [F, F] = 0 &\quad \text{if and only if} \quad b(X) = a(FX), \\ [G, G] = 0 &\quad \text{if and only if} \quad c(X) = -a(GX), \\ [H, H] = 0 &\quad \text{if and only if} \quad c(X) = b(HX). \end{aligned} \tag{2.3}$$

Proof: In this case, we have $[X, Y] = \nabla_X Y - \nabla_Y X$, and ∇F, ∇G and ∇H are given by (2.2). We give the details of the computation for $[F, F]$:

$$\begin{aligned}\frac{1}{2}[F, F](X, Y) &= [FX, FY] - [X, Y] - F[FX, Y] - F[X, FY] \\ &= (\nabla_{FX} FY - \nabla_{FY} FX) - (\nabla_X Y - \nabla_Y X) \\ &\quad - F(\nabla_{FX} Y - \nabla_Y FX) - F(\nabla_X FY - \nabla_{FY} X) \\ &= (\nabla_{FX} F)Y - (\nabla_{FY} F)X + F((\nabla_Y F)X) - F((\nabla_X F)Y). \end{aligned} \tag{2.4}$$

By using the conditions of Theorem 2.1, we get

$$\begin{aligned}\frac{1}{2}[F, F](X, Y) &= [a(FX)G(Y) - b(FX)H(Y)] - [a(FY)G(X) - b(FY)H(X)] \\ &\quad + F[a(Y)G(X) - b(Y)H(X)] - F[a(X)G(Y) - b(X)H(Y)] \\ &= a(FX)G(Y) - b(FX)H(Y) - a(FY)G(X) + b(FY)H(X) \\ &\quad + a(Y)H(X) + b(Y)G(X) - a(X)H(Y) - b(X)G(Y) \\ &= [b(Y) - a(FY)]G(X) + [a(FX) - b(X)]G(Y) \\ &\quad + [b(FY) + a(Y)]H(X) + [-a(X) - b(FX)]H(Y). \end{aligned} \tag{2.5}$$

If X and Y are any linearly independent vector fields, then $\{GX, GY, HX, HY\}$ is a linearly independent set and their coefficients should vanish. Hence, $b(X) = a(FX)$. We can perform the similar computations for G and H, but we do not reproduce them here. ■

Similarly, it can be easily shown that the vanishing of any mixed Nijenhuis bracket is equivalent to the vanishing of all six, but we omit the proof here. We show that for a quaternionic Kahler manifold, the vanishing of $[V, V]$ is equivalent to the vanishing of all Nijenhuis brackets.

Theorem 2.2: Let M be a quaternionic Kahler manifold, ∇ be a torsion-free connection reducible to $\mathcal{H}(M)$ and a, b and c be as in Theorem 2.1. Then,

$$[F, F] + [G, G] + [H, H] = 0 \quad \text{if and only if} \quad b(X) = a(FX), \quad c(X) = -a(GX).$$

Proof: By computing $[F, F]$, $[G, G]$, and $[H, H]$ we obtain

$$\begin{aligned}\frac{1}{2}([F, F] + [G, G] + [H, H])(X, Y) &= [a(GY) + 2c(Y) - b(HY)](FX) \\ &\quad + [-a(GX) - 2c(X) + b(HX)](FY) \\ &\quad + [-a(FY) + 2b(Y) + c(HY)](GX) \\ &\quad + [a(FX) - 2b(X) - c(HX)](GY) \\ &\quad + [-c(GY) + 2a(Y) + b(FY)](HX) \\ &\quad + [c(GX) - 2a(X) - b(FX)](HY). \end{aligned} \tag{2.6}$$

In dimensions $4n > 4$, if X and Y are linearly independent vector fields, then $\{FX, FY, GX, GY, HX, HY\}$ is a linearly independent set, and the result follows by algebraic calculation. In 4-dimensions, the results can be obtained by direct computation. Thus, $[V, V] = 0$ implies $b(X) = a(FX)$ and $c(X) = -a(GX)$.

Conversely, if the conditions $b(X) = a(FX)$ and $c(X) = -a(GX)$ hold, then by using the Lemma (2.1), we obtain $[V, V] = 0$. ■

It then follows from Lemma 2.1 that all Nijenhuis brackets vanish.

In the next section, we study the implications of $[V, V] = 0$ on an arbitrary almost quaternionic manifold, and we also show that $[V, V] = 0$ implies the vanishing of the individual Nijenhuis brackets, if either of F, G and H has no Nijenhuis torsion.

3. Triviality of almost Quaternionic Structures on Complex Manifolds

We first write down explicitly the action of $[V, V]$ on pairs of vectors belonging to an orthonormal basis such as given in Eq.(2.1). If M is a $4n$ dimensional manifold, then the set $\{X_i, FX_i, GX_i, HX_i\}$, $i = 1, \cdots, n$ can be chosen as a local basis for TM. Then, the tensor $[V, V]$ is determined by its action on the sets S_1 and S_2 given by

$$S_1 = \{(X_i, FX_i), (X_i, GX_i), (X_i, HX_i), (FX_i, GX_i), (FX_i, HX_i), (GX_i, HX_i)\}, \quad i = 1, \cdots, n \tag{3.1a}$$

$$\begin{aligned} S_2 = \{&(X_i, X_j), (X_i, FX_j), (X_i, GX_j), (X_i, HX_j), \\ &(FX_i, X_j), (FX_i, FX_j), (FX_i, GX_j), (FX_i, HX_j), \\ &(GX_i, X_j), (GX_i, FX_j), (GX_i, GX_j), (GX_i, HX_j), \\ &(HX_i, X_j), (HX_i, FX_j), (HX_i, GX_j), (HX_i, HX_j)\} \\ &\quad i, j = 1 \ldots n, \quad i < j. \end{aligned} \tag{3.1b}$$

For any X and Y, we first compute the actions $[F, F]$, $[G, G]$ and $[H, H]$ on the pairs of vectors

$$\begin{aligned} &(X, Y), (X, FY), (X, GY), (X, HY), (FX, Y), (FX, FY), (FX, GY), (FX, HY), \\ &(GX, Y), (GX, FY), (GX, GY), (GX, HY), (HX, Y), (HX, FY), \\ &\quad (HX, GY), (HX, HY). \end{aligned} \tag{3.2}$$

These actions are given in the tables below.

