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An Initial-Boundary Value Problem for a Generalized Boussinesq Water System in a Ball[1]

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Abstract

In this paper, we consider an initial-boundary value problem for the following generalized Boussinesq water equation defined in a unit ball

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \eta\Delta(u^2).$$

The existence of mild solutions is established in the space $C^0([0,\infty), H_0^\kappa(B))$ where $\kappa < 5/2$, and the solutions are constructed in the form of series of the small parameter present in the initial conditions. For $-1/2 < \kappa < 5/2$, the uniqueness is proved. In addition, the long-time asymptotics is obtained in explicit form.

keywords: Generalized Boussinesq water equation; initial-boundary value problem; long-time asymptotics.
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1 Introduction

Recently, the studies of nonlinear waves in water with dispersion have attracted attention of many mathematicians and physicists. One of the equations describing such processes is the Boussinesq one which was derived in [2], and the classific Boussinesq equation can be written as

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.1}$$

where $u(x,t)$ is the elevation of the free surface of fluid, the subscripts denote partial derivatives, and the constant coefficients α and β depend on the depth of fluid and the characteristic speed of long waves. The spatially one dimension Boussinesq equation and its generalization

$$u_{tt} = -u_{xxxx} + u_{xx} + (f(u))_{xx} \tag{1.2}$$

were studied from various points of view in the papers [1, 3, 5, 7, 8, 14]. In [7], an abstract Cauchy problem for the generalization of (1.2) was considered in a Banach space, and sufficient conditions for the blow up in finite time were established. By means of the variational approach, Liu [10] established the sufficient conditions for the blow up in terms of the energy of the ground state. In [15, 16] Varlamov considered a damped version of (1.1) as follows

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}. \tag{1.3}$$

In [16], a similar version of equation (1.3) was studied in a ball by means of the eigenfunction expansion method and the perturbation theory.
In the present paper, we consider an initial-boundary value problem for the following generalized Boussinesq water equation defined in a ball

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \eta\Delta(u^2), \tag{1.4}$$

where a, b, α, β, η and ε are positive constants, and Δ is the Laplace operator. We construct the mild solution and obtain the long-time asymptotics in explicit form.

2 Preliminaries

Denote by B a ball of a unit radius and put the origin of the coordinate system in its centre, so that in spherical coordinates $B = \{(r,\theta,\varphi) : 0 \le r < 1, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$. Denote by $L_2(B)$ the space of real functions product $\langle f, g\rangle = \int_0^1 \int_0^{2\pi} \int_0^{\pi} f(r,\theta,\varphi) g(r,\theta,\varphi) r^2 \sin\theta d\theta d\varphi dr$, and the corresponding norm $\|\cdot\|$.
A function $f(r,\theta,\varphi) \in L_2(B)$ can be represented as

$$f(r,\theta,\varphi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \hat{f}_{mn}\chi_{mn}(r,\theta,\varphi), \quad \hat{f}_{mn} = \langle f, \chi_{mn}\rangle, \tag{2.5}$$

where $\chi_{mn}(r,\theta,\varphi)$ is the eigenfunction of the Laplace operator in ball B.
From [16], it follows

$$\chi_{mn} = j_m(\lambda_{mn}r)Y_m(\theta,\varphi), \quad j_m(\lambda_{mn}r) = \sqrt{\frac{\pi}{2r}}J_{m+\frac{1}{2}}(\lambda_{mn}r), \tag{2.6}$$

$$Y_m(\theta,\varphi) = \sum_{l=0}^{m}[C_{lm}^{(1)}\cos(l\varphi) + C_{lm}^{(2)}\sin(l\varphi)]P_m^l(\cos\theta),$$

where $Y_m(\theta,\varphi)$ is represented by the linear combination of tesseral harmonics, $j_m(\lambda_{mn}r)$ is the spherical Bessel function (see[6]), $\lambda_{m1}, \lambda_{m2}, \cdots$ are the possitive zeros of the Bessel function $J_{m+\frac{1}{2}}(z)$ numbered in increasing order $m \in \mathbf{N}\bigcup\{\mathbf{0}\}$. We note that $\lambda_{0n} = n\pi$.
Introducing the real space $L_{2,r}(0,1)$ with the norm $\|f\|_r^2 = \int_0^1 rf^2(r)dr$, we can obtain that for $\nu \geq 0$,

$$\frac{C_1}{\lambda} \leq \|J_\nu(\lambda r)\|_r^2 \leq \frac{C_2}{\lambda}, \tag{2.7}$$

where C_1,C_2 are some positive constants. In the sequal we denote by C generic positive constants independent of m, n, ε, r, θ, φ, t.
Introducing the real space $L_{2,r^2}(0,1)$ with the scalar product

$$(f,g) = \int_0^1 r^2 f(r)g(r)dr$$

and the corresponding norm, from (2.7), it follows that for sufficiently large $\lambda_{mn} > 0$,

$$\frac{C_3}{\lambda_{mn}} \leq \|j_m\|_{(n)}^2 = \int_0^1 r^2 j_m^2(\lambda_{mn}r)dr \leq \frac{C_4}{\lambda_{mn}}. \tag{2.8}$$

For bounded m and large possitive zeros of $J_m(z)$, we have the asymptotic expansion uniform in m (McMahon's expansion, see [6])

$$\lambda_{mn} = \mu_{mn} + O(\frac{1}{\mu_{mn}}), \quad \mu_{mn} = (m + 2n - \frac{1}{2})\frac{\pi}{2}, \quad n \to +\infty. \tag{2.9}$$

