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Continuity of solutions to operator equations with respect to a parameter.

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Abstract Let $A(k)u(k) = f(k)(1)$ be an operator equation, X and Y are Banach spaces, $k \in \Delta \subset \mathbb{C}$ is a parameter, $A(k) : X \to Y$ is a map, possibly nonlinear. Sufficient conditions are given for continuity of $u(k)$ with respect to k. Necessary and sufficient conditions are given for the continuity of $u(k)$ with respect to k in the case of linear operators $A(k)$.

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1 Introduction

Let X and Y be Banach spaces, $k \in \Delta \subset \mathbb{C}$ be a parameter, Δ be an open bounded set on a complex plane $\mathbb{C}$, $A(k) : X \to Y$ be a map, possibly nonlinear, $f := f(k) \in Y$ be a function.

Consider an equation

$$A(k)u(k) = f(k). \tag{1.1}$$

We are interested in conditions, sufficient for the continuity of $u(k)$ with respect to $k \in \Delta$. There is a large literature (see eg. [1], [2]) on this subject. The novel points in our paper include necessary and sufficient conditions for continuity of the solution to equation (1.1) and sufficient conditions for its continuity when the operator $A(k)$ is nonlinear.

Consider separately the cases when $A(k)$ is a linear map and when $A(k)$ is a nonlinear map.

Assumption A_1. *$A(k) := X \to Y$ is a linear bounded operator, and*

a) equation (1.1) *is uniquely solvable for any $k \in \Delta_0 := \{k : |k - k_0| \le r\}$, $k_0 \in \Delta$, $\Delta_0 \subset \Delta$,*

b) $f(k)$ is continuous with respect to $k \in \Delta_0$, $\sup_{k \in \Delta_0} \|f(k)\| \le c_0$;

c) $\lim_{h\to 0} \sup_{\substack{k\in\Delta_0\\ v\in M}} \|[A(k+h)-A(k)]v\| = 0$, *where* $M\subset X$ *is an arbitrary bounded set,*

d) $\sup_{\substack{k\in\Delta_0\\ f\in N}} \|A^{-1}(k)f\| \le c_1$, *where* $N\subset Y$ *is an arbitrary bounded set, and* c_1 *may depend on* N.

Theorem 1.1. *If Assumptions* A_1 *hold, then*

$$\lim_{h\to 0} \|u(k+h)-u(k)\| = 0. \tag{1.2}$$

Proof. One has

$$\begin{aligned} u(k+h)-u(k) &= A^{-1}(k+h)f(k+h) - A^{-1}(k)f(k) \\ &= A^{-1}(k+h)f(k+h) - A^{-1}(k)f(k+h) + A^{-1}(k)f(k+h) - A^{-1}(k)f(k). \end{aligned} \tag{1.3}$$

$$\|A^{-1}(k)[f(k+h)-f(k)]\| \le c_1\|f(k+h)-f(k)\| \to 0 \quad \text{as} \quad h\to 0. \tag{1.4}$$

$$\begin{aligned} \|A^{-1}(k+h)-A^{-1}(k)\| &= \|A^{-1}(k+h)[A(h+k)-A(k)]A^{-1}(k)\| \\ &\le c_1^2\|A(k+h)-A(k)\| \to 0 \quad \text{as} \quad h\to 0. \end{aligned} \tag{1.5}$$

From (1.3)–(1.5) and Assumptions A_1 the conclusion of Theorem 1 follows. □

Remark 1.2. *Assumptions* A_1 *are not only sufficient for the continuity of the solution to* (1.1)*, but also necessary if one requires the continuity of* $u(k)$ *uniform with respect to* f *running through arbitrary bounded sets. Indeed, the necessity of the assumption a) is clear; that of the assumption b) follows from the case* $A(k)=I$*, where* I *is the identity operator; that of the assumption c) follows from the case* $A(k)=I$*,* $A(k+h)=2I$*,* $\forall h\neq 0$*,* $f(k)=g$ $\forall k\in\Delta_0$*. Indeed, in this case assumption c) fails and one has* $u(k)=g$*,* $u(k+h)=\frac{g}{2}$*, so* $||u(k+h)-u(k)||=\frac{||g||}{2}$ *does not tend to zero as* $h\to 0$*.*

