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On semi-invariant submanifolds of nearly trans-Sasakian manifolds

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Abstract Semi-invariant submanifolds of a nearly trans-Sasakian manifold are studied. Nijenhuis tensor in a nearly trans-Sasakian manifold is calculated. Integrability conditions for some distributions on a semi-invariant submanifold of a nearly trans-Sasakian manifold are investigated. Totally umbilical, totally contact umbilical, totally geodesic and totally contact geodesic submanifolds are also studied.

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1 Introduction

In [26], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they could be divided into three classes: **(1)** homogeneous normal contact Riemannian manifolds with $c > 0$, **(2)** global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and **(3)** a warped product space $\mathbb{R}\times_f\mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class **(1)** are characterized by admitting a Sasakian structure. The manifolds of class **(2)** are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu ([11]) characterized the differential geometric properties of the manifolds of class **(3)**; the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian ([11]).

In the Gray-Hervella classification of almost Hermitian manifolds ([9]), there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $\bar{M}$

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is called a *trans-Sasakian structure* ([18]) if the product manifold $\bar{M} \times \mathbb{R}$ belongs to the class $\mathcal{W}_4$. The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ ([15]) coincides with the class of trans-Sasakian structures of type (α, β). We note that trans-Sasakian structures of type $(0, 0)$ are cosymplectic ([3]), trans-Sasakian structures of type $(0, \beta)$ are β-Kenmotsu ([10]) and trans-Sasakian structures of type $(\alpha, 0)$ are α-Sasakian ([10]). In [29], it is proved that an almost contact metric manifold is trans-Sasakian if and only if it is generalised quasi-Sasakian ([17]), trans-K-contact and normal.

Recently, C. Gherghe ([8]) introduced a nearly trans-Sasakian structure of type (α, β), which generalize trans-Sasakian structure in the same sense as nearly Sasakian structure generalize Sasakian ones. A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly-Sasakian ([5]) or nearly Kenmotsu ([33]) or nearly cosymplectic ([3]) according as $\beta = 0$ or $\alpha = 0$ or $\alpha = 0 = \beta$.

The study of the differential geometry of semi-invariant or contact CR submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was initiated by Bejancu [2] and was followed by several geometers (see [2], [34] and references cited there). Several authors studied semi-invariant submanifolds of different classes of almost contact metric manifolds given in references of this paper.

In the present paper we study semi-invariant submanifolds of a nearly trans-Sasakian manifold. The rest of this paper is organized as follows. In Section 2 we recall some necessary details about the nearly trans-Sasakian manifolds. In section 3 we give some basic results and examples about semi-invariant submanifolds of a nearly trans-Sasakian manifold. In section 4 the Nijenhuis tensor $[\phi, \phi]$ of ϕ in a of a nearly trans-Sasakian manifold is calculated. In section 5 some basic results are given. Integrability conditions for some distributions on a semi-invariant submanifold of a of a nearly trans-Sasakian manifold are the subject matter of the sections 6, 7, 8. In section 9, the non-integrability of the distribution $\boldsymbol{\mathcal{D}^1}$ is proved. Section 10 deals with totally umbilical and totally geodesic submanifolds. In section 11, totally contact umbilical and totally contact geodesic submanifolds are studied. In the last section, we discuss some parallel operators.

2 Nearly trans-Sasakian manifolds

Let $\boldsymbol{\widetilde{M}}$ be an almost contact metric manifold ([3]) with an almost contact metric structure $\boldsymbol{(\phi, \xi, \eta, g)}$, that is, $\boldsymbol{\phi}$ is a $\boldsymbol{(1,1)}$ tensor field, $\boldsymbol{\xi}$ is a vector field; $\boldsymbol{\eta}$ is $\boldsymbol{1}$-form and $\boldsymbol{g}$ is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g\,(X, \xi) = \eta(X) \tag{3}$$

for all $\boldsymbol{X, Y \in T\widetilde{M}}$.

There are two known classes of almost contact metric manifolds, namely Sasakian and Kenmotsu manifolds. Sasakian manifolds are characterized by the tensorial relation

$$(\widetilde{\nabla}_X\phi)Y = g(X,Y)\xi - \eta(Y)X,$$

while Kenmotsu manifolds are given by the tensor equation

$$(\widetilde{\nabla}_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X.$$

An almost contact metric structure (ϕ,ξ,η,g) on $\widetilde{M}$ is called a *trans-Sasakian structure* ([18]) if $(\widetilde{M}\times\mathbb{R},J,G)$ belongs to the class $\mathcal{W}_4$ of the Gray-Hervella classification of almost Hermitian manifolds ([9]), where J is the almost complex structure on $\widetilde{M}\times\mathbb{R}$ defined by

$$J(X,ad/dt) = (\phi X - a\xi, \eta(X)d/dt)$$

for all vector fields X on $\widetilde{M}$ and smooth functions a on $\widetilde{M}\times\mathbb{R}$ and G is the product metric on $\widetilde{M}\times\mathbb{R}$. This may be expressed by the condition ([4])

$$(\widetilde{\nabla}_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \qquad (4)$$

for some smooth functions α and β on $\widetilde{M}$, and we say that the trans-Sasakian structure is of type (α,β). From the formula (4) it follows that ([4])

$$\widetilde{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \qquad (5)$$

The class $\mathcal{C}_6\oplus\mathcal{C}_5$ ([15]) coincides with the class of trans-Sasakian structures of type (α,β). We note that trans-Sasakian structures of type $(0,0)$ are cosymplectic ([3]), trans-Sasakian structures of type $(0,\beta)$ are β-Kenmotsu ([10]) and trans-Sasakian structures of type $(\alpha,0)$ are α-Sasakian ([10]).

Recently, C. Gherghe ([8]) introduced a nearly trans-Sasakian structure of type (α,β). An almost contact metric structure (ϕ,ξ,η,g) on $\widetilde{M}$ is called a *nearly trans-Sasakian structure* ([8]) if

$$\begin{aligned}(\widetilde{\nabla}_X\phi)Y + (\widetilde{\nabla}_Y\phi)X &= \alpha\,(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y)\\ &\quad - \beta\,(\eta(Y)\phi X + \eta(X)\phi Y)\end{aligned} \qquad (6)$$

A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α,β) is nearly-Sasakian ([5]) or nearly Kenmotsu ([33]) or nearly cosymplectic ([3]) according as $\beta = 0$, $\alpha = 1$; or $\alpha = 0$, $\beta = 1$; or $\alpha = 0 = \beta$ respectively.

A nearly trans-Sasakian structure of type (α,β) will be called *nearly α-Sasakian* (resp. *nearly β-Kenmotsu*) if $\beta = 0$ (resp. $\alpha = 0$).

