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Nonhomogeneous boundary value problems in Sobolev spaces with variable exponent

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Abstract

We study a nonlinear boundary value problem involving the $p(x)$ - Laplace operator. Our main result establishes the existence of a nontrivial weak solution and our main arguments rely on the Mountain Pass Theorem of Ambrosetti and Rabinowitz.

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1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions have received more and more interest in recent years. The specific attention accorded to such kind of problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomenon which arise in elastic mechanics or electrorheological fluids (sometimes referred to as "smart fluids").

In this paper we discuss the existence of weak solutions for the following problem:

$$\begin{cases} -\Delta_{p(x)}u(x) + a(x)|u(x)|^{p(x)-2}u = |u(x)|^{q(x)-1}u, & \text{for } x \in \Omega \\ u \not\equiv 0, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $p, q : \overline{\Omega} \to \mathbb{R}$ are continuous functions with $2 \leq \min_{\overline{\Omega}} p(x) < \max_{\overline{\Omega}} p(x) < N$, $\max_{\overline{\Omega}} p(x) < \min_{\overline{\Omega}} q(x)+1$, $q(x) \leq N$, $q(x)+1 < Np(x)/(N-p(x))$ for all $x \in \overline{\Omega}$ and $a : \overline{\Omega} \to \mathbb{R}$ satisfies the condition:

(A) $a \in L^\infty(\overline{\Omega})$ and $a(x) \geq 0$, for any $x \in \Omega$;

We denoted by $\Delta_{p(x)}$ the $p(x)$-Laplace operator, i.e.

$$\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u(x)|^{p(x)-2} \nabla u(x)\right).$$

Equation (1.1) will be studied in the framework of the variable Lebesgue and Sobolev spaces $L^{p(x)}$ and $W_0^{1,p(x)}$ which will be described briefly in the next section.

2. Preliminary results

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context we refer to the book of Musielak [12] and the papers of Kovacik and Rakosnik [8] and Fan et al. [4, 6].

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \text{is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are separable and Banach spaces [8, Theorem 2.5; Corollary 2.7] and the Hölder inequality holds [8, Theorem 2.1]. The inclusion between Lebesgue spaces also generalizes naturally [8, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x)=1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big|\le\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \tag{2.1}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)}$: $L^{p(x)}(\Omega)\to\mathbb{R}$ defined by

$$\rho_{p(x)}(u)=\int_\Omega |u|^{p(x)}\,dx.$$

If $(u_n),\ u,\ |\nabla|u\in L^{p(x)}(\Omega)$ and $p^+<\infty$ then the following relations hold

$$\begin{aligned}|\nabla u|_{p(x)}>1&\Rightarrow|\nabla u|^{p^-}_{p(x)}\le\rho_{p(x)}(\nabla u)\le|\nabla u|^{p^+}_{p(x)},\\ |\nabla u|_{p(x)}<1&\Rightarrow|\nabla u|^{p^+}_{p(x)}\le\rho_{p(x)}(\nabla u)\le|\nabla u|^{p^-}_{p(x)},\end{aligned} \tag{2.2}$$

$$\begin{aligned}|u|_{q(x)+1}>1&\Rightarrow|u|^{q^-+1}_{q(x)+1}\le\rho_{q(x)+1}(u)\le|u|^{q^++1}_{q(x)+1},\\ |u|_{q(x)+1}<1&\Rightarrow|u|^{q^++1}_{q(x)+1}\le\rho_{q(x)+1}(u)\le|u|^{q^-+1}_{q(x)+1},\end{aligned} \tag{2.3}$$

$$\begin{gathered}|u|_{p(x)}\to0\ \Leftrightarrow\ \rho_{p(x)}(u)\to0,\\ \lim_{n\to\infty}|u_n-u|_{p(x)}=0\ \Leftrightarrow\ \lim_{n\to\infty}\rho(u_n-u)=0,\\ |u|_{p(x)}\to\infty\ \Leftrightarrow\ \rho_{p(x)}(u)\to\infty.\end{gathered} \tag{2.4}$$

We define also $W_0^{1,p(x)}(\Omega)$ as the closure of C_0^∞ under the norm

$$\|u\|=|\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega),\|.\|)$ is a separable and reflexive Banach space. Next, we remember some embedding results regarding variable exponent Lebesgue-Sobolev spaces. We note that if $s(x)\in C_+(\overline{\Omega})$, $p(x)<N$ and $s(x)<Np(x)/(N-p(x))$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous. We refer to [8] for more properties of Lebesgue and Sobolev spaces with variable exponent. We also refer to the recent papers [4, 5, 6, 9, 10, 11] for the treatment of nonlinear boundary value problems in Lebesque-Sobolev spaces with variable exponent.