To prove the necessity of the assumption d), assume that $\sup_{k\in\Delta_0}\|A^{-1}(k)\|=\infty$*. Then, by the Banach-Steinhaus theorem, there is an element* f *such that* $\sup_{k\in\Delta_0}\|A^{-1}(k)f\|=\infty$*, so that* $\lim_{j\to\infty}\|A^{-1}(k_j)f\|=\infty$*,* $k_j\to k\in\Delta_0$*. Then* $\|u(k_j)\|=\|A^{-1}(k_j)f\|\to\infty$*, so* $u(k_j)$ *does not converge to* $u:=u(k)=A^{-1}(k)f$*, although* $k_j\to k$*.*

Assumption A_2**.** $A(k): X\to Y$ *is a nonlinear map and a), b), c) and d) of Assumption* A_1 *hold, and the following assumption holds:*

e) $A^{-1}(k)$ *is a homeomorphism of* X *onto* Y *for each* $k\in\Delta_0$*.*

Remark 1.3. *Assumption e) is included in d) in the case of a linear operator* $A(k)$ *because if* $\|A(k)\|\le c_2$ *and assumption d) holds, then* $\|A^{-1}(k)\|\le c_1$ *and* $A(k)$*,* $k\in\Delta_0$*, is an isomorphism of* X *onto* Y*.*

Theorem 1.4. *If* A_2 *hold, then* (1.2) *holds for the solution* $u(k)$ *to* (1.1)*.*

Remark 1.5. *Let us introduce the following assumption:*

Assumption A_d. *:*

Assumptions A_2) hold and

f) $\dot f(k) := \frac{df(k)}{dk}$ is continuous in Δ_0,

g) $\dot A(u,k) := \frac{\partial A(u,k)}{\partial k}$ is continuous in Δ_0,

j) $\sup_{k\in\Delta_0}\|[A'(u,k)]^{-1}\| \le c_3$, where $A'(u,k)$ is the Fréchet derivative of $A(u,k)$.

Claim: *If Assumption A_d holds, then*

$$\lim_{h\to 0}\|\dot u(k+h)-\dot u(k)\| = 0. \tag{1.6}$$

Remark 1.6. *If Assumptions A_1 hold except one: $A(k)$ is not necessarily a bounded linear operator, $A(k)$ may be unbounded, closed, densely defined operator-function, then the conclusion of Theorem 1.1 still holds and its proof is the same. For example, let $A(k) = L+B(k)$, where $B(k)$ is a bounded linear operator continuous with respect to $k\in\Delta_0$, and L is a closed, linear, densely defined operator from X into Y. Then*

$$\|A(k+h)-A(k)\| = \|B(k+h)-B(k)\| \to 0 \quad \textit{as} \quad h\to 0,$$

although $A(k)$ and $A(k+h)$ are unbounded.

In Section 2 proofs of Theorem 1.4 and of Remark 1.5 are given.

2 Proofs

Proof of Theorem 1.4. One has:

$$A(k+h)u(k+h) - A(k)u(k) = f(k+h) - f(k) = o(1) \quad \text{as} \quad h\to 0.$$

Thus

$$A(k)u(k+h) - A(k)u(k) = o(1) - [A(k+h)u(k+h) - A(k)u(k+h)].$$

Since $\sup_{\{u(k+h):\|u(k+h)\|\le c\}}\|A(k+h)u(k+h) - A(k)u(k+h)\| \underset{h\to 0}{\to} 0$, one gets

$$A(k)u(k+h) \to A(k)u(k) \quad \text{as} \quad h\to 0. \tag{2.1}$$

By the Assumption A_2, item e), the operator $A(k)$ is a homeomorphism. Thus (2.1) implies (1.2).

Theorem 1.4 is proved. □

Proof of Remark 1.5. First, assume that $A(k)$ is linear. Then

$$\frac{d}{dk}A^{-1}(k) = -A^{-1}(k)\dot A(k)A^{-1}(k), \quad \dot A := \frac{dA}{dk}. \tag{2.2}$$

Indeed, differentiate the identity $A^{-1}(k)A(k) = I$ and get $\frac{dA^{-1}(k)}{dk}A(k)+A^{-1}(k)\dot A(k) = 0$. This implies (2.2). This argument proves also the existence of the deriviative $\frac{dA^{-1}(k)}{dk}$. Formula $u(k) = A^{-1}(k)f(k)$ and the continuity of $\dot f$ and of $\frac{dA^{-1}(k)}{dk}$ yield the existence and continuity of $\dot u(k)$. Remark 1.5 is proved for linear operators $A(k)$.
□