Thus the structural equations for nearly α-Sasakian, nearly Sasakian, nearly β-Kenmotsu, nearly Kenmotsu and nearly cosymplectic manifolds are given by

$$(\widetilde{\nabla}_X\phi)Y + (\widetilde{\nabla}_Y\phi)X = \alpha\,(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y)\,, \qquad (7)$$

$$(\widetilde{\nabla}_X\phi)Y + (\widetilde{\nabla}_Y\phi)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y, \qquad (8)$$

$$(\widetilde{\nabla}_X\phi)Y + (\widetilde{\nabla}_Y\phi)X = -\beta\left(\eta(Y)\phi X + \eta(X)\phi Y\right), \tag{9}$$

$$(\widetilde{\nabla}_X\phi)Y + (\widetilde{\nabla}_Y\phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{10}$$

$$(\widetilde{\nabla}_X\phi)Y + (\widetilde{\nabla}_Y\phi)X = 0 \tag{11}$$

respectively.

3 Semi-invariant submanifolds

Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (X, Y \in TM), \tag{12}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \quad (N \in T^{\perp}M), \tag{13}$$

where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to A by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{14}$$

Moreover, if ϕ is a $(1,1)$ tensor field on $\widetilde{M}$, for $X \in TM$ and $N \in T^{\perp}M$ we have

$$\begin{aligned} (\widetilde{\nabla}_X\phi)Y &= ((\nabla_X P)Y - A_{FY}X - th(X,Y)) + \\ &\qquad ((\nabla_X F)Y + h(X,PY) - fh(X,Y)), \qquad (15)\\ (\widetilde{\nabla}_X\phi)N &= ((\nabla_X t)N - A_{fN}X - PA_N X)) + \\ &\qquad ((\nabla_X f)N + h(X,tN) - FA_N X)), \qquad (16) \end{aligned}$$

where

$$\begin{aligned} \phi X &\equiv PX + FX, \qquad (PX \in TM, FX \in T^{\perp}M), \qquad (17)\\ \phi N &\equiv tN + fN, \qquad (tN \in TM, fN \in T^{\perp}M), \qquad (18)\\ (\nabla_X P)Y &\equiv \nabla_X PY - P\nabla_X Y, \qquad (\nabla_X F)Y \equiv \nabla^{\perp}_X FY - F\nabla_X Y,\\ (\nabla_X t)N &\equiv \nabla_X tN - t\nabla^{\perp}_X N, \qquad (\nabla_X f)N \equiv \nabla^{\perp}_X fN - f\nabla^{\perp}_X N. \end{aligned}$$

The submanifold M is known to be *totally geodesic* in $\widetilde{M}$ if $h = 0$, *minimal* in $\widetilde{M}$ if $\mathrm{H} = \mathrm{trace}(h)/\dim(M) = 0$, and *totally umbilical* in $\widetilde{M}$ if $h(X,Y) = g(X,Y)\mathrm{H}$.

For a distribution $\mathcal{D}$ on M, M is said to be $\mathcal{D}$*-totally geodesic* if for all $X, Y \in \mathcal{D}$ we have $h(X,Y) = 0$. If for all $X, Y \in \mathcal{D}$ we have $h(X,Y) = g(X,Y)\mathrm{K}$ for some normal vector K, then M is called $\mathcal{D}$*-totally umbilical.* For two distributions $\mathcal{D}$ and $\mathcal{E}$ defined on M, M is said to be $(\mathcal{D},\mathcal{E})$*-mixed totally geodesic* if for all $X \in \mathcal{D}$ and $Y \in \mathcal{E}$ we have $h(X,Y) = 0$.

Let $\mathcal{D}$ and $\mathcal{E}$ be two distributions defined on a manifold M. We say that $\mathcal{D}$ is $\mathcal{E}$*-parallel* if for all $X \in \mathcal{E}$ and $Y \in \mathcal{D}$ we have $\nabla_X Y \in \mathcal{D}$. If $\mathcal{D}$ is $\mathcal{D}$-parallel then it is called *autoparallel.* $\mathcal{D}$ is called X*-parallel* for some $X \in TM$ if for all $Y \in \mathcal{D}$ we have $\nabla_X Y \in \mathcal{D}$. $\mathcal{D}$ is said to be *parallel* if for all $X \in TM$ and $Y \in \mathcal{D}$, $\nabla_X Y \in \mathcal{D}$.

If a distribution $\mathcal{D}$ on M is autoparallel, then it is clearly integrable, and by Gauss formula $\mathcal{D}$ is totally geodesic in M. If $\mathcal{D}$ is parallel then the orthogonal complementary distribution $\mathcal{D}^\perp$ is also parallel, which implies that $\mathcal{D}$ is parallel if and only if $\mathcal{D}^\perp$ is parallel. In this case M is locally the product of the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$.

Let M be a submanifold of an almost contact metric manifold. If $\xi \in TM$ then we write $TM = \{\xi\} \oplus \{\xi\}^\perp$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^\perp$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then one gets

$$P\xi = 0 = F\xi, \qquad \eta \circ P = 0 = \eta \circ F, \tag{19}$$

$$P^2 + tF = -I + \eta \otimes \xi, \qquad FP + fF = 0, \tag{20}$$

$$f^2 + Ft = -I, \qquad tf + Pt = 0. \tag{21}$$

A submanifold M of an almost contact metric manifold $\widetilde{M}$ with $\xi \in TM$ is called a *semi-invariant submanifold* (Bejancu, [2]) of $\widetilde{M}$ if there exists two differentiable distributions $\mathcal{D}^1$ and $\mathcal{D}^0$ on M such that

(1) $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\}$,

(2) the distribution $\mathcal{D}^1$ is invariant by ϕ, that is, $\phi(\mathcal{D}^1) = \mathcal{D}^1$, and

(3) the distribution $\mathcal{D}^0$ is anti-invariant by ϕ, that is, $\phi(\mathcal{D}^0) \subseteq T^\perp M$.

Here

$$\mathcal{D}^1_x = \ker(F|_{\{\xi\}^\perp})_x = \{X_x \in \{\xi\}^\perp_x : \| X_x \| = \| PX_x \|\},$$

$$\mathcal{D}^0_x = \ker(P|_{\{\xi\}^\perp})_x = \{X_x \in \{\xi\}^\perp_x : \| X_x \| = \| FX_x \|\}.$$

In fact, we have

$$\mathcal{D}^1_x = T_x M \cap \phi(T_x M), \qquad \mathcal{D}^0_x = T_x M \cap \phi(T^\perp_x M).$$

For $X \in TM$ we can write

$$X = U^1 X + U^0 X + \eta(X)\xi \tag{22}$$

where U^1 and U^0 are projection operators of TM on $\mathcal{D}^1$ and $\mathcal{D}^0$ respectively. A semi-invariant submanifold of an almost contact metric manifold becomes an *invariant submanifold* ([2], [34]) (resp. *anti-invariant submanifold* ([2], [34]) if $\mathcal{D}^0 = \{0\}$ (resp. $\mathcal{D}^1 = \{0\}$). Moreover, we have

$$T^\perp M = \widetilde{\mathcal{D}}^1 \oplus \widetilde{\mathcal{D}}^0$$

where $\widetilde{\mathcal{D}}^1 = \ker(t) = T^\perp M \cap \phi(T^\perp M)$, $\widetilde{\mathcal{D}}^0 = \ker(f) = T^\perp M \cap \phi(TM)$, $F\mathcal{D}^0 = \widetilde{\mathcal{D}}^0$, and $t\widetilde{\mathcal{D}}^0 = \mathcal{D}^0$.