3. The main result and an auxiliary result

Definition 1. *We say that* $u\in W_0^{1,p(x)}(\Omega)$ *is a weak solution of* (1.1) *if*

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\right)dx = \int_\Omega |u|^{q(x)-1}uvdx,$$

for any $v \in W_0^{1,p(x)}(\Omega)$.

The main result of this paper is given by the following theorem:

Theorem 1. *Suppose* $p, q : \Omega \to \mathbb{R}$ *are continuous functions with* $2 \le p^- < p^+ < N$, $p^+ < q^- + 1$, $q(x) \le N$, $q(x) + 1 < Np(x)/(N - p(x))$ *for all* $x \in \overline{\Omega}$ *and* $a : \overline{\Omega} \to \mathbb{R}$ *satisfies the condition(A). Then the problem* (1.1) *has a non-trivial weak solution.*

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. The energy functional corresponding to problem (1.1) is defined as $J : E \to \mathbb{R}$,

$$J(u) := \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right)dx - \int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}dx.$$

Proposition 1. *The functional* J *is well-defined on* E.

Proof. Using the above hypothesis and the condition (A) we get:

$$J(u) \le \frac{1}{p^-}\Big(\int_\Omega |\nabla u|^{p(x)}dx + \|a\|_{L^\infty}\int_\Omega |u|^{p(x)}dx\Big) - \frac{1}{q^-+1}\int_\Omega |u|^{q(x)+1}dx.$$

The proof, then holds by the sobolev embeddings $E \hookrightarrow L^{p(x)}$, $E \hookrightarrow L^{q(x)+1}$ and the relation (2.2). □

We set

$$J_1(u) := \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right)dx$$

and

$$J_2(u) := \int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}dx.$$

We first prove an auxiliary result:

Theorem 2. *Suppose we are under hypotheses of Theorem 1. Then* $J : E \to \mathbb{R}$ *is a continuously differentiable function and* $u \in E$ *is a critical point of* J *if and only if* u *is a weak solution for the problem* (1.1).

Proof. To show that $I : H \to \mathbb{R}$ is continuously differentiable it is sufficient to show that for all $\varphi \in H$,

$$\lim_{t\to 0^+} \frac{I(u+t\varphi) - I(u)}{t} = \langle dI(u), \varphi\rangle,$$

with $dI : H \to H^*$ continuous, where we denote by H^* the dual space of H.

For all $\varphi \in E$ we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{J_2(u+t\varphi)-J_2(u)}{t} &= \frac{d}{dt}J_2(u+t\varphi)|_{t=0} = \frac{d}{dt}\int_\Omega \frac{1}{q(x)+1}|u+t\varphi|^{q(x)+1}dx|_{t=0} = \\ &= \int_\Omega \frac{\partial}{\partial t}\left(\frac{1}{q(x)+1}|u+t\varphi|^{q(x)+1}\right)|_{t=0}dx = \int_\Omega |u+t\varphi|^{q(x)}sgn(u+t\varphi)\varphi|_{t=0}dx = \\ &= \int_\Omega |u+t\varphi|^{q(x)-1}(u+t\varphi)\varphi|_{t=0}dx = \int_\Omega |u|^{q(x)-1}u\varphi dx = \\ &= \langle dJ_2(u),\varphi\rangle.\end{aligned}$$

The differentiation under the integral is allowed if for t close to zero, $|u+t\varphi|^{q(x)-1}(u+t\varphi)\varphi$ can be dominated by one fixed function $g \in L^1(\Omega)$ which does not depend on t. Indeed we have, say if $|t|<1$ that

$$||u+t\varphi|^{q(x)-1}(u+t\varphi)\varphi| = |u+t\varphi|^{q(x)}|\varphi| \le (|u|+|t||\varphi|)^{q(x)}|\varphi| \le (|u|+|\varphi|)^{q(x)}|\varphi| \in L^1$$

because $u,\varphi \in E$ imply

$$|u|,|\varphi| \in E \hookrightarrow L^{q(x)+1}(\Omega),$$

and then

$$(|u|+|\varphi|)^{q(x)} \in L^{\frac{q(x)+1}{q(x)}}(\Omega), |\varphi| \in L^{q(x)+1}(\Omega) = \left(L^{\frac{q(x)+1}{q(x)}}(\Omega)\right)^*.$$

