

## Minimality of certain contact slant submanifolds in Sasakian space forms

Ion MIHAI and Valentin GHIȘOIU

**Abstract** One of the present authors established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Sasakian space form.

In this article, we derive a sharp estimate of the maximum Ricci curvature function for a special contact slant submanifold  $M$  in a Sasakian space form  $\widetilde{M}(c)$ , in terms of the main extrinsic invariant, namely the squared mean curvature. If  $M$  satisfies the equality case identically, then it is minimal. Moreover, in this case,  $M$  is a ruled submanifold.

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### 1 Sasakian manifolds

A  $(2m + 1)$ -dimensional Riemannian manifold  $(\widetilde{M}, g)$  is said to be a *Sasakian manifold* if it admits an endomorphism  $\phi$  of its tangent bundle  $T\widetilde{M}$ , a vector field  $\xi$  and a 1-form  $\eta$ , satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \\ (\widetilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad \widetilde{\nabla}_X \xi = -\phi X, \end{cases}$$

for any vector fields  $X, Y$  on  $T\widetilde{M}$ , where  $\widetilde{\nabla}$  denotes the Riemannian connection with respect to  $g$ . A plane section  $\pi$  in  $T_p\widetilde{M}$  is called a  $\phi$ -section if it is spanned by  $X$  and  $\phi X$ , where  $X$  is a unit tangent vector orthogonal to  $\xi$ . The sectional curvature of a  $\phi$ -section is called a  $\phi$ -sectional curvature. A Sasakian manifold with constant  $\phi$ -sectional curvature  $c$  is said to be a *Sasakian space form* and is denoted by  $\widetilde{M}(c)$ .

The curvature tensor  $\widetilde{R}$  of a Sasakian space form  $\widetilde{M}(c)$  is given by [1]:

$$\widetilde{R}(X, Y)Z = \frac{c+3}{4}(g(Y, Z)X - g(X, Z)Y) +$$

$$\begin{aligned}
& + \frac{c-1}{4} (\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \\
& \quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z),
\end{aligned}$$

for any tangent vector fields  $X, Y, Z$  to  $\widetilde{M}(c)$ .

As examples of Sasakian space forms we mention  $\mathbf{R}^{2m+1}$  and  $S^{2m+1}$  with standard Sasakian structures (see [1], [12]).

Let  $M$  be an  $n$ -dimensional submanifold of a Sasakian space form  $\widetilde{M}(c)$  of constant  $\phi$ -sectional curvature  $c$ . We denote by  $K(\pi)$  the sectional curvature of  $M$  associated with a plane section  $\pi \subset T_p M, p \in M$ , and  $\nabla$  the Riemannian connection of  $M$ , respectively. Also, let  $h$  be the second fundamental form and  $R$  the Riemann curvature tensor of  $M$ . Then the equation of Gauss is given by

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

for any vectors  $X, Y, Z, W$  tangent to  $M$ .

Let  $p \in M$  and  $\{e_1, \dots, e_{2m+1}\}$  an orthonormal basis of the tangent space  $T_p \widetilde{M}$ , such that  $e_1, \dots, e_n$  are tangent to  $M$  at  $p$ . We denote by  $H$  the mean curvature vector, that is,  $H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$ . Also, we set  $h_{ij}^r = g(h(e_i, e_j), e_r), i, j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m+1\}$ .

For any vector field  $X$  tangent to  $M$ , we set  $\phi X = PX + FX$ , where  $PX$  and  $FX$  are the tangential and normal components of  $\phi X$  respectively.

## 2 Contact slant submanifolds

One of the present authors established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in Sasakian space forms.

**Theorem 2.1** [8]. *Let  $\widetilde{M}(c)$  be a  $(2m+1)$ -dimensional Sasakian space form and  $M$  an  $n$ -dimensional submanifold tangent to  $\xi$ . Then:*

i) *For each unit vector  $X \in T_p M$  orthogonal to  $\xi$ , we have*

$$(2.1) \quad \text{Ric}(X) \leq \frac{1}{4} \{ (n-1)(c+3) + \frac{1}{2} (3\|PX\|^2 - 2)(c-1) + n^2 \|H\|^2 \}.$$

ii) *A unit tangent vector  $X$  orthogonal to  $\xi$  at  $p$  satisfies the equality case of (2.1) if and only if*

$$(2.2) \quad \begin{cases} h(X, Y) = 0, & \forall Y \in T_p M \text{ orthogonal to } X, \\ 2h(X, X) = nH(p). \end{cases}$$

A submanifold  $M$  tangent to  $\xi$  in a Sasakian space form  $\widetilde{M}(c)$  is said to be a *contact slant* submanifold [2] if for any  $p \in M$  and any  $X \in T_p M$  linearly independent on  $\xi$ , the angle between  $\phi X$  and  $T_p M$  is a constant  $\theta$ , called the slant angle of  $M$ . A contact slant submanifold which is neither invariant nor anti-invariant is called a *proper slant* submanifold. For other results in this respect, we refer to [10], [11].

In [2] a particular case of contact slant submanifolds, which we call *special contact slant* submanifolds, was considered. A proper contact  $\theta$ -slant submanifold is said to be *special contact slant* if

$$(\nabla_X P)Y = (\cos^2 \theta)[g(X, Y)\xi - \eta(Y)X], \quad \forall X, Y \in \Gamma(TM).$$

Any 3-dimensional proper contact slant submanifold of a Sasakian manifold is a special contact slant submanifold (see [2]).