Introducing the scalar product in the real space $L_2(S)$ by the formula $(f,g)_S = \int_S fgdS$ and denoting by $\|\cdot\|_S$ the corresponding norm, we can write that (see [12])

$$(Y_m, Y_k) = \int_S Y_m(Q)P_k[\cos\gamma(P,Q)]dS_Q = 0, \quad m \neq k,$$

$$\|Y_m\|_S^2 = \frac{4\pi}{2m+1}, \tag{2.10}$$

$$Y_m(P) = \frac{2m+1}{4\pi}\int_S Y_m(Q)P_k[\cos\gamma(P,Q)]dS_Q,$$

where P and Q are two variable points on the unit sphere S and $\gamma(P,Q)$ is the angle (between 0 and π) formed by two vector radius OP and OQ (O is the centre of the unit sphere). The spherical harmonic expressed as a symmetric function of two points P and Q is called Laplace's coefficient [12], the name coming from the expansion of a function $f(P)$ into the Laplace series

$$f(P) \sim \sum_{m=0}^{\infty} Y_m(P), \quad Y_m(P) = \frac{2m+1}{4\pi} \int_S f(Q) P_m[\cos\gamma(P,Q)] dS_Q, \quad m = 0, 1, 2, \cdots.$$

If we direct the z-axis of the coordinate system through P, the spherical harmonics will turn out to be zonal and the constants will be determined. Then from the last formula in (2.10), it follows

$$\frac{2m+1}{4\pi} \int_0^{2\pi} \int_0^{\pi} [P_m(\cos\gamma)]^2 \sin\gamma d\gamma d\chi = Y_m(P) = 1,$$

where (γ, χ) is the spherical coordinates in the system with the north pole at the point P. Moreover, if $P = (\theta', \varphi')$ and $Q = (\theta, \varphi)$, then $\theta' = 0$ and $\cos\gamma(P,Q) = \cos\theta$.
From [16], a function $f(r,Q) \in L_1(B), Q \in S$ can be written as

$$f(r,Q) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \hat{f}_{kn} j_k(\lambda_{kn} r) Y_k(Q).$$

Taking a fixed point $P \in S$, multiplying both sides of this formula by $P_m[\cos\gamma(P,Q)]$, and integrating over S, we have by means of (2.10) that

$$\int_S f(r,Q) P_m[\cos\gamma(P,Q)] dS_Q = \frac{4\pi}{2m+1} Y_m(P) \sum_{n=1}^{\infty} \hat{f}_{mn} j_m(\lambda_{mn} r).$$

Since

$$f(r,Q) = \sum_{n=1}^{\infty} \frac{(f, j_m)_{(n)}(Q)}{\|j_m\|^2_{(n)}} j_m(\lambda_{mn} r),$$

we can get that

$$\hat{f}_{mn} Y_m(P) = \frac{1}{\|j_m\|^2_{(n)} \|Y_m\|^2_S} \int_0^1 r^2 j_m(\lambda_{mn} r) dr \int_S f(r,Q) P_m[\cos\gamma(P,Q)] dS_Q. \tag{2.11}$$

If we combine the north pole of our coordinate system with the point P, then $Y_m(P) = 1$, $P_m[\cos\gamma(P,Q)] = P_m(\cos\theta)$, and

$$\hat{f}_{mn} = \frac{((f, j_m)_{(n)}(Q), Y_m(Q))_S}{\|j_m\|^2_{(n)} \|Y_m\|^2_S}.$$

Now we give some facts concerning Legendre Polynomials $P_m(x)$, for $-1 \le x \le 1$, (see [16])

$$|P_m(x)| < 1, x \in (-1,1), \quad P_m(-1) = (-1)^m, P_m(1) = 1,$$

$$\int_{-1}^{1} P_m^2(x)dx = \frac{2}{2m+1}. \tag{2.12}$$

First Theorem Stieltjes: For $\theta \in (0,\pi), m \in \mathbf{N}$,

$$|P_m(\cos\theta)| \le \frac{4\sqrt{2}}{\sqrt{\pi}\sqrt{m}\sqrt{\sin\theta}}. \tag{2.13}$$

Second Theorem Stieltjes: For $x \in [-1,1], m \in \mathbf{N}\bigcup\{\mathbf{0}\}$,

$$|P_{m+2}(x) - P_m(x)| \le \frac{4}{\sqrt{\pi}\sqrt{m+1}}. \tag{2.14}$$

From the relation

$$P'_{m+1}(x) - P'_{m-1}(x) = (2m+1)P_m(x), m \le 1, \tag{2.15}$$

it follows

$$\int_{-1}^{x} P_m(\xi)d\xi = \frac{P_{m+1}(x) - P_{m-1}(x)}{2m+1}. \tag{2.16}$$

Combining (2.14) and (2.16), we get

$$|\int_{-1}^{x} P_m(\xi)d\xi| \le \frac{4}{\sqrt{\pi}\sqrt{m+1}(2m+1)}. \tag{2.17}$$

Making use of the relation $\int_{-1}^{1} P_k(x)P_m(x)dx = 0, k \ne m$, we can obtain that

$$\begin{aligned} &\int_{-1}^{1} P_m(\xi)d\xi = 0, \qquad m \ge 1, \\ &\int_{-1}^{1} \xi P_m(\xi)d\xi = 0, \qquad m \ge 2, \\ &\int_{-1}^{1} \xi^2 P_m(\xi)d\xi = 0, \qquad m \ge 3. \end{aligned} \tag{2.18}$$

We introduce the Sobolev space $H^\kappa(B)$ endowed with the equivalent norm (see [9, 16])

$$\|f\|_\kappa^2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn}^{2\kappa}|\hat{f}_{mn}|^2\|\chi_{mn}\|^2,$$

and set $H_0^\kappa(B) = H^\kappa(B)\bigcap\{u : u|_S = 0\}$.

