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A further asymptotic behavior result on certain non-autonomous differential equations of fifth order

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Abstract

In the current paper, we present some sufficient conditions under which all solutions of a certain fifth order differential equation are uniformly bounded and tend to zero together with their derivatives of the fourth order as $t \to \infty$. Our result includes and improves some well-known results in the literature ([2], [3], [15]).

AMS subject classification: 34D05.
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1. Introduction

We are interested in obtaining a result on the asymptotic behaviour of solutions of the fifth order differential equations of the form

$$\begin{aligned} x^{(5)} &+ f_1(t,x,\dot{x},\ddot{x},\dddot{x},x^{(4)})x^{(4)} + b(t)f_2(x,\dot{x},\ddot{x},\dddot{x}) + c(t)f_3(x,\dot{x},\ddot{x}) \\ &+ d(t)f_4(x,\dot{x}) + e(t)f_5(x) = p(t,x,\dot{x},\ddot{x},\dddot{x},x^{(4)}) \end{aligned} \tag{1.1}$$

or its equivalent system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= u \\ \dot{u} &= -f_1(t,x,y,z,w,u)u - b(t)f_2(x,y,z,w) - c(t)f_3(x,y,z) \\ &\quad -d(t)f_4(x,y) - e(t)f_5(x) + p(t,x,y,z,w,u), \end{aligned} \tag{1.2}$$

where $b, ..., e$, $f_1, ..., f_5$ and p are continuous functions for the arguments displayed explicitly in (1.2). It is assumed that the functions $b, ..., e$ are positive definite and differentiable in $R^+ = [0, \infty)$ and the derivatives $\frac{\partial}{\partial x} f_2(x, y, z, w)$, $\frac{\partial}{\partial y} f_2(x, y, z, w)$, $\frac{\partial}{\partial z} f_2(x, y, z, w)$, $\frac{\partial}{\partial x} f_3(x, y, z)$, $\frac{\partial}{\partial y} f_3(x, y, z)$, $\frac{\partial}{\partial x} f_4(x, y)$, $\frac{\partial}{\partial y} f_4(x, y)$ and $f_5'(x)$ exist and are continuous for all x, y, z and w. All solutions considered are also assumed to be real valued.

In recent years, the qualitative behavior of solutions for various classes of certain fifth order non-linear differential equations have been widely discussed in the literature (see, for example,Abou-El-Ela and Sadek [1], Adesina [4], Chukwu ([5], [6]), Tunç ([12], [13], [14], [18]) and Yuan-hong [19] and the references quoted therein). In general, the investigators have given sufficient conditions for stability and boundedness of solutions. However, according to our observations in the relevant literature, only a few researches have been carried out on the asymptotic behavior of solutions of certain fifth-order ordinary differential equations such as Abou-El-Ela and Sadek ([2], [3]), Sadek [11] and Tunç ([15], [17], [18]). More recently, the authors in ([2], [3]) and ([15], [17]) discussed the same subject for the differential equation (1.1) and the equations as follows:

$$x^{(5)} + a(t)f_1(\ddot{x}, \dddot{x})x^{(4)} + b(t)f_2(\ddot{x}, \dddot{x}) + c(t)f_3(\ddot{x}) + d(t)f_4(\dot{x}) + e(t)f_5(x)$$

$$= p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})$$

and

$$x^{(5)} + f_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + b(t)f_2(\ddot{x}, \dddot{x}) + c(t)f_3(\dot{x}, \ddot{x}) + d(t)f_4(\dot{x}) + e(t)f_5(x)$$

$$= p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}),$$

$$x^{(5)} + f_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + \phi(t, \ddot{x}, \dddot{x}) + \psi(t, \dot{x}, \ddot{x}) + g(t, x, \dot{x}) + e(t)h(x)$$

$$= p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}),$$

respectively.

The motivation for the present work, especially, has come from the papers of Abou-El-Ela and Sadek ([2], [3]), Hara ([7], [8], [9], [10]), Tunç ([15], [17], [18]), C. Tunç and E. [16], Zhernovyi and Kostenko [20] and the papers mentioned above It should be noted that in spite of the equation considered in [18] and (1.1) are the same as one another, the conditions established here and the method of proof partially different than that in [18].

2. The main result

The following result is established.