For $u \in E$ chosen we show that $dJ_2(u) \in W^{-1,p'(x)}(\Omega) = E^*$, where $1/p(x)+1/p'(x)=1$. We observe that $dJ_2(u)$ is linear. Indeed, for all $\alpha\ ,\beta \in \mathbb{R}$ and $\varphi\ ,\psi \in E$ we have

$$\begin{aligned}\langle dJ_2(u),\alpha\varphi+\beta\psi\rangle &= \int_\Omega |u|^{q(x)-1}u(\alpha\varphi+\beta\psi)dx = \alpha\int_\Omega |u|^{q(x)-1}u\varphi dx + \beta\int_\Omega |u|^{q(x)-1}u\psi dx = \\ &= \alpha\langle dJ_2(u),\varphi\rangle + \beta\langle dJ_2(u),\psi\rangle.\end{aligned}$$

We know $p,q:\Omega\to\mathbb{R}$ are continuous and $2\le p^- < p^+ < N$, so for any p,q with $p(x)\le q(x)+1\le \frac{Np(x)}{N-p(x)}$, there is a continuous embedding $E\hookrightarrow L^{q(x)+1}(\Omega)$. Then there exists a constant $M>0$ such that

$$|v|_{q(x)+1} \le M\|v\|, \quad \text{for all } v\in E. \tag{3.1}$$

Using (2.1) and (3.1) we obtain

$$|\langle dJ_2(u),\varphi\rangle| = |\int_\Omega |u|^{q(x)-1}u\varphi dx| \le \int_\Omega |u|^{q(x)}|\varphi|dx \le ||u|^{q(x)}|_{\frac{q(x)+1}{q(x)}}|\varphi|_{q(x)+1} \le M||u|^{q(x)}|_{\frac{q(x)+1}{q(x)}}\|\varphi\|.$$

Hence there exists $M_1 = M||u|^{q(x)}|_{\frac{q(x)+1}{q(x)}} > 0$ such that

$$|\langle dJ_2(u),\varphi\rangle| \le M_1\|\varphi\|.$$

Using the linearity of $dJ_2(u)$ and the above formula we deduce that $dJ_2(u)\in E^* = W^{-1,p'(x)}(\Omega)$.

For the Fréchet differentiability we need the following lemma:

Lemma 1. *The map* $u \in L^{q(x)+1}(\Omega) \to |u|^{q(x)-1}u \in L^{\frac{q(x)+1}{q(x)}}(\Omega)$ *is continuous.*

Proof. We have that $1 \leq q(x) \leq N$, so $\frac{q(x)+1}{q(x)} < q(x)+1$, therefore $L^{q(x)+1}(\Omega) \hookrightarrow L^{\frac{q(x)+1}{q(x)}}(\Omega)$ continuous. Then there exists a constant $M_0 > 0$ such that

$$|u|_{\frac{q(x)+1}{q(x)}} \leq M_0 |u|_{q(x)+1}, \quad \text{for all } u \in L^{q(x)+1}(\Omega).$$

Hence

$$||u|^{q(x)-1}u - |v|^{q(x)-1}v|_{\frac{q(x)+1}{q(x)}} \leq M_0 ||u|^{q(x)-1}u - |v|^{q(x)-1}v|_{q(x)+1}. \tag{3.2}$$

Our intention is to prove

$$\int_\Omega ||u|^{q(x)-1}u - |v|^{q(x)-1}v|^{q(x)+1} dx \to 0$$

and then, using relations (2.4) and (3.2), we obtain

$$||u|^{q(x)-1}u - |v|^{q(x)-1}v|_{\frac{q(x)+1}{q(x)}} \to 0.$$

Fix $x \in \Omega$. By Lagrange Theorem applied to $f(u) = |u|^{q(x)-1}u$, there exists $C_0(x)$ somewhere between $u(x)$ and $v(x)$ such that

$$\frac{f(u(x)) - f(v(x))}{u(x) - v(x)} = f'(C_0(x)).$$

Thus, using (2.1) and (2.4) we obtain

$$\int_\Omega ||u|^{q(x)-1}u - |v|^{q(x)-1}v|^{q(x)+1} dx \leq \int_\Omega \left[|u-v| N \max(|u|,|v|)^{q(x)-1}\right]^{q(x)+1} dx \to 0.$$