3 Main results

Consider the first initial-boundary value problem for the following damped Boussinesq equation defined in the ball B

$$\begin{cases} u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \eta\Delta(u^2), \quad (r,\theta,\varphi) \in B, t > 0, \\ u(r,\theta,\varphi,0) = \varepsilon^2\phi(r,\theta,\varphi), u_t(r,\theta,\varphi,0) = \varepsilon^2\psi(r,\theta,\varphi), \quad (r,\theta,\varphi) \in B, \\ u|_S = \Delta u|_S = \Delta^2 u|_S = 0, \quad t > 0, \end{cases} \tag{3.19}$$

where $\phi(r,\theta,\varphi)$ and $\psi(r,\theta,\varphi)$ are real-valued 2π periodic functions with respect to φ; and a, b, α, β, η and ε are positive constants.

Definition 3.1: The function $u(t)$ is called a mild solution of problem (3.19) if it satisfies the following integral equation

$$u(t) = \varepsilon^2 \exp(-\frac{bA}{1+aA}t)\{[C(t) + \frac{aA}{1+bA}S(t)]\phi + \psi\}$$

$$+\frac{1}{1+aA}\int_0^t \exp[-\frac{bA}{1+aA}(t-\tau)]S(t-\tau)A(u^2(\tau))d\tau,$$

where

$$A = -\Delta, \quad C(t) = \cos(\sigma(A)t), \quad S(t) = \frac{\sin(\sigma(A)t)}{\sigma(A)},$$

$$\sigma(A) = \frac{\sqrt{a\alpha A^4 + (\alpha + a\beta)A^3 + (a+\beta-b^2)A^2 + A}}{1+aA},$$

in the Banach space $C^0([0,\infty), H_0^\kappa(B))$.

Denote by $V_0^1(f(r,Q))$ the total variation of the function $f(r,Q)$, $Q = (\theta,\varphi)$, $r \in [0,1]$. Let $D_\theta = -(1/\sin\theta)\partial_\theta$. Now we state some assumptions on a sufficient smooth function $f(r,Q)$, $r \in [0,1]$,$Q \in S$.

Assumption A.

$$V_0^1(rf(r,Q)) = V_{0,0}(Q) \in L_1(S), \quad \lim_{r\to 0^+} rf(r,Q) = f_{0,0}(Q) \in L_1(S),$$

$$V_0^1(rD_\theta f(r,Q)) = V_{0,1}(Q) \in L_1(S), \quad \lim_{r\to 0^+} rD_\theta f(r,Q) = f_{0,1}(Q) \in L_1(S).$$

Assumption B.

$$f(0,Q) = \partial_r f(0,Q) = f(1,Q) = \partial_r f(1,Q) = 0,$$

$$D_\theta^3 f(0,Q) = \partial_r D_\theta^3 f(0,Q) = D_\theta^3 f(1,Q) = \partial_r D_\theta^3 f(1,Q) = 0,$$

$$V_0^1(r\partial_r^2 f(r,Q)) = V_{2,0}(Q) \in L_1(S), \quad \lim_{r\to 0^+} r\partial_r^2 f(r,Q) = f_{2,0}(Q) \in L_1(S),$$

$$V_0^1(r\partial_r^2 D_\theta^3 f(r,Q)) = V_{2,3}(Q) \in L_1(S), \quad \lim_{r\to 0^+} r\partial_r^2 D_\theta^3 f(r,Q) = f_{2,3}(Q) \in L_1(S).$$

Theorem 3.2: If the positive constants a, b, α, β and the initial functions satisfy the followinfg assumptions:

(H_1) $\quad a+\beta-b^2 \geq 0,\ 0 < a < \dfrac{\lambda_{11}^2 - 2\lambda_{01}^2}{\lambda_{11}^2\lambda_{01}^2}$;

(H_2) $\quad \psi(r,\theta,\varphi)$ and $\phi(r,\theta,\varphi)$ satisfy the Assumptions A and B respectively.

Then there exists a $\varepsilon_0 > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$ problem (3.19) has mild solutions in the space $C^0([0,\infty), H_0^\kappa(B))$ for $\kappa < 5/2$, which can be represented as

$$u(r,\theta,\varphi,t) = \sum_{m\geq 0, n\geq 1} \hat{u}_{mn}(t) j_{mn}(\lambda_{mn} r) Y_m(\theta,\varphi), \tag{3.20}$$

where the coefficient $\hat{u}_{mn}(t)$ is defined in the proof (see (5.39) and (5.40). If $-1/2 < \kappa < 5/2$, this solution is unique.

Theorem 3.3: Under the assumptions of Theorem 3.2, the asymptotic expansion can be represented as follows as $t \to +\infty$

$$\begin{aligned} u(r,\theta,\varphi,t) &= \exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t\right)\{[A_{01}\cos(\sigma_{01}t) + B_{01}\sin(\sigma_{01}t)] j_0(\lambda_{01} r) \\ &+ O\left(\exp\left(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t\right)\right)\}, \end{aligned} \tag{3.21}$$

where

$$\sigma_{01} = \frac{\lambda_{01}\sqrt{a\alpha\lambda_{01}^6 + (\alpha + a\beta)\lambda_{01}^4 + (a+\beta-b^2)\lambda_{01}^2 + 1}}{1+a\lambda_{01}^2}$$

the coefficients A_{01} and B_{01} are defined in the proof (see (6.49) and (6.51)) and the estimate of the residual term in (3.21) is uniform with respect to $(r,\theta,\varphi) \in B$, $\varepsilon \in [0,\varepsilon_0]$.

4 Technical Lemmas

Let the function $f(r,Q)$ be defined in the unit ball B, and Q be a point on the unit sphere S. Consider the integral

$$\Im_m(\lambda, Q) = \int_0^1 r^2 j_m^2(\lambda r) f(r,Q) dr,$$

for integer $m \geq 0$, real $\lambda > 0$, and $Q \in S$.