Theorem: *In addition to the basic assumptions imposed on the functions* $b, ..., e, f_1, ..., f_5$, *and* p, *suppose that the following conditions are satisfied* ($\alpha_1, ..., \alpha_5$- *some arbitrary positive constants and* $\varepsilon, \varepsilon_0, \varepsilon_1, ..., \varepsilon_5$ *are some sufficiently small positive constants):*

(i) $B \geq b(t) \geq b_0 \geq 1, C \geq c(t) \geq c_0 \geq 1, D \geq d(t) \geq d_0 \geq 1, E \geq e(t) \geq e_0 \geq 1$ *for all* $t \in R^+$ *(here* B, C, D, E, b_0, c_0, d_0 *and* e_0 *are some constants).*

(ii)

$$\alpha_1 > 0, \alpha_1\alpha_2 - \alpha_3 > 0, (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0, \tag{2.1}$$

$$\delta_0 := \alpha_1 > 0, (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \alpha_5 > 0, \tag{2.2}$$

$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \left[\alpha_1 d(t)\frac{\partial}{\partial y}f_4(x, y) - \alpha_5\right] \geq 2\varepsilon\alpha_2 \tag{2.3}$$

for all $t \in R^+$ *and all* x *and* y,

$$\Delta_2 := \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5).\gamma.d(t)}{\alpha_4(\alpha_1\alpha_2 - \alpha_5)} - \frac{\varepsilon}{\alpha_1} > 0 \tag{2.4}$$

for all $t \in R^+$ *and all* x *and* y, *where*

$$\gamma := \begin{cases} \frac{f_4(x,y)}{y}, y \neq 0 \\ \\ \frac{\partial}{\partial y}f_4(x, 0), y = 0. \end{cases} \tag{2.5}$$

(iii) $\varepsilon_0 \leq f_1(t, x, y, z, w, u) - \alpha_1 \leq \varepsilon_1$ *for all* $t \in R^+$ *and all* x, y, z, w *and* u.

(iv)

$$f_2(x, y, z, 0) = 0, 0 \leq \frac{f_2(x, y, z, w)}{w} - \alpha_2 \leq \varepsilon_2 \;\; \textit{for all} \;\; x, y, z \textit{and } w \neq 0, \tag{2.6}$$

$$yw\frac{\partial}{\partial x}f_2(x, y, z, w) \leq 0, zw\frac{\partial}{\partial y}f_2(x, y, z, w) \leq 0 \text{ and } \frac{\partial}{\partial z}f_2(x, y, z, w) \leq 0 \tag{2.7}$$

for all x, y, z *and* w.

(v)

$$f_3(x, y, 0) = 0, 0 \leq \frac{f_3(x, y, z)}{z} - \alpha_3 \leq \varepsilon_3 \;\; \textit{for all} \;\; x, y \textit{ and } z \neq 0, \tag{2.8}$$

$$yz\frac{\partial}{\partial x}f_3(x, y, z) \leq 0 \textit{ and } \frac{\partial}{\partial y}f_3(x, y, z) \leq 0 \textit{ for all } x, y \textit{ and } z. \tag{2.9}$$

(vi)

$$f_4(x, 0) = 0, \frac{f_4(x, y)}{y} \geq \frac{E\alpha_4}{d_0} \textit{ for all } x \textit{ and } y \neq 0, \tag{2.10}$$

$$\left|\alpha_4 - \frac{\partial}{\partial y} f_4(x,y)\right| \le \varepsilon_4 \ \textit{ for all } x \textit{ and } y, \tag{2.11}$$

$$\frac{\partial}{\partial y} f_4(x,y) - \frac{f_4(x,y)}{y} \le \frac{\alpha_5 \delta_0}{D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \ \textit{ for all } x \textit{ and } y \ne 0, \tag{2.12}$$

$$\left[\frac{\partial}{\partial x} f_4(x,y)\right]^2 \le \min\left[\frac{\varepsilon^2\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{16D(\alpha_1\alpha_4-\alpha_5)}, \frac{\Delta_1 \varepsilon \alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{16D\alpha_1^2(\alpha_1\alpha_4-\alpha_5)}\right] \ \textit{ for all } x \textit{ and } y, \tag{2.13}$$

and

$$\frac{1}{y}\int_0^y \frac{\partial}{\partial x} f_4(x,\eta)d\eta \le -\frac{\varepsilon\alpha_4}{2} \ \textit{ for all } x \textit{ and } y \ne 0. \tag{2.14}$$

(vii)

$$f_5(0) = 0, f_5(x) sgn x > 0 \ (x \ne 0), F_5(x) = \int_0^x f_5(\xi)d\xi \to \infty \ \textit{ as } |x| \to \infty, \tag{2.15}$$

and

$$0 \le \alpha_5 - f_5'(x) \le \varepsilon_5 \ \textit{ for all } x. \tag{2.16}$$