The proof of Lemma 1 is complete. □

We conclude that J_2 is Fréchet differentiable.
Now we show that J_1 is Fréchet differentiable. For all $\varphi \in E$ we have

$$\begin{aligned}\lim_{t \to 0^+} \frac{J_1(u+t\varphi) - J_1(u)}{t} &= \frac{d}{dt} J_1(u+t\varphi)|_{t=0} = \frac{d}{dt} \int_\Omega \frac{1}{p(x)} \left[|\nabla u + t\nabla\varphi|^{p(x)} + a(x)|u+t\varphi|^{p(x)}\right] dx|_t \\ &= \int_\Omega \frac{\partial}{\partial t} \left[\frac{1}{p(x)} \left(|\nabla u + t\nabla\varphi|^{p(x)} + a(x)|u+t\varphi|^{p(x)}\right)\right] |_{t=0} dx.\end{aligned}$$

But

$$\frac{\partial}{\partial t}|u+t\varphi|^{p(x)} = p(x)|u+t\varphi|^{p(x)-1} sgn(u+t\varphi)\varphi = p(x)|u+t\varphi|^{p(x)-2}(u+t\varphi)\varphi,$$

$$\frac{\partial}{\partial t}|\nabla u + t\nabla\varphi|^{p(x)} = p(x)|\nabla(u+t\varphi)|^{p(x)-2}\nabla(u+t\varphi)\nabla\varphi.$$

So

$$\begin{aligned}\lim_{t\to 0^+}\frac{J_1(u+t\varphi)-J_1(u)}{t} &= \int_\Omega \left(|\nabla(u+t\varphi)|^{p(x)-2}\nabla(u+t\varphi)\nabla\varphi + a(x)|u+t\varphi|^{p(x)-2}(u+t\varphi)\varphi\right)|_{t=0} \\ &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + a(x)|u|^{p(x)-2}u\varphi\right)dx.\end{aligned}$$

The differentiation under the integral is allowed if for t close to zero

$$|\nabla(u+t\varphi)|^{p(x)-2}\nabla(u+t\varphi)\nabla\varphi + a(x)|u+t\varphi|^{p(x)-2}(u+t\varphi)\varphi$$

can be dominated by one fixed function $g \in L^1(\Omega)$ which does not depend on t. Indeed we have, say if $|t|<1$ that

$$\begin{aligned}&||\nabla(u+t\varphi)|^{p(x)-2}\nabla(u+t\varphi)\nabla\varphi + a(x)|u+t\varphi|^{p(x)-2}(u+t\varphi)\varphi| \leq \\ \leq\ &|\nabla u + t\nabla\varphi|^{p(x)-1}|\nabla\varphi| + \|a\|_{L^\infty}|u+t\varphi|^{p(x)-1}|\varphi| \leq \\ \leq\ &(|\nabla u| + |\nabla\varphi|)^{p(x)-1}|\nabla\varphi| + \|a\|_{L^\infty}(|u|+|\varphi|)^{p(x)-1}|\varphi|.\end{aligned}$$

Set

$$g(x) := (|\nabla u| + |\nabla\varphi|)^{p(x)-1}|\nabla\varphi| + \|a\|_{L^\infty}(|u|+|\varphi|)^{p(x)-1}|\varphi|.$$

We observe that $g \in L^1(\Omega)$ because $u,\ \varphi \in L^{p(x)},$ so

$$|u|,\ |\varphi| \in L^{p(x)},\ \text{yield}\ (|u|+|\varphi|)^{p(x)-1} \in L^{\frac{p(x)}{p(x)-1}} = \left(L^{p(x)}\right)^*$$

and

$$|\nabla u|,\ |\nabla\varphi| \in L^{p(x)},\ \text{yield}\ (|\nabla u|+|\nabla\varphi|)^{p(x)-1} \in L^{\frac{p(x)}{p(x)-1}} = \left(L^{p(x)}\right)^*.$$