Lemma 4.1: Suppose that for each fixed $Q \in S$ the function $rf(r,Q)$ has a bounded total variation in $r \in [0,1]$. Moreover, assume that this variation is absolutely integrable over S. Then for $m \geq 0$, $\lambda > 0$, and $Q \in S$,

$$|\Im_m(\lambda, Q)| \leq \frac{C(Q)}{\lambda^{\frac{3}{2}}},$$

where $C(Q) \in L_1(S)$ and is independent of m and λ.
Proof. See [16].
Lemma 4.2: Assume that $f(r,Q)$ has partial derivatives in $r \in (0,1)$, $Q \in S$, through second order, $f(0,Q) = \partial_r f(0,Q) = 0$ (in case $m = 0$, only $\partial_r f(0,Q) = 0$) and $f(1,Q) = \partial_r f(1,Q) = 0$. Moreover, suppose that for any fixed $Q \in S$, the function $r\partial_r^2 f(r,Q)$ has a bounded total variation in $r \in [0,1]$, which is absolutely integrable in $Q \in S$. Then for $m \geq 0$, $\lambda > 0$,

$$|\Im_m(\lambda, Q)| \leq \frac{C(Q)(m+1)^2}{\lambda^{\frac{7}{2}}},$$

where $C(Q) \in L_1(S)$ and is independent of m and λ.
Proof. Integrating two times by parts in $\Im_m(\lambda, Q)$, expanding around $r_0 = 0$ by Taylor's formula, and applying Lemma 4.1, we deduce the necessary estimate.
Lemma 4.3: (a) If $f(r,Q)$, $r \in (0,1)$, $Q \in S$, satisfies Assumption A, then there exists such a constant c independent of m, n, that for all integers $m \geq 0$, $n \geq 1$

$$|\hat{f}_{mn}| \leq \frac{c}{\lambda_{mn}^{\frac{1}{2}}\sqrt{m+1}}. \tag{4.22}$$

(b) If $f(r,Q)$, $r \in (0,1)$, $Q \in S$, satisfies Assumption B, then there exists such a constant c independent of m, n, that for all integers $m \geq 0$, $n \geq 1$

$$|\hat{f}_{mn}| \leq \frac{c}{\lambda_{mn}^{\frac{5}{2}}\sqrt{m+1}}. \tag{4.23}$$

Proof. In the chosen coordinate system with the pole at the point P,

$$\hat{f}_{mn} = \frac{1}{\|j_m\|_{(n)}^2 \|Y_m\|_S^2} \int_0^{2\pi} \int_0^{\pi} P_m(\cos\theta)\sin\theta d\theta d\varphi \int_0^1 r^2 j_m(\lambda_{mn} r) f(r,\theta,\varphi) dr.$$

Firstly, we consider the case $m = 0, 1, 2$.
In case (a), by lemma 4.1 and (2.8), it follows

$$\hat{f}_{mn} \leq c\lambda_{mn} \int_0^{2\pi} d\varphi \int_0^{\pi} |P_m(\cos\theta)| \sin\theta d\theta | \int_0^1 r^2 j_m(\lambda_{mn} r) f(r,\theta,\varphi) dr|$$

$$\leq \frac{c}{\lambda_{mn}^{\frac{1}{2}}} \leq \frac{c}{\lambda_{mn}^{\frac{1}{2}}\sqrt{m+1}}.$$

In case (b), by lemma 4.2 and (2.8), we get (4.23).
Now we consider $m \geq 3$. By using the primitive Legendre polynomials

$$\wp_m^{(1)}(z) = \int_{-1}^{z} P_m(\xi) d\xi,$$

$$\wp_m^{(2)}(z) = \int_{-1}^{z} d\xi \int_{-1}^{\xi} P_m(\tau)d\tau = \int_{-1}^{z} (z-\xi)P_m(\xi)d\xi, \tag{4.24}$$

$$\wp_m^{(3)}(z) = \int_{-1}^{z} d\xi \int_{-1}^{\xi} d\tau \int_{-1}^{\tau} P_m(\varsigma)d\varsigma = \int_{-1}^{z} \frac{1}{2}(z-\xi)^2 P_m(\xi)d\xi.$$

From (2.18), it follows that

$$\wp_m^{(1)}(1) = \wp_m^{(2)}(1) = \wp_m^{(3)}(1) = 0, \quad m \geq 3.$$

Applying (2.14) and (2.16), it follows

$$|\wp_m^{(3)}(z)| \leq \frac{c}{(2m+1)m^{\frac{5}{2}}}, \qquad m \geq 3. \tag{4.25}$$

Let

$$F(r, \cos\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r,\theta,\varphi)d\varphi = F(r,z),$$

where $z = \cos\theta$. Consider the integral

$$\Gamma_m(r) = \int_{0}^{\pi} F(r,\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \int_{-1}^{1} F(r,z)P_m(z)dz.$$

Integrating by parts we get from (4.24) that

$$\Gamma_m(r) = \int_{0}^{\pi} \wp_m^{(1)}(\cos\theta) D_\theta F(r,\cos\theta)\sin\theta d\theta, \quad m \geq 3, \tag{4.26}$$

$$\Gamma_m(r) = -\int_{0}^{\pi} \wp_m^{(3)}(\cos\theta) D_\theta^3 F(r,\cos\theta)\sin\theta d\theta \quad m \geq 3. \tag{4.27}$$

In case (a), combining (4.26), (4.27) and Lemma 4.1, we get that

$$|\hat{f}_{mn}| \leq \frac{c}{\lambda_{mn}^{\frac{1}{2}}\sqrt{m+1}}, \quad m \geq 3. \tag{4.28}$$