Example 3.1 Let $\mathbb{R}^{2n+1} = C^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Euclidean space endowed with the almost contact metric structure (ϕ, ξ, η, g) defined by

$$\phi(x^1, \dots, x^{2n}, t) = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1}, 0), \ \eta = dt, \ \xi = \frac{\partial}{\partial t},$$

with $1 < h < n$. The product $M_1 \times M_2 \times \mathbb{R}$, where M_1 is a complex submanifold of C^h and M_2 is a totally real submanifold of C^{n-h}, is a semi-invariant submanifold of $\mathbb{R}^{2n+1}$.

Let $\widetilde{M}$ be an almost contact metric manifold. Consider the Riemannian product manifold $\widetilde{M} \times \mathbb{R}$ and define J on its tangent bundle, by

$$JX = \phi X, \quad X \in \{\xi\}^{\perp}, \quad J\xi = \frac{\partial}{\partial t}, \quad J\left(\frac{\partial}{\partial t}\right) = -\xi.$$

Then J defines an almost Hermitian structure on $\widetilde{M} \times \mathbb{R}$. Let M be a submanifold of $\widetilde{M}$ tangent to ξ. Then we have

Proposition 3.2 *The submanifold M is a semi-invariant submanifold of the almost contact metric manifold $\widetilde{M}$ if and only if M is a CR-submanifold* ([2]) *of the almost Hermitian manifold $\widetilde{M} \times \mathbb{R}$.*

It is known that ([33]) if M is a semi-invariant submanifold of a normal almost contact metric manifold with a non-trivial invariant distribution then M possesses a *CR–structure* ([2], pp. 128-129).

Thus, for a semi-invariant submanifold of an almost contact metric manifold $\widetilde{M}$ with $\mathcal{D}^1 \neq \{0\}$ to be a *CR*–manifold, it is sufficient that $\widetilde{M}$ is normal. However, this is not necessary, and here we construct an example of a semi invariant submanifold M of an almost contact metric manifold $\widetilde{M}$ such that M is a *CR*–manifold and $\widetilde{M}$ is not normal.

Example 3.3 ([30]) Consider the Euclidean space $\mathbb{R}^5$ and denote its points by $x = (x^1, \dots, x^5)$. Let (e_j), $j = 1, ..., 5$ be the natural basis defined by $e_j = \partial/\partial x^j$, and g be the canonical metric defined by $g(e_i, e_j) = \delta_{ij}, \quad i, j = 1, \dots, 5$. For every $x \in \mathbb{R}^5$, the set (E_j) defined by

$$E_1 = e_1, \quad E_4 = e_4, \quad E_5 = e_5$$

$$E_2 = \cos(x^1)e_2 + \sin(x^1)e_3,$$

$$E_3 = -\sin(x^1)e_2 + \cos(x^1)e_3,$$

forms an orthonormal basis, i.e $g(E_i, E_j) = \delta_{ij}$. As the point x varies in $\mathbb{R}^5$ the above set of equations defines five vector fields also denoted by (E_j). Now, we define a vector field ξ by $\xi = \partial/\partial x^5$, a 1-form η by $\eta = dx^5$ and a $(1,1)$ tensor field ϕ by

$$\phi(E_1) = E_2, \ \phi(E_2) = -E_1, \ \phi(E_3) = E_4, \ \phi(E_4) = -E_3, \ \phi(E_5) = 0.$$

Then (ϕ, ξ, η, g) define an almost contact metric structure on $\mathbb{R}^5$. Since $N^{(1)}(E_1, E_4) = E_1 \neq 0$, this structure is not normal.
The hypersurface $\mathbb{R}^4 = \{x \in \mathbb{R}^5 \mid x^5 = 0\}$ is a semi-invariant submanifold of $\mathbb{R}^5$ with $\mathcal{D}^1 = \mathit{Span}\{E_1, E_2\}$ and $\mathcal{D}^0 = \mathit{Span}\{E_3\}$ such that $(\mathcal{D}^1, \phi)$ is a CR–structure on $\mathbb{R}^4$. Moreover, $\mathcal{D}^1$ is not integrable as $[E_1, E_2] = E_3$.

In view of the above example we propose the following

Problem 3.4 ([30]) *Does a non-normal almost contact metric manifold have a semi-invariant submanifold which is not a* CR*–manifold?*

4 Nijenhuis tensor

An almost contact metric manifold is said to be *normal* ([3]) if the torsion tensor $N^{(1)}$ vanishes :

$$N^{(1)} \equiv [\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{23}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ and d denotes the exterior derivative operator.

In this section we obtain expression for the Nijenhuis tensor $[\phi, \phi]$ of the structure tensor field ϕ given by

$$[\phi, \phi](X, Y) \equiv \Big((\widetilde{\nabla}_{\phi X}\phi)Y - (\widetilde{\nabla}_{\phi Y}\phi)X\Big) - \phi\Big((\widetilde{\nabla}_X\phi)Y - (\widetilde{\nabla}_Y\phi)X\Big) \tag{24}$$

in a nearly trans-Sasakian manifold. In particular, we derive the expressions for the Nijenhuis tensor $[\phi, \phi]$ in nearly Sasakian manifold and nearly Kenmotsu manifolds.

First, we need the following Lemma.

Lemma 4.1 *In an almost contact metric manifold we have*

$$(\widetilde{\nabla}_Y\phi)\phi X = -\phi(\widetilde{\nabla}_Y\phi)X + ((\widetilde{\nabla}_Y\eta)X)\xi + \eta(X)\widetilde{\nabla}_Y\xi\,. \tag{25}$$

Proof. For $X, Y \in T\widetilde{M}$, we have

$$\begin{aligned}(\widetilde{\nabla}_Y\phi)\phi X &= \widetilde{\nabla}_Y(\phi^2X) - \phi(\widetilde{\nabla}_Y\phi X) + \phi(\phi\widetilde{\nabla}_YX) - \phi^2\widetilde{\nabla}_YX\\ &= \widetilde{\nabla}_Y(-X + \eta(X)\xi) - \phi(\widetilde{\nabla}_Y\phi X) + \phi(\phi\widetilde{\nabla}_YX)\\ &\qquad - (-\widetilde{\nabla}_YX + \eta(\widetilde{\nabla}_YX)\xi),\end{aligned}$$

which gives the equation (25). □

Now, we prove the following theorem.