For all $u \in E$ we have

$$\langle dJ_1(u),\varphi\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + a(x)|u|^{p(x)-2}u\varphi\right)dx$$

and $dJ_1(u) : E \to \mathbb{R}$ is a continuous function. It is easy to see that $dJ_1(u)$ is a linear function. Indeed, for all $\alpha,\beta \in \mathbb{R}$ and $v_1, v_2 \in E$ we have:

$$\begin{aligned}\langle dJ_1(u),\alpha v_1+\beta v_2\rangle &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla(\alpha v_1+\beta v_2) + a(x)|u|^{p(x)-2}u(\alpha v_1+\beta v_2)\right)dx = \\ &= \alpha\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v_1 + a(x)|u|^{p(x)-2}uv_1\right)dx + \\ &+ \beta\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v_2 + a(x)|u|^{p(x)-2}uv_2\right)dx = \\ &= \alpha\langle dJ_1(u),v_1\rangle + \beta\langle dJ_1(u),v_2\rangle.\end{aligned}$$

We also have the continuous embedding $E \hookrightarrow L^{p(x)}(\Omega)$, which implies that there exists $M' > 0$ such that, for all $u \in E$,

$$|u|_{p(x)} \leq M'\|u\|. \tag{3.3}$$

Using (2.1) and (3.3) we obtain

$$\begin{aligned} |\langle dJ_1(u), \varphi\rangle| &= |\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + a(x)|u|^{p(x)-2}u\varphi\right) dx| \\ &\leq \int_\Omega |\nabla u|^{p(x)-1}|\nabla\varphi| + \|a\|_{L^\infty}\int_\Omega |u|^{p(x)-1}|\varphi| dx \\ &\leq ||\nabla u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}|\nabla\varphi|_{p(x)} + \|a\|_{L^\infty}||u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}|\varphi|_{p(x)} \\ &\leq ||\nabla u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}\|\varphi\| + M'\|a\|_{L^\infty}||u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}\|\varphi\|. \end{aligned}$$

Thus there exists $M_2 = ||\nabla u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}} + M'\|a\|_{L^\infty}||u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}} > 0$ such that

$$|\langle dJ_1(u), \varphi\rangle| \leq M_2\|\varphi\|, \text{ for all } \varphi \in E.$$

The above relation and the linearity of $dJ_1(u)$ imply that

$$dJ_1(u) \in E^*,\ dJ_1 : E \to E^*$$

and

$$\langle dJ_1(u), \varphi\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + a(x)|u|^{p(x)-2}u\varphi\right) dx.$$

We conclude that J_1 is Frechet differentiable.

We remark that $J \in C^1(E, \mathbb{R})$ because $J_1,\ J_2 \in C^1(E, \mathbb{R})$. Moreover

$$\begin{aligned} \langle dJ(u), v\rangle &= \langle dJ_1(u) - dJ_2(u), v\rangle = \langle dJ_1(u), v\rangle - \langle dJ_2(u), v\rangle = \\ &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\right)dx - \int_\Omega |u|^{q(x)-1}uvdx. \end{aligned}$$

Let u be a critical point of J. Then we have $dJ(u) = 0_{E^*}$ that is

$$\langle dJ(u), v\rangle = 0, \text{ for all } v \in E$$

which yields

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\right)dx = \int_\Omega |u|^{q(x)-1}uvdx, \quad \forall v \in E.$$

It follows that u is a weak solution for the problem (1.1).

Now we assume that u is a weak solution of the problem (1.1). By Definition 1 it results that

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\right)dx = \int_\Omega |u|^{q(x)-1}uvdx, \quad \forall v \in E.$$

that is $\langle dJ(u), v\rangle = 0, \forall v \in E$. We obtain $dJ(u) = 0_{E^*}$. Hence u is a critical point of J. This completes the proof of Theorem 2.

□

4. Proof of Main Theorem

Our idea is to prove Theorem 1 applying the Mountain Pass Theorem (see e.g. [1]). With that end in view, we prove some auxiliary results which show that the functional J has a mountain pass geometry.