In case (b), combining (4.25), (4.27) and Lemma 4.2, we obtain that

$$|\hat{f}_{mn}| \leq \frac{c}{\lambda_{mn}^{\frac{5}{2}}\sqrt{m+1}}, \quad m \geq 3. \tag{4.29}$$

This completes the proof.
Set

$$H_{mnpqks} = \int_{0}^{1} r^2 j_m(\lambda_{mn}r) j_p(\lambda_{pq}r) j_k(\lambda_{ks}r)dr.$$

From [16, p.687], it follows that for m, p, $k \geq 0$, n, q, $s \geq 1$,

$$|H_{mnpqks}| \leq \frac{c}{\lambda_{mn}^{\frac{1}{2}}} \begin{cases} \frac{1}{\lambda_{pq}^{\frac{3}{2}}\lambda_{ks}^{\frac{1}{2}}}, & \lambda_{pq} > \lambda_{ks}, \\ \frac{1}{\lambda_{pq}^{\frac{1}{2}}\lambda_{ks}^{\frac{3}{2}}}, & \lambda_{pq} < \lambda_{ks}, \\ \frac{1}{\lambda_{pq}}, & \lambda_{pq} = \lambda_{ks}, \end{cases} \tag{4.30}$$

$$|H_{mnpqks}| \leq \frac{c}{\lambda_{mn}^{\frac{3}{2}}} \begin{cases} \frac{1}{\lambda_{pq}^{\frac{1}{2}}\lambda_{ks}^{\frac{1}{2}}}, & \lambda_{mn} \neq \lambda_{ks}, \\ \frac{\lambda_{ks}^{\frac{1}{2}}}{\lambda_{pq}^{\frac{1}{2}}}, & \lambda_{mn} = \lambda_{ks}, \end{cases} \tag{4.31}$$

where c is independent of m, n, p, q, k and s.
Consider the integral

$$U_{pkm} = (Y_p Y_k, Y_m)_S = \int_S Y_p(Q)Y_k(Q)Y_m(Q)dS_Q.$$

From [16, p.687-688], it follows that

$$|U_{pkm}| \leq \frac{c}{\sqrt{(m+1)(p+1)(k+1)}}, \quad p, k, m \geq 0, \tag{4.32}$$

$$|U_{pkm}| \leq \frac{16\sqrt{\pi}}{(2m+1)\sqrt{m+1}}[\sqrt{p+1}\ln(p+1) + \sqrt{k+1}\ln(k+1) + \frac{\ln 2}{2}(\sqrt{p+1} + \sqrt{k+1} + 2)], \quad p, k, m \geq 1. \tag{4.33}$$

If $m = 0$, $n \geq 2$ or $m, n \geq 1$, we have $\lambda_{mn}^2 \geq \lambda_{11}^2 > 2\lambda_{01}^2$. Since $b > 0$ and $0 < a < \frac{\lambda_{11}^2 - 2\lambda_{01}^2}{\lambda_{11}^2\lambda_{01}^2}$, we can get that for $m = 0$, $n \geq 2$ or $m, n \geq 1$

$$\frac{2b\lambda_{01}^2}{1 + a\lambda_{01}^2} < \frac{b\lambda_{11}^2}{1 + a\lambda_{11}^2} \leq \frac{b\lambda_{mn}^2}{1 + a\lambda_{mn}^2} < \frac{b}{a}. \tag{4.34}$$

5 Proof of Theorem 3.2

5.1 Existence and construction of solutions

In order to satisfy the boundary conditions, we seek mild solutions of (3.19) in the form

$$u(r, \theta, \varphi, t) = \sum_{m \geq 0, n \geq 1} \hat{u}_{mn}(t)\chi_{mn}(r, \theta, \varphi), \tag{5.35}$$

where $\hat{u}_{mn}(t) = \frac{\langle u, \chi_{mn}\rangle(t)}{\|\chi_{mn}\|^2}$, $\chi_{mn}(r, \theta, \varphi) = j_m(\lambda_{mn}r)Y_m(\theta, \varphi)$.
We expand the initial functions in the same type of (5.35)

$$\phi(r, \theta, \varphi) = \sum_{m \geq 0, n \geq 1} \hat{\phi}_{mn}\chi_{mn}(r, \theta, \varphi), \quad \hat{\phi}_{mn} = \frac{\langle \phi, \chi_{mn}\rangle(t)}{\|\chi_{mn}\|^2},$$

$$\psi(r, \theta, \varphi) = \sum_{m \geq 0, n \geq 1} \hat{\psi}_{mn}\chi_{mn}(r, \theta, \varphi), \quad \hat{\psi}_{mn} = \frac{\langle \psi, \chi_{mn}\rangle(t)}{\|\chi_{mn}\|^2}, \tag{5.36}$$

$$u^2(r,\theta,\varphi,t) = \sum_{m\ge 0, n\ge 1} \widehat{(u^2)}_{mn}(t)\chi_{mn}(r,\theta,\varphi),$$

$$\widehat{(u^2)}_{mn}(t) = \sum_{p,k\ge 0,q,s\ge 1} \frac{H_{mnpqks}U_{pkm}}{\|j_m\|^2_{(n)}\|Y_m\|^2_S}\hat{u}_{pq}(t)\hat{u}_{ks}(t).$$

Substituting (5.35) and (5.36) into (3.19) we obtain that for $m \ge 0$, $n \ge 1$,

$$\begin{aligned}(a\lambda^2_{mn}+1)\hat{u}''_{mn}(t) + 2b\lambda^2_{mn}\hat{u}'_{mn}(t) \quad &+ \quad (\alpha\lambda^6_{mn}+\beta\lambda^4_{mn}+\lambda^2_{mn})\hat{u}_{mn}(t) \\ &= \quad -\eta\lambda^2_{mn}\widehat{(u^2)}_{mn}(t),\end{aligned} \tag{5.37}$$

$$\hat{u}_{mn}(0) = \varepsilon^2\hat{\phi}_{mn}, \qquad \hat{u}'_{mn}(0) = \varepsilon^2\hat{\psi}_{mn}.$$