Theorem 4.2 *In a nearly trans-Sasakian manifold the Nijenhuis tensor* $[\phi, \phi]$ *of* ϕ *is given by*

$$\begin{aligned}[\phi, \phi](X, Y) &= 4\phi(\widetilde{\nabla}_Y\phi)X + 2d\eta(X, Y)\xi - \eta(X)\widetilde{\nabla}_Y\xi + \eta(Y)\widetilde{\nabla}_X\xi\\ &\quad + \alpha\left(\eta(Y)\phi X + 3\eta(X)\phi Y + 4g(\phi X, Y)\xi\right)\\ &\quad + \beta\left(\eta(Y)\phi^2X + 3\eta(X)\phi^2Y\right).\end{aligned} \tag{26}$$

Proof. Using Lemma 3.1 and $\eta \circ \phi = 0$ in (6) we get

$$\begin{aligned}(\widetilde{\nabla}_{\phi X}\phi)Y &= \phi(\widetilde{\nabla}_Y\phi)X - ((\widetilde{\nabla}_Y\eta)X)\xi - \eta(X)\widetilde{\nabla}_Y\xi \\ &+ \ \alpha\,(2g(\phi X, Y)\xi - \eta(Y)\phi X) - \beta\eta(Y)\phi^2 X. \qquad (27)\end{aligned}$$

Thus

$$\begin{aligned}[\phi,\phi](X,Y) &= \Big((\widetilde{\nabla}_{\phi X}\phi)Y + \phi((\widetilde{\nabla}_Y\phi)X))\Big) \\ &- \ \Big((\widetilde{\nabla}_{\phi Y}\phi)X + \phi((\widetilde{\nabla}_X\phi)Y))\Big) \\ &= 2\phi(\widetilde{\nabla}_Y\phi)X - ((\widetilde{\nabla}_Y\eta)X))\xi - \eta(X)\widetilde{\nabla}_Y\xi \\ &+\alpha\,(2g(\phi X, Y)\xi - \eta(Y)\phi X) - \beta\eta(Y)\phi^2 X \\ &- 2\phi(\widetilde{\nabla}_X\phi)Y + ((\widetilde{\nabla}_X\eta)Y))\xi + \eta(Y)\widetilde{\nabla}_X\xi \\ &- \alpha\,(2g(\phi Y, X)\xi - \eta(X)\phi Y) + \beta\eta(X)\phi^2 Y \\ &= 2\phi((\widetilde{\nabla}_Y\phi)X - (\widetilde{\nabla}_X\phi)Y) \\ &+ 2d\eta(X,Y)\xi - \eta(X)\widetilde{\nabla}_Y\xi + \eta(Y)\widetilde{\nabla}_X\xi \\ &+ \alpha\,(4g(\phi X, Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y) \\ &- \beta\left(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y\right) \\ &= 2\phi\left((\widetilde{\nabla}_Y\phi)X + (\widetilde{\nabla}_Y\phi)X\right) \\ &- 2\alpha\phi\,(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) \\ &+ 2\beta\phi\,(\eta(Y)\phi X + \eta(X)\phi Y) \\ &+ 2d\eta\,(X,Y)\,\xi - \eta(X)\widetilde{\nabla}_Y\xi + \eta(Y)\widetilde{\nabla}_X\xi \\ &+ \alpha\,(4g(\phi X, Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y) \\ &- \beta\left(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y\right) \\ &= 4\phi(\widetilde{\nabla}_Y\phi)X + \beta\left(\eta(Y)\phi^2 X + 3\eta(X)\phi^2 Y\right) \\ &+ \alpha\,(\eta(Y)\phi X + 3\eta(X)\phi Y + 4g(\phi X, Y)\xi) \\ &+ 2d\eta(X,Y)\xi - \eta(X)\widetilde{\nabla}_Y\xi + \eta(Y)\widetilde{\nabla}_X\xi\end{aligned}$$

which implies the equation (26). □

From equation (26) we get

$$\eta(N^{(1)}(X,Y)) = 4d\eta(X,Y) - 4\alpha g(X,\phi Y). \qquad (28)$$

In particular, if X and Y are perpendicular to ξ then (26) gives

$$[\phi,\phi](X,Y) = 4\phi(\widetilde{\nabla}_Y\phi)X - 2\eta([X,Y])\xi. \qquad (29)$$

Corollary 4.3 *In a nearly Sasakian manifold the Nijenhuis tensor* $[\phi,\phi]$ *of* ϕ *is given by*

$$\begin{aligned}[\phi,\phi](X,Y) &= 4\phi(\widetilde{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi \\ &- \eta(X)\widetilde{\nabla}_Y\xi + \eta(Y)\widetilde{\nabla}_X\xi \\ &+ \eta(Y)\phi X + 3\eta(X)\phi Y - 4g(X,\phi Y)\xi. \qquad (30)\end{aligned}$$

Consequently,

$$\eta(N^{(1)}(X,Y)) = 4d\eta(X,Y) - 4g(X,\phi Y), \tag{31}$$

$$[\phi,\phi](X,Y) = 4\phi(\widetilde{\nabla}_Y\phi)X - 2\eta([X,Y])\xi - 4g(X,\phi Y)\xi, \quad X,Y\perp\xi. \tag{32}$$

The equation (31) is the first equation in Theorem 3.2 of [5].

Corollary 4.4 *In a nearly Kenmotsu manifold the Nijenhuis tensor* $[\phi,\phi]$ *of* ϕ *is given by*

$$\begin{aligned}[\phi,\phi](X,Y) &= 4\phi(\widetilde{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi \\ &\quad - \eta(X)\widetilde{\nabla}_Y\xi + \eta(Y)\widetilde{\nabla}_X\xi \\ &\quad + \eta(Y)\phi^2X + 3\eta(X)\phi^2Y. \end{aligned}\tag{33}$$

Consequently,

$$\eta(N^{(1)}(X,Y)) = 4d\eta(X,Y), \tag{34}$$

$$[\phi,\phi](X,Y) = 4\phi(\widetilde{\nabla}_Y\phi)X - 2\eta([X,Y])\xi, \qquad X,Y\perp\xi. \tag{35}$$

5 Some basic results

Let M be a submanifold of a nearly trans-Sasakian manifold. Using (15) and (17) in (6), for $X,Y\in TM$ we get

$$\begin{aligned} &\alpha\,(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) \\ &-\beta\,(\eta(Y)PX + \eta(Y)FX + \eta(X)PY + \eta(X)FY) \\ = \;&(\nabla_XP)Y + (\nabla_YP)X - A_{FY}X - A_{FX}Y - 2th(X,Y) \\ &+(\nabla_XF)Y + (\nabla_YF)X + h(X,PY) + h(PX,Y) - 2fh(X,Y). \end{aligned}$$

Consequently, we have

Proposition 5.1 *Let* M *be a submanifold of a nearly trans-Sasakian manifold. Then for all* $X,Y\in TM$ *we have*

$$\begin{aligned} &(\nabla_XP)Y + (\nabla_YP)X - A_{FY}X - A_{FX}Y - 2th(X,Y) \\ &\quad = \alpha\,(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) \\ &\quad - \beta\,(\eta(Y)PX + \eta(X)PY) \end{aligned}\tag{36}$$

$$\begin{aligned} &(\nabla_XF)Y + (\nabla_YF)X + h(X,PY) + h(PX,Y) - 2fh(X,Y) \\ &\quad = -\beta\,(\eta(Y)FX + \eta(X)FY)\,. \end{aligned}\tag{37}$$

Now, we state the following proposition.