Lemma 2. *Suppose we are under hypotheses of Theorem 1. Then there exist $\varrho > 0$ and $r > 0$ such that for all $u \in E$ with $\|u\| = r$*

$$J(u) \geq \varrho > 0.$$

Proof.
Let us assume that $\|u\| < \min(1, 1/M)$, where M is the positive constant from above. Then, we have $|u|_{q(x)+1} < 1$. Using relations (2.2), (2.3), (3.1), and (A) we obtain:

$$\begin{aligned} J(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right) dx - \frac{1}{q^-+1}\int_\Omega |u|^{q(x)+1}dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{1}{q^-+1}\int_\Omega |u|^{q(x)+1}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{1}{q^-+1}\int_\Omega |u|^{q(x)+1}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{M^{q^-+1}}{q^-+1}\|u\|^{q^-+1}, \end{aligned}$$

Let $h(t) = \frac{1}{p^+}t^{p^+} - \frac{M^{q^-+1}}{q^-+1}t^{q^-+1}$, $t > 0$. Then, $h'(t) = t^{p^+-1} - M^{q^-+1}t^{q^-}$. So, $h'(t_0) = 0$ implies that $t_0 = \left(\frac{1}{M^{q^-+1}}\right)^{\frac{1}{q^-+1-p^+}}$. We calculate $h''(t) = (p^+-1)t^{p^+-2} - q^-M^{q^-+1}t^{q^--1}$ and we observe that $p^+ - 1 < q^-$ implies $h''(t_0) < 0$. Therefore t_0 is a maximum point of the function h and $h(0) = 0$ implies $h(t) > 0$ for all $r \in (0, t_1)$, where $t_1 = \left(\frac{(p^+-1)}{q^-M^{q^-+1}}\right)^{\frac{1}{q^-+1-p^+}}$. So we can choose $r > 0$ and $\varrho > 0$ such that $J(u) \geq \varrho > 0$ for all $u \in E$ with $\|u\| = r$.

□

Lemma 3. *Suppose we are under hypotheses of Theorem 1. Then there exists $v_0 \in E$ with $\|v_0\| > r$ (r given in Lemma 2) such that $J(v_0) < 0$.*

Proof. We choose $v_0 = tu_0$, where u_0 is a positive function in E with $\|u_0\| > 1$ and t is large.

$$\begin{aligned} J(tu_0) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla(tu_0)|^{p(x)} + a(x)|tu_0|^{p(x)}\right) dx - \int_\Omega \frac{1}{q(x)+1}|tu_0|^{q(x)+1}dx \\ &\leq \frac{1}{p^-}\int_\Omega t^{p(x)}\left(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}\right) dx - \frac{1}{q^++1}\int_\Omega t^{q(x)+1}|u_0|^{q(x)+1}dx. \end{aligned}$$

Thus, for $t > 1$, using (2.2) we have:

$$\begin{aligned}
J(tu_0) &\leq \frac{t^{p^+}}{p^-}\int_\Omega \left(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}\right) dx - \frac{t^{q^-+1}}{q^+ + 1}\int_\Omega |u_0|^{q(x)+1}dx \\
&\leq \frac{t^{p^+}}{p^-}\left(\int_\Omega |\nabla u_0|^{p(x)}dx + \|a\|_{L^\infty}\int_\Omega |u_0|^{p(x)}dx\right) - \frac{t^{q^-+1}}{q^+ + 1}\int_\Omega |u_0|^{q(x)+1}dx \\
&\leq \frac{t^{p^+}}{p^-}\left(\|u_0\|^{p^+} + \|a\|_{L^\infty}\int_\Omega |u_0|^{p(x)}dx\right) - \frac{t^{q^-+1}}{q^+ + 1}\int_\Omega |u_0|^{q(x)+1}dx.
\end{aligned}$$

On the other hand, using (3.3) we obtain

$$J(tu_0) \leq \frac{t^{p^+}(1 + \|a\|_{L^\infty})}{p^-}\|u_0\|^{p^+} - \frac{t^{q^-+1}}{q^+ + 1}\int_\Omega |u_0|^{q(x)+1}dx.$$

Being under condition $p^+ < q^- + 1$, we see that $J(tu_0) \to -\infty$ as $t \to \infty$. Choosing t large enough we observe that there exists $v_0 = tu_0$ in E with $\|v_0\| > r$ such that $J(v_0) < 0$. □

Proof of Theorem 1. *Proof.* We set

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = v_0\},$$

where $v_0 \in E$ is determined by Lemma 3, and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

According to Lemma 3 we know that $\|v_0\| > r$, so every path $\gamma \in \Gamma$ intersects the sphere $\|w\| = r$. Then Lemma 2 implies

$$c \geq \inf_{\|u\|=r} J(u) \geq \varrho, \tag{4.1}$$

with the constant $\varrho > 0$ in Lemma 2, thus $c > 0$.