Setting $\hat{\Phi}_{mn} = \varepsilon\hat{\phi}_{mn}$, $\hat{\Psi}_{mn} = \varepsilon\hat{\psi}_{mn}$, and

$$\sigma_{mn} = \frac{\lambda_{mn}\sqrt{a\alpha\lambda^6_{mn} + (\alpha + a\beta)\lambda^4_{mn} + (a+\beta-b^2)\lambda^2_{mn} + 1}}{1 + a\lambda^2_{mn}},$$

we integrate (5.37) with respect to t and obtain

$$\begin{aligned}\hat{u}_{mn}(t) &= \varepsilon\exp(-\frac{b\lambda^2_{mn}}{1+a\lambda^2_{mn}}t)\{[\cos(\sigma_{mn}t) + \frac{b\lambda^2_{mn}}{1+a\lambda^2_{mn}}\cdot\frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}]\hat{\Phi}_{mn} \\ &+ \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}\hat{\Psi}_{mn}\} \\ &- \frac{\eta\lambda^2_{mn}}{(1+a\lambda^2_{mn})\sigma_{mn}}\int_0^t \exp[-\frac{b\lambda^2_{mn}}{1+a\lambda^2_{mn}}(t-\tau)]\sin[\sigma_{mn}(t-\tau)]\widehat{(u^2)}_{mn}(\tau)d\tau.\end{aligned} \tag{5.38}$$

In order to solve (5.38), we apply the perturbation theory. We represent $\hat{u}_{mn}(t)$ as a formal series in ε

$$\hat{u}_{mn}(t) = \sum_{N=0}^{\infty}\varepsilon^{N+1}\hat{v}^{(N)}_{mn}(t). \tag{5.39}$$

Substituting the above into equation (5.38) and comparing the coefficients of equal powers of ε, we get that for $m \ge 0$, $n \ge 1$, $t > 0$:

$$\begin{aligned}\hat{v}^{(0)}_{mn}(t) \quad &= \quad \exp(-\frac{b\lambda^2_{mn}}{1+a\lambda^2_{mn}}t)\{[\cos(\sigma_{mn}t) + \frac{b\lambda^2_{mn}}{1+a\lambda^2_{mn}}\cdot\frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}]\hat{\Phi}_{mn} \\ &+ \quad \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}\hat{\Psi}_{mn}\},\end{aligned}$$

$$\begin{aligned}\hat{v}^{(N)}_{mn}(t) \quad &= \quad -\frac{\eta\lambda^2_{mn}}{(1+a\lambda^2_{mn})\sigma_{mn}}\int_0^t \exp[-\frac{b\lambda^2_{mn}}{1+a\lambda^2_{mn}}(t-\tau)]\sin[\sigma_{mn}(t-\tau)] \\ &\times \quad F^{(N)}_{mn}(\hat{v}(\tau))d\tau, \qquad N \ge 1,\end{aligned} \tag{5.40}$$

where

$$F_{mn}^{(N)}(\hat{v}(t)) = \sum_{p,k\geq 0; q,s\geq 1} \frac{H_{mnpqks}U_{pkm}}{\|j_m\|_{(n)}^2 \|Y_m\|_S^2} \sum_{j=1}^{N} \hat{v}_{pq}^{(j-1)}(t)\hat{v}_{ks}^{(N-j)}(t).$$

Using the induction on the number N we establish the following estimate for $m \geq 0$, $n \geq 1$, $N \geq 0$, $t \geq 0$:

$$|\hat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} \lambda_{mn}^{-5/2}(m+1)^{-1/2} \exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t), \tag{5.41}$$

where $c = c(b) \to \infty$ as $b \to 0^+$.
Since $\phi(r,\theta,\varphi)$ and $\psi(r,\theta,\varphi)$ satisfy Assumption B and Assumption A respectively, by lemma 4.3, we obtain that

$$|\hat{\phi}_{mn}| \leq \frac{c}{\lambda_{mn}^{\frac{5}{2}}\sqrt{m+1}}, \quad |\hat{\psi}_{mn}| \leq \frac{c}{\lambda_{mn}^{\frac{1}{2}}\sqrt{m+1}}.$$

Noting $\frac{\lambda_{mn}^2}{\sigma_{mn}} \leq c < +\infty$ and choosing sufficiently small ε, we get (5.41) for $N = 0$. Assuming that (5.41) holds for $0 \leq l < N$, we shall prove that (5.41) is valid for $l = N$. Note that the following inequality is valid for $0 \leq l \leq N$ (see [11])

$$j^{-2}(N+1-j)^{-2} \leq 2^2(N+1)^{-2}[j^{-2} + (N+1-j)^{-2}].$$

Combining (2.8), (2.9), (4.31) and (5.40), it follows

$$|F_{mn}^{(N)}(\hat{v}(t))| \leq c \cdot c^{N-1}(N+1)^{-2}(m+1)^{-1/2}\lambda_{mn}^{-1/2} \exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t). \tag{5.42}$$

Hence it follows

$$|\hat{v}_{mn}^{(N))}(t)| \leq c \cdot c^{N-1}(N+1)^{-2}(m+1)^{-1/2}\lambda_{mn}^{-5/2} S_{mn}(t),$$

where

$$\begin{aligned} S_{mn}(t) &= \exp(-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2}t) \int_0^t \exp[(\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} - 2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2})\tau]d\tau \\ &= \exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t) \cdot \frac{\exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t) - \exp[-(\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} - \frac{b\lambda_{01}^2}{1+a\lambda_{01}^2})t]}{\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} - 2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}} \\ &= \exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t) \cdot \frac{1 - \exp[-(\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} - 2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2})t]}{\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2} - 2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}}. \end{aligned}$$