Proposition 5.2 *Let M be a submanifold of a nearly trans-Sasakian manifold. Then for all $X, Y \in TM$ we get*

$$\begin{aligned}
&\widetilde{\nabla}_X\phi Y - \widetilde{\nabla}_Y\phi X - \phi[X,Y] \\
&= 2\left((\nabla_X P)Y - A_{FY}X - th(X,Y)\right) \\
&+ 2\left((\nabla_X F)Y + h(X,PY) - fh(X,Y)\right) \\
&- \alpha\left(\eta(Y)X + \eta(X)Y - 2g(X,Y)\xi\right) \\
&+ \beta\left(\eta(Y)PX + \eta(X)PY\right) + \beta\left(\eta(Y)FX + \eta(X)FY\right). \qquad (38)
\end{aligned}$$

Consequently,

$$\begin{aligned}
P[X,Y] \;=\; &- \nabla_X PY - \nabla_Y PX + A_{FX}Y + A_{FY}X \\
&+ 2P\nabla_X Y + 2th(X,Y) \\
&+ \alpha\left(\eta(Y)X + \eta(X)Y - 2g(X,Y)\xi\right) \\
&- \beta\left(\eta(Y)PX + \eta(X)PY\right). \qquad (39)
\end{aligned}$$

$$\begin{aligned}
F[X,Y] \;=\; &- \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - h(X,PY) - h(PX,Y) \\
&+ 2F\nabla_X Y + 2fh(X,Y) \\
&- \beta\left(\eta(Y)FX + \eta(X)FY\right) \qquad (40)
\end{aligned}$$

The proof is straightforward and hence omitted.

Proposition 5.3 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. Then (P, ξ, η, g) is a nearly trans-Sasakian structure on the distribution $\mathcal{D}^1 \oplus \{\xi\}$ if $th(X,Y) = 0$ for all $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$.*

Proof. From $\mathcal{D}^1 \oplus \{\xi\} = \ker(F)$ and (20) we have $P^2 = -I + \eta \otimes \xi$ on $\mathcal{D}^1 \oplus \{\xi\}$. We also get $P\xi = 0,\ \eta(\xi) = 2,\ \eta \circ P = 0$. Using $\mathcal{D}^1 \oplus \{\xi\} = \ker(F)$ and $th(X,Y) = 0$ in (36) we get

$$(\nabla_X P)Y + (\nabla_Y P)X = -\eta(Y)PX - \eta(X)PY, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}.$$

This completes the proof. □

Theorem 5.4 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. We have*
(1) *if $\mathcal{D}^0 \oplus \{\xi\}$ is autoparallel then*

$$A_{FX}Y + A_{FY}X + 2th(X,Y) = 0, \quad X, Y \in \mathcal{D}^0 \oplus \{\xi\},$$

(6) *if $\mathcal{D}^1 \oplus \{\xi\}$ is autoparallel then*

$$h(X,PY) + h(PX,Y) = 2fh(X,Y), \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}.$$

Proof. In view of (36) and autoparallelness of $\mathcal{D}^0 \oplus \{\xi\}$ we get (1), while in view of (37) and appropriateness of $\mathcal{D}^1 \oplus \{\xi\}$ we get (2). □

In view of Proposition 5.3 and Theorem 5.4(2) we get

Theorem 5.5 *Let M be a submanifold of a nearly trans-Sasakian manifold with $\xi \in TM$. If M is invariant then M is nearly trans-Sasakian. Moreover*

$$h(X,PY) + h(PX,Y) - 2fh(X,Y) = 0, \quad X, Y \in TM.$$

6 Integrability of the distribution $\mathcal{D}^1 \oplus \{\xi\}$

We begin with a lemma.

Lemma 6.1 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. For $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$ we get*

$$F[X,Y] = -h(X,PY) - h(PX,Y) + 2F\nabla_X Y + 2fh(X,Y) \tag{41}$$

or equivalently

$$-h(X,PX) + F\nabla_X X + fh(X,X) = 0. \tag{42}$$

Proof. Equation (41) follows from $\mathcal{D}^1 \oplus \{\xi\} = \ker(F)$ and (40). Equivalence of (41) and (42) is obvious. □

In view of (41) and $\mathcal{D}^1 \oplus \{\xi\} = \ker(F)$ we can state the following theorem.

Theorem 6.2 *The distribution $\mathcal{D}^1 \oplus \{\xi\}$ on a semi-invariant submanifold of a nearly trans-Sasakian manifold is integrable if and only if*

$$h(X,PY) + h(PX,Y) = 2\left(F\nabla_X Y + fh(X,Y)\right). \tag{43}$$

Now, we need the following

Definition 6.3 ([28]) Let M be a Riemannian manifold with the Riemannian connection ∇. A distribution $\mathcal{D}$ on M will be called *nearly autoparallel* if for all $X, Y \in \mathcal{D}$ we have $(\nabla_X Y + \nabla_Y X) \in \mathcal{D}$ or equivalently $\nabla_X X \in \mathcal{D}$.

Thus, we have the following flow chart ([28]) :
Parallel $\Longrightarrow$ Autoparallel $\Longrightarrow$ Nearly autoparallel,
Parallel $\Longrightarrow$ Integrable,
Autoparallel $\Longrightarrow$ Integrable, and
Nearly autoparallel + Integrable $\Longrightarrow$ Autoparallel.

Theorem 6.4 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. Then the following four statements*
(a) *the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is autoparallel,*
(b) $h(X,PY) + h(PX,Y) = 2fh(X,Y), \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}$,
(c) $h(X,PX) = fh(X,X), \quad X \in \mathcal{D}^1 \oplus \{\xi\}$,
(d) *the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is nearly autoparallel,*
are related by $(a) \Rightarrow (b) \Leftrightarrow (c) \Rightarrow (d)$. In particular, if $\mathcal{D}^1 \oplus \{\xi\}$ is integrable then the above four statements are equivalent.

The proof is similar to that Theorem 4.4 of [28].