$[F, F]$	Y	FY	GY	HY
X	A_1	$-FA_1$	A_3	$-FA_3$
FX	$-FA_1$	$-A_1$	$-FA_3$	$-A_3$
GX	A_2	$-FA_2$	A_4	$-FA_4$
HX	$-FA_2$	$-A_2$	$-FA_4$	$-A_4$

$[G, G]$	Y	FY	GY	HY
X	B_1	B_3	$-GB_1$	GB_3
FX	B_2	B_4	$-GB_2$	GB_4
GX	$-GB_1$	$-GB_3$	$-B_1$	B_3
HX	GB_2	GB_4	B_2	$-B_4$

$[H, H]$	Y	FY	GY	HY
X	C_1	C_3	$-HC_3$	$-HC_1$
FX	C_2	C_4	$-HC_4$	$-HC_2$
GX	$-HC_2$	$-HC_4$	$-C_4$	$-C_2$
HX	$-HC_1$	$-HC_3$	$-C_3$	$-C_1$

From these tables, we obtain the action of $[V, V]$ on the set given by (3.2) as

$$\begin{array}{rlll}
\text{[V,V] (X,Y)=} & A_1 & +B_1 & +C_1 \\
\text{[V,V] (FX,Y)=} & -FA_1 & +B_2 & +C_2 \\
\text{[V,V] (GX,Y)=} & A_2 & -GB_1 & -HC_2 \\
\text{[V,V] (HX,Y)=} & -FA_2 & +GB_2 & -HC_1 \\
\text{[V,V] (X,FY)=} & -FA_1 & +B_3 & +C_3 \\
\text{[V,V] (FX,FY)=} & -A_1 & +B_4 & +C_4 \\
\text{[V,V] (GX,FY)=} & -FA_2 & -GB_3 & -HC_4 \\
\text{[V,V] (HX,FY)=} & -FA_2 & +GB_4 & -HC_3 \\
\text{[V,V] (X,GY)=} & A_3 & -GB_1 & -HC_3 \\
\text{[V,V] (FX,GY)=} & -FA_3 & -GB_2 & -HC_4 \\
\text{[V,V] (GX,GY)=} & A_4 & -B_1 & -C_4 \\
\text{[V,V] (HX,GY)=} & -FA_4 & +B_2 & -C_3 \\
\text{[V,V] (X,HY)=} & -FA_3 & +GB_3 & -HC_1 \\
\text{[V,V] (FX,HY)=} & -A_3 & +GB_4 & -HC_2 \\
\text{[V,V] (GX,HY)=} & -FA_4 & +B_3 & -C_2 \\
\text{[V,V] (HX,HY)=} & -A_4 & -B_4 & -C_1.
\end{array} \tag{3.3}$$

These linear equations can be solved as

$$\begin{aligned}
B_1 &= \frac{1}{2}\,[-A_1 - GA_2 - GA_3 + A_4], \\
B_2 &= \frac{1}{2}\,[FA_1 + HA_2 - HA_3 + FA_4], \\
B_3 &= \frac{1}{2}\,[FA_1 - HA_2 + HA_3 + FA_4], \\
B_4 &= \frac{1}{2}\,[A_1 - GA_2 - GA_3 - A_4],
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
C_1 &= \frac{1}{2}\,[-A_1 + GA_2 + GA_3 - A_4], \\
C_2 &= \frac{1}{2}\,[FA_1 - HA_2 + HA_3 - FA_4], \\
C_3 &= \frac{1}{2}\,[FA_1 + HA_2 - HA_3 - FA_4], \\
C_4 &= \frac{1}{2}\,[A_1 + GA_2 + GA_3 + A_4].
\end{aligned} \tag{3.5}$$

Thus, the action of $[V, V]$ on the set given by (3.2) is determined by the action of $[F, F]$ (actually any one of $[F, F]$, $[G, G]$ or $[H, H]$) on the same set.

Note that, to see the effect of $[V, V]$ on the set S_1, we set $X = Y = X_i$ in the tables and it can be seen that $A_1 = A_4 = B_1 = B_4 = C_1 = C_4 = 0$, $A_2 = -A_3$, $B_2 = -B_3$,

$C_2 = -C_3$. Then, equations (3.3) reduce to

$$B_3 + C_3 = 0, \qquad A_3 - HC_3 = 0, \qquad -FA_3 + CB_3 = 0,$$

which imply that

$$B_3 = HA_3, \qquad C_3 = -HA_3.$$

On the other hand, on the set S_2, we have the full set of equations (3.4-3.5) for each i, j. As a result, if in particular $[F, F] = 0$, then $[V, V] = 0$ implies that $[G, G] = [H, H] = 0$. Thus, we have proved the following theorem.

Theorem 3.1: Let M be a $4n$ dimensional almost quaternionic manifold with a local basis $\{F, G, H\}$ for its almost quaternionic structure and let $[V, V] = [F, F] + [G, G] + [H, H]$. Then if any of the Nijenhuis tensors of F, G and H vanishes and $[V, V] = 0$, then the other two also vanishes.

In particular, from previous section, it follows that V is flat. If M is a complex manifold with almost complex structure say F, then $[V, V] = 0$ implies that V is a trivial bundle.

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