From (4.34) it follows that

$$S_{mn}(t) \leq c\exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}), \quad m \geq 0, n \geq 1, \tag{5.43}$$

$$S_{mn}(t) \leq c\exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}), \quad m=0, n\geq 2 \quad \text{or} \quad m,n\geq 1. \tag{5.44}$$

Combining (5.42) and (5.43), we get (5.41) for $l = N$.
Similarly, combining (4.34) and (5.44), we can get that for $m = 0$, $n \geq 2$ or $m, n \geq 1$,

$$|\hat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2}\lambda_{mn}^{-5/2}(m+1)^{-1/2}\exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t). \tag{5.45}$$

To prove that the formally constructed function in (5.35) is really a mild solution of (3.19), we should investigate the convergence of the series (5.35). For sufficiently small ε, we can get the following estimate from (5.41)

$$|\hat{u}_{mn}(t)| \leq c\lambda_{mn}^{-5/2}(m+1)^{-1/2}\exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t), \quad m\geq 0, n\geq 1. \tag{5.46}$$

Combining (2.8), (2.9), (2.10) and (5.46), we can obtain that the series

$$\|u(t)\|_\kappa^2 = \sum_{m\geq 0, n\geq 1}\lambda_{mn}^{2\kappa}|\hat{u}_{mn}|^2\|Y_m\|_S^2\|j_m\|_{(n)}^2.$$

converges absolutely and uniformly with respect to $t \geq 0$, for $\kappa < 5/2$.

5.2 Uniqueness of the solution

In order to prove the uniqueness of solution we assume that there exist two solutions $u^{(1)}(r,\theta,\varphi,t)$ and $u^{(2)}(r,\theta,\varphi,t)$ to problem (3.19). Therefore both of them can be expanded into series (5.35) and the coefficients $\hat{u}_{mn}^{(1)}$ and $\hat{u}_{mn}^{(2)}$ have the integral representations (5.38) and satisfy (5.47). Setting $w(r,\theta,\varphi,t) = u^{(1)}(r,\theta,\varphi,t) - u^{(2)}(r,\theta,\varphi,t)$, and expanding it into series (5.35), we can get that for $m \geq 0$, $n \geq 1$, $t > 0$,

$$\hat{w}_{mn}(t) = \frac{-\eta\lambda_{mn}^2}{(1+a\lambda_{mn}^2)\sigma_{mn}}\int_0^t \exp[-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2}(t-\tau)]\sin[\sigma_{mn}(t-\tau)]$$
$$\times \sum_{p,k\geq 0, q,s\geq 1}\frac{H_{mnpqks}U_{pkm}}{\|j_m\|_{(n)}^2\|Y_m\|_S^2}(\hat{u}_{pq}^{(1)}\hat{w}_{ks} + \hat{u}_{ks}^{(2)}\hat{w}_{pq})d\tau, \tag{5.47}$$

From (2.10),(4.30) and (4.32), we can get that for $-1/2 < \varsigma < 5/2$,

$$|\sum_{p,k\geq 0, q,s\geq 1}\frac{H_{mnpqks}U_{pkm}}{\|j_m\|_{(n)}^2\|Y_m\|_S^2}(\hat{u}_{pq}^{(1)}\hat{w}_{ks} + \hat{u}_{ks}^{(2)}\hat{w}_{pq})| \leq c\|w(t)\|_\varsigma.$$

According to (5.47), it follows that for $-1/2 < \varsigma < 5/2$,

$$|\hat{w}_{mn}| \leq \frac{c\sqrt{2m+1}}{\lambda_{mn}^{\frac{3}{2}}} \int_0^t \exp[-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2}(t-\tau)]\|w(\tau)\|_\varsigma d\tau,$$

that is

$$|\hat{w}_{mn}|^2 \leq \frac{c(2m+1)}{\lambda_{mn}^3}\{\int_0^t \exp[-\frac{b\lambda_{mn}^2}{1+a\lambda_{mn}^2}(t-\tau)]\|w(\tau)\|_\varsigma d\tau\}^2.$$

Then we can get that for some $T > 0$, $t \in [0, T]$,

$$|\hat{w}_{mn}|^2 \leq \frac{c(2m+1)}{\lambda_{mn}^3}(\sup_{t\in[0,T]} \|w(t)\|_\varsigma)^2[1-\exp(-\frac{b}{a}T)]^2.$$

Hence, we have that

$$\|w(t)\|_\varsigma^2 = \sum_{m\geq 0, n\geq 1} \lambda_{mn}^{2\varsigma}|\hat{w}_{mn}|^2\|Y_m\|_S^2\|j_m\|_{(n)}^2$$

$$\leq (\sup_{t\in[0,T]} \|w(t)\|_\varsigma)^2[1-\exp(-\frac{b}{a}T)]^2 \sum_{m\geq 0, n\geq 1} \frac{c}{\lambda_{mn}^{4-2\varsigma}}$$

when $-1/2 < \varsigma < 1$ we get $\sum_{m\geq 0, n\geq 1} \frac{c}{\lambda_{mn}^{4-2\varsigma}} < C_5 < +\infty$. Let $G(T) = [1 - \exp(-\frac{b}{a}T)](C_5)^{1/2}$. We can obtain that for $t \in [0, T]$,

$$\|w(t)\|_\varsigma \leq G(T) \sup_{t\in[0,T]} \|w(t)\|_\varsigma.$$

Then it follows

$$\sup_{t\in[0,T]} \|w(t)\|_\varsigma \leq G(T) \sup_{t\in[0,T]} \|w(t)\|_\varsigma.$$