Let $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$. Using (1) and (17) in (23) and we get

$$\begin{aligned} N^{(1)}(X,Y) &= 2d\eta(X,Y)\xi + [\phi X, \phi Y] - [X,Y] + \eta([X,Y])\xi \\ &\quad - P([X,\phi Y] + [\phi X, Y]) - F([X,\phi Y] + [\phi X, Y]). \end{aligned} \tag{44}$$

On the other hand from equation (27) we have

$$\begin{aligned}(\widetilde{\nabla}_{\phi X}\phi)Y &= \phi(\widetilde{\nabla}_{Y}\phi)X - g(\nabla_Y\xi, X)\xi - \eta(X)\nabla_Y\xi\\ &\quad - \eta(Y)\phi^2X - \eta(X)h(Y,\xi),\end{aligned}$$

which implies that

$$\begin{aligned}&(\widetilde{\nabla}_{\phi X}\phi)Y - (\widetilde{\nabla}_{\phi Y}\phi)X \\ =\ & \phi((\widetilde{\nabla}_{Y}\phi)X - (\widetilde{\nabla}_{X}\phi)Y) - \eta(Y)\phi^2X - \eta(X)\phi^2Y\\ & -\eta(X)h(Y,\xi) + \eta(Y)h(X,\xi) + 2d\eta(X,Y)\xi\\ & -\eta(X)U^1\nabla_Y\xi + \eta(Y)U^1\nabla_X\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)U^0\nabla_X\xi\,. \end{aligned} \tag{45}$$

Next we easily can get

$$\begin{aligned}\phi(\widetilde{\nabla}_{Y}\phi)X &= \phi\widetilde{\nabla}_{Y}\phi X - \phi^2\widetilde{\nabla}_{Y}X\\ &= \phi(\nabla_Y\phi X + h(Y,\phi X)) + \widetilde{\nabla}_{Y}X - \eta(\widetilde{\nabla}_{Y}X)\xi\end{aligned}$$

so that

$$\begin{aligned}&\phi\left((\widetilde{\nabla}_{Y}\phi)X - (\widetilde{\nabla}_{X}\phi)Y\right)\\ =\ & -[X,Y] + \eta([X,Y])\xi + P(\nabla_Y\phi X - \nabla_X\phi Y)\\ & +F(\nabla_Y\phi X - \nabla_X\phi Y) + \phi(h(Y,\phi X) - h(X,\phi Y))\end{aligned} \tag{46}$$

In view of (45) and (46) we get

$$\begin{aligned}N^{(1)}(X,Y) &= -2[X,Y]+2\eta([X,Y])\xi+2P(\nabla_Y\phi X-\nabla_X\phi Y)\\ &\quad +2F(\nabla_Y\phi X-\nabla_X\phi Y)+2\phi(h(Y,\phi X)-h(X,\phi Y))\\ &\quad -\eta(Y)\phi^2X-\eta(X)\phi^2Y-\eta(X)h(Y,\xi)+\eta(Y)h(X,\xi)\\ &\quad -\eta(X)U^1\nabla_Y\xi+\eta(Y)U^1\nabla_X\xi\\ &\quad -\eta(X)U^0\nabla_Y\xi+\eta(Y)U^0\nabla_X\xi+4d\eta(X,Y)\xi\end{aligned} \tag{47}$$

Theorem 6.5 *The distribution $\mathcal{D}^1\oplus\{\xi\}$ is integrable on a semi-invariant submanifold M of a nearly trans-Sasakian manifold if and only if for all $X, Y \in \mathcal{D}^1\oplus\{\xi\}$*

$$N^{(1)}(X,Y) \in \mathcal{D}^1\oplus\{\xi\}\,, \tag{48}$$

$$\begin{aligned}&2\,(h(X,\phi Y) - h(Y,\phi X))\\ &= \eta(X)(\phi U^0\nabla_Y\xi + fh(Y,\xi)) - \eta(Y)(\phi U^0\nabla_X\xi + fh(X,\xi))\,.\end{aligned} \tag{49}$$

Proof. Let $X, Y \in \mathcal{D}^1\oplus\{\xi\}$. If $\mathcal{D}^1\oplus\{\xi\}$ is integrable, then (48) is true and from (47) we get

$$\begin{aligned}0 &= 2F(\nabla_Y\phi X - \nabla_X\phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y))\\ &\quad +\eta(Y)U^0\nabla_X\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)h(X,\xi) - \eta(X)h(Y,\xi).\end{aligned}$$

Applying ϕ to the above equation we get

$$\begin{aligned} 0 &= -2U^0(\nabla_Y\phi X - \nabla_X\phi Y) - 2(h(Y,\phi X) - h(X,\phi Y)) \\ &\quad +\eta(Y)\phi U^0\nabla_X\xi - \eta(X)\phi U^0\nabla_Y\xi + \eta(Y)th(X,\xi) \\ &\quad +\eta(Y)fh(X,\xi) - \eta(X)th(Y,\xi) - \eta(X)fh(Y,\xi). \end{aligned}$$

Hence taking the normal part we get (49).

Conversely, let (48) and (49) be true. Using (49) in (47) we get

$$\begin{aligned} 0 &= -2U^0[X,Y]+2F(\nabla_Y\phi X-\nabla_X\phi Y)+2\phi(h(Y,\phi X)-h(X,\phi Y)) \\ &\quad +\eta(Y)U^0\nabla_X\xi-\eta(X)U^0\nabla_Y\xi+\eta(Y)h(X,\xi)-\eta(X)h(Y,\xi). \end{aligned}$$

Applying ϕ to the above equation and using (49) we get $\phi U^0[X,Y] = 0$, from which we get $U^0[X,Y] = 0$, and hence $\mathcal{D}^1 \oplus \{\xi\}$ is integrable. □

If $\widetilde{M}$ is a trans-Sasakian manifold then for all $X \in \mathcal{D}^1 \oplus \{\xi\}$ it is known that $h(X,\xi) = 0$ and $U^0\nabla_X\xi = 0$. Hence in view of the previous theorem we have

Corollary 6.6 *If M is a semi-invariant submanifold of a trans-Sasakian manifold, then the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is integrable if and only if for all $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$*

$$h(X,\phi Y) = h(Y,\phi X).$$

7 Integrability of the distribution $\mathcal{D}^0\oplus\{\xi\}$

Lemma 7.1 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. Then*

$$3\,(A_{FX}Y - A_{FY}X) = P[X,Y], \qquad X, Y \in \mathcal{D}^0\oplus\{\xi\} \tag{50}$$

Proof. Let $X, Y \in \mathcal{D}^0\oplus\{\xi\}$ and $Z \in TM$. We have

$$\begin{aligned} -A_{\phi X}Z + \nabla^{\perp}_Z\phi X &= \tilde\nabla_Z\phi X = (\tilde\nabla_Z\phi)X + \phi\tilde\nabla_Z X \\ &= -(\tilde\nabla_X\phi)Z - \eta(X)\phi Z - \eta(Z)\phi X \\ &\quad + \phi\nabla_Z X + \phi h(Z,X) \end{aligned}$$

so that

$$\begin{aligned} [\phi h(Z,X) &= -A_{\phi X}Z + \nabla^{\perp}_Z\phi X + \eta(X)\phi Z \\ &\quad - \eta(Z)\phi X - \phi\nabla_Z X + (\tilde\nabla_X\phi)Z \end{aligned}$$

and hence we have

$$\begin{aligned} g(\phi h(Z,X),Y) &= -g(A_{\phi X}Z,Y) + g((\tilde\nabla_X\phi)Z,Y) \\ &= -g(A_{\phi X}Y,Z) - g((\tilde\nabla_X\phi)Y,Z). \end{aligned}$$