Assume now that $A(k)$ is nonlinear, $A(k)u := A(k,u)$. Then one can differentiate (1.1) with respect to k and get

$$\dot{A}(k,u) + A'(k,u)\dot{u} = \dot{f}, \tag{2.3}$$

where A' is the Fréchet derivative of $A(k,u)$ with respect to u. Formally one assumes that $\dot{u}$ exists, when one writes (2.3), but in fact (2.3) proves the existence of $\dot{u}$, because $\dot{f}$ and $\dot{A}(k,u) := \frac{\partial A(k,u)}{\partial k}$ exist by the Assumption A_d and $[A'(k,u)]^{-1}$ exists and is an isomorphism by the Assumption A_d, item j). Thus, (2.3) implies

$$\dot{u} = [A'(k,u)]^{-1}\dot{f} - [A'(k,u)]^{-1}\dot{A}(k,u). \tag{2.4}$$

Formula (2.4) and Assumption A_d imply (1.6).

Remark 1.5 is proved. □

3 Applications

3.1 Fredholm equations depending on a parameter

Let

$$Au := u - \int_D b(x,y,k)u(y)dy := [I - B(k)]u = f(k), \tag{3.1}$$

where $D \subset R^n$ is a bounded domain, $b(x,y,k)$ is a function on $D \times D \times \Delta_0$, $\Delta_0 := \{k : |k - k_0| < r\}$, $k_0 > 0$, $r > 0$ is a sufficiently small number. Assume that $A(k_0)$ is an isomorphism of $H := L^2(D)$ onto H, for example, $\int_D \int_D |b(x,y,k_0)|^2 dxdy < \infty$ and $N(I - B(k_0)) = \{0\}$, where $N(A)$ is the null-space of A. Then, $A(k_0)$ is an isomprohism of H onto H by the Fredholm alternative, and Assumption A_1 hold if $f(k)$ is continuous with respect to $k \in \Delta_0$ and

$$\lim_{h\to 0} \int_D \int_D |b(x,y,k+h) - b(x,y,k)|^2 dxdy = 0 \qquad k \in \Delta_0. \tag{3.2}$$

Condition (3.2) implies that if $A(k_0)$ is an isomorphism of H onto H, then so is $A(k)$ for all $k \in \Delta_0$ if $|k - k_0|$ is sufficiently small.

Remark 1.5 implies to (3.1) if $\dot{f}$ is continuous with respect to $k \in \Delta_0$, and $\dot{b} := \frac{\partial b}{\partial k}$ is continuous with respect to $k \in \Delta_0$ as an element of $L^2(D \times D)$. Indeed, under these assumptions $\dot{u} = [I - B(k)]^{-1}(\dot{f} - \dot{B}(k)u)$ and the right-hand side of this formula is continuous in Δ_0.

3.2 Semilinear elliptic problems

Let

$$A_1(k)u := Lu + g(u,k) = f_1(k), \tag{3.3}$$

where $L \geq m > 0$ is an elliptic, second order, self-adjoint, positive-definite operator with real-valued coefficients in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary, and $g(u,k)$ is a smooth real-valued function on $\mathbb{R} \times \Delta_0$. Then problem (3.3) is equivalent to (1.1) with

$$A(k)u := u + L^{-1}g(u,k) = f(k) := L^{-1}f_1(k). \tag{3.4}$$

The operator $L^{-1}g(u,k)$ is compact in $C(D)$. Therefore equation (3.4) is solvable in $C(D)$ by the Schauder principle if the map $A(k)$ maps a ball $B(0,R) := B_R$ into itself for some $R > 0$. This happens if $g' := g'_u > 0$ for $u > 0$ and $\inf_{R>0} \frac{g(R)}{R} \leq m^{-1}$, where $g(R) := \max_{k\in\Delta_0} |g(R,k)|$ and $||L^{-1}||_{C(D)\to C(D)} \leq m$. Equation (3.4) has at most one solution if $g' > 0$. Assumptions A_2 can be verified, for example, if $g(u,k)$ is a smooth function on $\mathbb{R} \times \Delta_0$ and $g' \geq 0$. In this case $\|A^{-1}(k,f)\| \leq c\|f\|$, and $A^{-1}(k)$ is a continuous operator defined on all of $H := H_0^2(D)$, where H is a real Hilbert space, for any fixed $k \in \Delta_0$. If, for example, $L = -\Delta + k^2$ is the Dirichlet operator in $D \subset \mathbb{R}^3$, then L^{-1} is a positive-definite integral operator with the kernel $0 \leq G(x,y) < \frac{\exp(-k|x-y|)}{4\pi|x-y|}$, and $m \leq \frac{\int_0^{ka} e^{-s} s ds}{k^2}$, where a is the radius of D, that is, $2a := \sup_{x,y\in D} |x-y|$.

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