Since $G(T)$ is a nondecreasing continuous function on $[0, +\infty)$ and $G(0) = 0$, we can make appropriate choice of T_1 such that $G(T_1) < 1$. Thus we obtain that $w(r, \theta, \varphi, t) = 0$ on $[0, T_1]$. In similar way, we get that $w(r, \theta, \varphi, t) = 0$ on $[T_1, 2T_1]$, $[2T_1, 3T_1]$,$\cdots$,$[nT_1, (n+1)T_1]$,$\cdots$ with $nT_1 \to \infty$ as $n \to +\infty$. Hence we establish the uniqueness of solution for all $t \geq 0$ and $-1/2 < \varsigma < 1$. Note the fact $\|w(t)\|_{k_1} \leq \|w(t)\|_{k_2}$ for $k_1 \leq k_2$ and all $t \geq 0$. Consequently $H^{k_2}(B) \subseteq H^{k_1}(B)$ for $t \geq 0$. Therefore, the uniqueness takes place for $-1/2 < \varsigma < 5/2$. This completes the proof of Theorem 1.

6 Proof of Theorem 3.3

In order to obtain the asymptotic expansion (3.21), we single out the terms $\hat{u}_{01}(t)$, $j_0(\lambda_{01}r)$, $Y_0(\theta,\varphi)$ in (5.35) and estimate the remaining series.
Using the fact that $j_0(\lambda_{0n}r) = \sqrt{\frac{\pi}{2r}}J_{1/2}(n\pi r)$ and $Y_0(\theta,\varphi) = 1$, it follows

$$u(r,\theta,\varphi,t) = \hat{u}_{01}(t)j_0(\lambda_{01}r) + (\sum_{m=0,n\geq 2} + \sum_{m,n\geq 1})\hat{u}_{mn}(t)\chi_{mn}. \tag{6.48}$$

From (5.39) and (5.40), we have that

$$\hat{u}_{01}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\hat{v}_{01}^{(N)}(t), \tag{6.49}$$

$$\hat{v}_{01}^{(0)}(t) = \exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t)[A_{01}^{(0)}\cos(\sigma_{01}t) + B_{01}^{(0)}\sin(\sigma_{01}t)],$$

$$\begin{aligned}\hat{v}_{01}^{(N)}(t) = \exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t)\{&[A_{01}^{(N)} + R_A^{(N)}(t)]\cos(\sigma_{01}t)\\ &+[B_{01}^{(N)} + R_B^{(N)}(t)]\sin(\sigma_{01}t)\}, N\geq 1,\end{aligned}$$

where

$$A_{01}^{(0)} = \varepsilon\hat{\phi}_{01}, \qquad B_{01}^{(0)} = \frac{\varepsilon}{\sigma_{01}}(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\hat{\phi}_{01} + \hat{\psi}_{01}),$$

$$A_{01}^{(N)} = \frac{\eta\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}}\int_0^{\infty}\exp(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau)\sin(\sigma_{01}\tau)F_{01}^{(N)}(\hat{v}(\tau))d\tau,$$

$$R_A^{(N)}(t) = -\frac{\eta\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}}\int_t^{\infty}\exp(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau)\sin(\sigma_{01}\tau)F_{01}^{(N)}(\hat{v}(\tau))d\tau,$$

$$B_{01}^{(N)} = -\frac{\eta\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}}\int_0^{\infty}\exp(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau)\cos(\sigma_{01}\tau)F_{01}^{(N)}(\hat{v}(\tau))d\tau,$$

$$R_B^{(N)}(t) = \frac{\eta\lambda_{01}^2}{(1+a\lambda_{01}^2)\sigma_{01}}\int_t^{\infty}\exp(\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}\tau)\cos(\sigma_{01}\tau)F_{01}^{(N)}(\hat{v}(\tau))d\tau.$$

From (5.42) it follows

$$|F_{01}^{(N)}(\hat{v}(t))| \leq \frac{c^N}{(N+1)^2\lambda_{01}^{1/2}}\exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2}t).$$

Noting that $\lambda_{01} = \pi$, we have

$$|R_{B,A}^{(N)}(t)| \le c^N (N+1)^{-2} O(\exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t)). \tag{6.50}$$

According to (6.49)-(6.51), it follows

$$\hat{u}_{01}(t) = \exp(-\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t)[A_{01}\cos(\sigma_{01}t) + B_{01}\sin(\sigma_{01}t)] + O(\exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t)), \tag{6.51}$$

where $A_{01} = \sum_{N=0}^{\infty} \varepsilon^{N+1} A_{01}^{(N)}$ and $B_{01} = \sum_{N=0}^{\infty} \varepsilon^{N+1} B_{01}^{(N)}$. The solution can be represented as follows

$$u(r,\theta,\varphi,t) = \hat{u}_{01}(t) j_0(\lambda_{01} r) + R_1(r,t) + R_2(r,\theta,\varphi,t), \tag{6.52}$$

where

$$R_1(r,t) = \sum_{n=2}^{\infty} \hat{u}_{0n}(t) j_0(\lambda_{0n} r),$$

$$R_2(r,\theta,\varphi,t) = \sum_{m,n\ge 1} \hat{u}_{mn}(t) j_m(\lambda_{mn} r) Y_m(\theta,\varphi).$$

Applying (5.39) and (5.45), it follows

$$|R_1(r,t)| \le c\exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t), \quad |R_2(r,\theta,\varphi,t)| \le c\exp(-2\frac{b\lambda_{01}^2}{1+a\lambda_{01}^2} t). \tag{6.53}$$

Combining formulas (6.52)-(6.53), we deduce (3.21).
The proof of Theorem 3.3 is complete.

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