On the other hand

$$g(\phi h(Z,X),Y) = -g(h(Z,X),\phi Y) = -g(A_{\phi Y}X,Z).$$

Thus from the above two relations we get

$$g(A_{\phi Y}X,Z)=g(A_{\phi X}Y,Z)+g((\widetilde{\nabla}_X\phi)Y,Z). \tag{51}$$

For $X,Y\in\mathcal{D}^0\oplus\{\xi\}$ we calculate $(\widetilde{\nabla}_X\phi)Y$ as follows. In view of

$$\widetilde{\nabla}_X\phi Y-\widetilde{\nabla}_Y\phi X=A_{\phi X}Y-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X$$

and

$$\widetilde{\nabla}_X\phi Y-\widetilde{\nabla}_Y\phi X=(\widetilde{\nabla}_X\phi)Y-(\widetilde{\nabla}_Y\phi)X+\phi[X,Y]$$

we have

$$(\widetilde{\nabla}_X\phi)Y-(\widetilde{\nabla}_Y\phi)X=A_{\phi X}Y-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X-\phi[X,Y],$$

which in view of (10) gives

$$\begin{aligned}(\widetilde{\nabla}_X\phi)Y &= \frac{1}{2}\Big(A_{\phi X}Y-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X\\ &\qquad -\phi[X,Y]-\eta(Y)\phi X-\eta(X)\phi Y\Big).\end{aligned}$$

Using this equation in the equation (51) we get (50). □

In view of $\ker(P)=\mathcal{D}^0\oplus\{\xi\}$, this lemma leads to the following

Theorem 7.2 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. Then the distribution $\mathcal{D}^0\oplus\{\xi\}$ is integrable if and only if*

$$A_{FX}Y=A_{FY}X,\qquad X,Y\in\mathcal{D}^0\oplus\{\xi\}.$$

Using (4) in (51) for $X,Y\in\mathcal{D}^0\oplus\{\xi\}$ we get $A_{FX}Y=A_{FY}X$. Hence in view of the above Theorem we get the following

Corollary 7.3 *Let M be a semi-invariant submanifold of a trans-Sasakian manifold. Then the distribution $\mathcal{D}^0\oplus\{\xi\}$ is integrable.*

8 Integrability of the distribution $\mathcal{D}^0$

We calculate the torsion tensor $N^{(1)}(Y,X)$ for $Y,X\in\mathcal{D}^0$. For $Y,X\in\mathcal{D}^0$, it can be verified that

$$\begin{aligned}\phi\left((\widetilde{\nabla}_X\phi)Y-(\widetilde{\nabla}_Y\phi)X\right) &= \phi(A_{\phi X}Y-A_{\phi Y}X)\\ &\quad+\phi(\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X)+[X,Y]-\eta([X,Y])\xi,\end{aligned} \tag{52}$$

$$\begin{aligned}&(\widetilde{\nabla}_{\phi Y}\phi)X-(\widetilde{\nabla}_{\phi X}\phi)Y\\ &=[X,Y]+\phi(A_{\phi X}Y-A_{\phi Y}X+\phi(\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X)).\end{aligned} \tag{53}$$

Using (52), (53) and (50) we get for $Y,X\in\mathcal{D}^0$

$$N^{(1)}(Y,X)=2[X,Y]+\frac{2}{3}\phi P[X,Y]+2\phi(\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X). \tag{54}$$

Theorem 8.1 *The distribution $\mathcal{D}^0$ is integrable on a semi-invariant submanifold M of a nearly trans-Sasakian manifold if and only if*

$$N^{(1)}(Y,X) \in \mathcal{D}^0 \oplus \widetilde{\mathcal{D}}^1 \qquad X, Y \in \mathcal{D}^0\,, \tag{55}$$

$$A_{FX}Y = A_{FY}X, \qquad X, Y \in \mathcal{D}^0. \tag{56}$$

Proof. If $\mathcal{D}^0$ is integrable, then in view of (53) and (54) the relations (55) and (56) follow easily. Conversely, let $X, Y \in \mathcal{D}^0$ and let the relations (55) and (56) be true. Then in view of (53) we get $P[X,Y] = 0$ and in view of (54) we get

$$0 = g(\xi, N^{(1)}(Y,X)) = g(\xi, 2[Y,X])\,.$$

Thus $[X,Y] \in \mathcal{D}^0$. □

9 Non-integrability of the distribution $\mathcal{D}^1$

Theorem 9.1 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with $\alpha \neq 0$. Then the non-zero invariant distribution $\mathcal{D}^1$ is not integrable.*

Proof. If $\mathcal{D}^1$ is integrable then for $X, Y \in \mathcal{D}^1$ it follows that $d\eta(X,Y) = 0$ and $[\phi,\phi](X,Y) \in \mathcal{D}^1$. Therefore, for $X \in \mathcal{D}^1$ in view of (28) we get

$$\begin{aligned} 0 &= \eta([\phi,\phi](X,PX) + 2d\eta(X,PX)\xi) \\ &= \eta(N^{(1)}(X,PX)) = 4\alpha g(\phi X, PX) = 4\alpha g(PX,PX), \end{aligned}$$

which is a contradiction. □

Remark 9.2 In particular, if M is a semi-invariant submanifold of a nearly Sasakian manifold then we get Theorem 3 of Kobayashi ([13]). It also makes Proposition 2.4 of Shahid ([24]) redundant because non-integrable $\mathcal{D}^1$ implies non-autoparallel $\mathcal{D}^1$ (parallel in the sense of [24]). The above Theorem 9.1 also makes Theorems 2.1, 3.1, 4.1, Corollaries 2.1, 2.2, 4.1, and Lemma 4.1 of Calin ([6]) redundant, where $\mathcal{D}^1$ has been assumed to be integrable.

10 Totally umbilical and totally geodesic submanifolds

On a nearly trans-Sasakian manifold it can be seen that

$$\widetilde{\nabla}_\xi \xi = 0, \tag{57}$$

$$\widetilde{\nabla}_\xi \eta = 0. \tag{58}$$

Let M be a submanifold of an almost contact metric manifold, tangent to the structure vector field ξ. We recall that in this case $TM = \{\xi\} \oplus \{\xi\}^\perp$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^\perp$ is the complementary orthogonal distribution of $\{\xi\}$ in M.