We denote by  $\mathcal{R}$  the maximum Ricci curvature function on  $M$  (see [6]), defined by

$$\mathcal{R}(p) = \max\{\text{Ric}(u) \mid u \in T_p^1 M\}, \quad p \in M,$$

where  $T_p^1 M = \{u \in T_p M \mid g(u, u) = 1\}$ .

If  $n = 3$ ,  $\mathcal{R}$  is the Chen first invariant  $\delta_M$  introduced in [4]. For  $n > 3$ ,  $\mathcal{R}$  is the Chen invariant  $\delta(n - 1)$  (see [7]).

In this section, we derive an inequality for the Chen invariant  $\mathcal{R}$  and prove that any special contact slant submanifold which satisfies the equality case, identically, is minimal. Corresponding result for  $C$ -totally real submanifolds in Sasakian space forms was obtained in [9].

**Theorem 2.2.** *Let  $M$  be an  $(n+1)$ -dimensional special contact slant submanifold in a  $(2n+1)$ -dimensional Sasakian space form  $\widetilde{M}(c)$  of constant  $\phi$ -sectional curvature  $c$ . Then*

$$(2.3) \quad \mathcal{R} \leq \frac{1}{4}\{(n+1)^2\|H\|^2 + n(c+3) + \frac{1}{2}(3\cos^2\theta - 2)(c-1)\},$$

If  $M$  satisfies the equality case of (2.3) identically, then  $M$  is a minimal submanifold.

*Proof.* The inequality (2.3) is an immediate consequence of the inequality (2.1).

We assume that  $M$  is a special contact slant submanifold of  $\widetilde{M}(c)$ , which satisfies the equality case of (2.3) at a point  $p \in M$ . Without loss of generality we may choose an orthonormal basis  $\{e_0 = \xi, e_1, \dots, e_n\}$  of  $T_p M$  such that  $\mathcal{R}(p) = \text{Ric}(e_n)$ . Then, by (2.2), it follows that

$$(2.4) \quad h(e_i, e_n) = 0, \quad \forall i \in \{0, \dots, n-1\}; \quad h(e_n, e_n) = \sum_{i=1}^{n-1} h(e_i, e_i),$$

where  $\{e_0 = \xi, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$  is the orthonormal basis with  $e_{n+j} = (\sec\theta)Pe_j$ ,  $j \in \{1, \dots, n\}$ , and  $e_{n+1}$  parallel to the mean curvature vector  $H(p)$ .

Let  $A$  denote the shape operator of  $M$  in  $\widetilde{M}(c)$ . It is easy to prove that

$$(2.5) \quad A_{\phi Y} X = A_{\phi X} Y + (\sin^2\theta)\widetilde{R}(X, Y)\xi,$$

for all vector fields  $X, Y$  tangent to  $M$  (see, for instance, [2]). Then we have  $h_{ij}^{n+k} = h_{ik}^{n+j} = h_{jk}^{n+i}$  for any  $i, j, k \in \{1, \dots, n\}$ .

Thus, using the equations (2.4), we find

$$(n+1)H(p) = \sum_{i=1}^n h_{ii}^{n+1} e_{n+1} = 2h_{nn}^{n+1} e_{n+1} = 2h_{1n}^{2n} e_{n+1} = 0.$$

Therefore  $M$  is a minimal submanifold. □

We have the following characterisation of  $n$ -dimensional special slant submanifolds in the sphere  $S^{2n+1}$  which satisfy the equality case of (2.3) identically.

**Theorem 2.3.** *Let  $M$  be an  $(n + 1)$ -dimensional special contact slant submanifold of the  $(2n + 1)$ -dimensional sphere  $S^{2n+1}$ . Then  $M$  satisfies the equality case of (2.3) identically if and only if  $M$  is minimal and ruled.*

*Proof.* Theorem 2.2 implies that  $M$  is a minimal submanifold.

By Theorem 2.1, it follows that  $M$  satisfies the equality case of (2.3) identically if and only if the relative null space

$$\mathcal{N}_p = \{X \in T_p M \mid h(X, Y) = 0, \forall Y \in T_p M\} \neq \{0\}$$

at any  $p \in M$ .

Let  $U_l$  denote the interior of the subset consisting of points in  $M$  such that the relative null space at  $p$  has dimension  $l$ . It follows that  $U_l \neq \emptyset$ , for some integer  $1 \leq l \leq n$ .

Let  $X, Y \in \mathcal{N}$  and  $Z$  any vector field tangent to  $M$ . By applying Codazzi equation  $(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Z h)(X, Y)$ , we get  $h(\nabla_X Y, Z) = 0$ . Then  $\mathcal{N}$  is integrable on  $U_l$  and each leaf of  $\mathcal{N}|_{U_l}$  is an  $l$ -dimensional totally geodesic submanifold of  $S^{2n+1}$ . Thus,  $M$  contains a geodesic of  $S^{2n+1}$  through each point  $p \in U_l$ . Since  $M$  is the union of the closure of all  $U_l$ , we conclude by continuity that  $M$  contains a geodesic of the ambient space through each point in  $M$ . Therefore,  $M$  is a ruled minimal submanifold.

The converse statement follows from straightforward calculations.

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Faculty of Mathematics  
University of Bucharest  
Str. Academiei 14  
010014 Bucharest  
Romania  
E-mail: imihai@math.math.unibuc.ro