By the Mountain-Pass Theorem (see, e.g., [1]) we obtain a sequence $(u_n)_n \subset E$ such that

$$J(u_n) \to c, \quad dJ(u_n) \to 0. \tag{4.2}$$

We claim that $(u_n)_n$ is bounded in E. Arguing by contradiction and passing to a subsequence, we have $\|u_n\| \to \infty$. Using (4.2) it follows that for n large enough, we have

$$c + 1 + \|u_n\| \ \geq J(u_n) - \frac{1}{q^- + 1}\langle dJ(u_n), u_n\rangle. \tag{4.3}$$

Since

$$J(u_n) := \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right) dx - \int_\Omega \frac{1}{q(x)+1}|u_n|^{q(x)+1}dx,$$

$$\langle dJ(u_n), u_n\rangle = \int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right)dx - \int_\Omega |u_n|^{q(x)+1}dx,$$

using (4.3) we obtain

$$\begin{aligned} c+1+\|u_n\| &\geq J(u_n) - \frac{1}{q^-+1}\langle dJ(u_n), u_n\rangle \\ &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right)dx - \frac{1}{q^-+1}\int_\Omega |u_n|^{q(x)+1}dx \\ &- \frac{1}{q^-+1}\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right)dx + \frac{1}{q^-+1}\int_\Omega |u_n|^{q(x)+1}dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-+1}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}\right)dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-+1}\right)\|u_n\|^{p^-}. \end{aligned}$$

Thus,

$$c+1+\|u_n\| \geq \left(\frac{1}{p^+} - \frac{1}{q^-+1}\right)\|u_n\|^{p^-} \tag{4.4}$$

Now dividing by $\|u_n\|$ in (4.4) and passing to the limit as $n \to \infty$ we obtain a contradiction. So, up to a subsequence, $(u_n)_n$ converges weakly in E to some $u \in E$. Since $q(x)+1 < Np(x)/(N-p(x))$ for all $x \in \overline{\Omega}$ we deduce that there exists a compact embedding $E \hookrightarrow L^{q(x)+1}(\Omega)$. Then $(u_n)_n$ converges strongly in $L^{q(x)+1}(\Omega)$. So we have:

Proposition 2. $\lim\limits_{n\to\infty}\int_\Omega |u_n|^{q(x)-1}u_n(u_n-u)dx = 0.$

Proof. Using (2.1) we have $\int_\Omega |u_n|^{q(x)-1}u_n(u_n-u)dx \leq ||u_n|^{q(x)-1}u_n|_{\frac{q(x)+1}{q(x)}}|u_n - u|_{q(x)+1}$, then if $||u_n|^{q(x)-1}u_n|_{\frac{q(x)+1}{q(x)}} > 1$, by (2.3), there exists $C > 0$ such that $||u_n|^{q(x)-1}u_n|_{\frac{q(x)+1}{q(x)}} \leq |u_n|^C_{q(x)+1}$ and this ends the proof. □

Note that relation (4.2) yields

$$\lim_{n\to\infty}\langle dJ(u_n), u_n - u\rangle = 0.$$

Combining proposition 2 and the last relation we deduce that

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u) + a(x)|u_n|^{p(x)-2}u_n(u_n-u)\right)dx = 0 \tag{4.5}$$

Since relation (4.5) holds true and $(u_n)_n$ converges weakly to u in E, by [5, Theorem 3.1], we deduce that $(u_n)_n$ converges strongly to u in E. Then since $J \in C^1(E, \mathbb{R})$ we conclude

$$dJ(u_n) \to dJ(u), \tag{4.6}$$

as $n \to \infty$.

Relations (4.2) and (4.6) show that $dJ(u) = 0$ and thus u is a weak solution for (1.1). Moreover, by relation (4.2) it follows that $J(u) > 0$ and thus, u is a nontrivial weak solution for (1.1). The proof of Theorem 1 is complete.

□

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