Using (12) in (57) we can state the following lemma.

Lemma 10.1 *Let M be a submanifold of a nearly trans-Sasakian manifold, tangent to ξ. Then*
the integral curve of ξ in M is a geodesic in M,
and ξ is an asymptotic direction.

In view of the second part of the above lemma we are able to state the following proposition.

Proposition 10.2 *Let $\mathcal{D}$ be a distribution on a submanifold M of a nearly trans-Sasakian manifold such that $\xi \in TM$. If M is $\mathcal{D}$-totally umbilical then M is $\mathcal{D}$-totally geodesic.*

The above proposition leads to the following theorem.

Theorem 10.3 *Every totally umbilical submanifold of a nearly trans-Sasakian manifold, tangent to ξ, is totally geodesic.*

Remark 10.4 In Proposition 3.2 of [23] it has been proved that if M is a non-totally geodesic, totally umbilical proper CR submanifold of a Kenmotsu manifold, then the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are non integrable. But, in view of the above theorem the assumption of the Proposition 3.2 of [23] is not possible.

Now, we prove

Theorem 10.5 *If M is a semi-invariant submanifold of a nearly trans-Sasakian manifold, such that M is $\mathcal{D}^1 \oplus \{\xi\}$-totally umbilical, then*
(a) the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is nearly autoparallel.
Consequently, the following two statements become equivalent:
(b) the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is integrable,
(c) the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is autoparallel.

Proof. From Proposition 10.2, M is $\mathcal{D}^1 \oplus \{\xi\}$-totally geodesic. Thus in view of (42), the statement (a) is true. Hence (b) is equivalent to (c). □

We have the following corollary.

Corollary 10.6 *In a totally umbilical semi-invariant submanifold of a nearly trans-Sasakian manifold, $\mathcal{D}^1 \oplus \{\xi\}$ is autoparallel.*

11 Totally contact umbilical and totally contact geodesic submanifolds

First, we recall the following definition.

Definition 11.1 ([28]) A submanifold M of an almost contact metric manifold, tangent to the structure vector field ξ, is called
(1) totally contact umbilical if it is $\{\xi\}^{\perp}$-totally umbilical, and
(2) totally contact geodesic if it is $\{\xi\}^{\perp}$-totally geodesic.

The conditions of totally contact umbilical and totally contact geodesic may be represented by

$$h(\phi^2 X, \phi^2 Y) = g(\phi^2 X, \phi^2 Y)\mathrm{K}, \qquad X, Y \in TM, \tag{59}$$

$$h(\phi^2 X, \phi^2 Y) = 0, \qquad X, Y \in TM \tag{60}$$

respectively, where $\mathbf{K}$ is some normal vector field. Thus the above definition is equivalent to that given in [35].

The equations (59) and (60) become

$$h(X,Y) = g(\phi X, \phi Y)\mathrm{K} + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \quad X, Y \in TM, \tag{61}$$

$$h(X,Y) = \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \qquad X, Y \in TM \tag{62}$$

respectively. It is also known that if M is a totally contact umbilical or totally contact geodesic submanifold of an almost contact metric manifold, tangent to ξ; then ξ is an asymptotic direction ([32]).

Theorem 11.2 *If M is a totally contact umbilical semi-invariant submanifold of a nearly trans-Sasakian manifold, then M is $\left(\mathcal{D}^1, \mathcal{D}^0\right)$-mixed totally geodesic.*

Proof. In this case we have $h(X,Y) = g(X,Y)\mathrm{K}$ for $X, Y \in \{\xi\}^\perp$, which in view of the definition of semi-invariant submanifolds implies the Theorem. □

Theorem 11.3 *If M is a totally contact umbilical semi-invariant submanifold of a nearly trans-Sasakian manifold, then either $\mathcal{D}^0 = \{0\}$ or $\dim\left(\mathcal{D}^0\right) = 1$ or the normal vector field K is orthogonal to $\phi\mathcal{D}^0$.*

Proof. If $\dim(\mathcal{D}^0) > 1$, for each $W \in \mathcal{D}^0$ there exists $X \in \mathcal{D}^0$ such that $g(X,W) = 0$ and $\|X\| = 1$. Then

$$\begin{aligned} g(\mathrm{K}, \phi W) &= g(h(X,X), \phi W) = g(A_{\phi W}X, X) \\ &= g(A_{\phi X}W, X) = g(h(X,W), \phi X) = 0, \end{aligned}$$

which completes the proof. □

At last, we present the following theorem.

Theorem 11.4 *Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold. If M is totally contact umbilical with $\mathcal{D}^1 \neq \{0\}$ and $\mathcal{D}^1 \oplus \{\xi\}$ is autoparallel, then M is totally contact geodesic.*

Proof. On a totally contact umbilical semi-invariant submanifold with $\mathcal{D}^1 \neq \{0\}$, in view of (26), for $X \in \mathcal{D}^1$ we get $(\widetilde{\nabla}_X \phi)(\phi X) = 0$ and thus

$$\begin{aligned} 0 &= g\left((\widetilde{\nabla}_X \phi)(\phi X), \mathrm{K}\right) \\ &= g\left((\widetilde{\nabla}_X \phi^2 X - \phi(\widetilde{\nabla}_X \phi^2 X), \mathrm{K}\right) \\ &= g(\widetilde{\nabla}_X X, \mathrm{K}) + g(\widetilde{\nabla}_X \phi X, \phi\mathrm{K}) \\ &= -g(h(X,X), \mathrm{K}) + g(\nabla_X \phi X, t\mathrm{K}) + g(h(X, \phi X), f\mathrm{K}). \end{aligned}$$

Now, if $\mathcal{D}^1 \oplus \{\xi\}$ is autoparallel, then

$$0 = -g(h(X,X),\mathrm{K}) + g(h(X,\phi X), f\mathrm{K}),$$

which on using (61) implies that $0 = -g(X,X)g(\mathrm{K},\mathrm{K})$, that is, $\mathrm{K} = 0$ or in other words M is totally contact geodesic. □

12 Certain parallel operators

Theorem 12.1 *Let M be a semi-invariant submanifold of a nearly α-Sasakian manifold ($\alpha \neq 0$). Then M is anti-invariant if and only if P is parallel.*

Proof. If M is anti-invariant then $P = 0$ which implies that $\nabla P = 0$, that is, P is parallel. Conversely, if $\nabla P = 0$ then in view of Theorem 6.3 of [33], $\mathcal{D}^1$ is parallel, which in view of Theorem 9.1 is a contradiction. □

Theorem 12.2 *If M is a non-invariant semi-invariant submanifold of a nearly α-Sasakian manifold ($\alpha \neq 0$) such that T is parallel, $T \in \{P, F, t, f\}$, then M is anti-invariant and $\mathcal{D}^0$ is parallel.*

Proof. In view of Theorem 6.3 of [33] and the non-integrability of $\mathcal{D}^1$ the proof follows. □

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