(viii)

$$\int_0^\infty \beta_0(t)dt < \infty, e'(t) \to 0 \ \textit{ as } t \to \infty,$$

where

$$\beta_0(t) := b'_+(t) + c'_+(t) + |d'(t)| + |e'(t)|,$$

$$b'_+(t) = \max\{b'(t), 0\}, c'_+(t) := \max\{c'(t), 0\}.$$

(ix) $|p(t,x,y,z,w,u)| \le p_1(t) + p_2(t)\left[F_5(x) + y^2 + z^2 + w^2 + u^2\right]^{\frac{\sigma}{2}} + \Delta(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}}$, *where* σ, Δ *are constants such that* $0 \le \sigma \le 1, \Delta \ge 0$ *(is sufficiently small), and* p_1, p_2 *are non-negative continuous functions satisfying*

$$\int_0^\infty p_i(t)dt < \infty \ \ (i = 1, 2).$$

Then all solution $x(t)$ *of (1.1) is uniformly bounded and satisfies*

$$x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), x^{(4)}(t) \to 0 \ \textit{ as } t \to \infty.$$

Remark 1: *It should be pointed out that in the special case where* $f_1(t,x,y,z,w,u) = \alpha_1$, $b(t)f_2(x,y,z,w) = \alpha_2 w$, $c(t)f_3(x,y,z) = \alpha_3 z$, $d(t)f_4(x,y) = \alpha_4 y$, $e(t)f_5(x) =$

$\alpha_5 x$ and $p(t,x,y,z,w,u) = 0$ in the equation (1.1) (so that all the assumptions (ii)-(vii) of the theorem are automatically fulfilled), the equation (1.1) reduces to the linear constant coefficient differential equation and the assumptions (ii)-(vii) of the theorem reduce to the corresponding Routh-Hurwitz criterion.

Remark 2: *It should be noted that the theorem stated just above includes the results in ([2], [3]) and [15] and improves the results in ([2], [3]); because the equation (1.1) is more general than those considered there and the result stated here can be proved without the restriction $A \geq a(t) \geq a_0 \geq 1$ in ([2], [3]).*

3. The Lyapunov Function $V_0(t,x,y,z,w,u)$

Our mail tool in the proof of the theorem is the function $V_0 = V_0(t,x,y,z,w,u)$ defined as follows:

$$\begin{aligned}
2V_0 = \; & u^2 + 2\alpha_1 uw + \tfrac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} uz + 2\delta uy + 2b(t)\int_0^w f_2(x,y,z,\rho)d\rho + \left[\alpha_1^2 - \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)}\right] w^2 \\
& +2\left[\alpha_3 + \tfrac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right] wz + 2\alpha_1\delta w + 2d(t)wf_4(x,y) + 2e(t)wf_5(x) \\
& +2\alpha_1 c(t)\int_0^z f_3(x,y,\zeta)d\zeta + \left[\tfrac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_4 - \alpha_1\delta\right] z^2 + 2\delta\alpha_2 yz + 2\alpha_1 d(t) z f_4(x,y) \\
& -2\alpha_5 zy + 2\alpha_1 e(t) z f_5(x) + \tfrac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} d(t)\int_0^y f_4(x,\eta)d\eta + (\delta\alpha_3 - \alpha_1\alpha_5)y^2 \\
& +\tfrac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} e(t) y f_5(x) + 2\delta e(t)\int_0^x f_5(\xi)d\xi + k,
\end{aligned} \tag{3.1}$$

where δ is a positive constant satisfying

$$\delta := \frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} + \varepsilon, \tag{3.2}$$

and k is a positive constant to be determined later in the proof.

In order to prove the theorem we need the following two lemmas.

Lemma 1: *Subject to the conditions (i) - (vii) of the theorem, there exist positive constants D_7 and D_8 such that*

$$D_7\left[F_5(x) + y^2 + z^2 + w^2 + u^2 + k\right] \leq V_0 \leq D_8\left[F_5(x) + y^2 + z^2 + w^2 + u^2 + k\right]. \tag{3.3}$$

Proof: The function $V_0 = V_0(t,x,y,z,w,u)$ defined in (3.1) can be arranged as follows:

$$\begin{aligned} 2V_0 = & \left[u+\alpha_1 w+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right]^2+\tfrac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z+\tfrac{\alpha_5}{\alpha_4}y\right)^2+\Delta_2(w+\alpha_1 z)^2 \\ & +\tfrac{\alpha_4(\alpha_1\alpha_4-\alpha_5)}{(\alpha_1\alpha_2-\alpha_3).\gamma.d(t)}\left[\tfrac{(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)f_5(x)+\tfrac{(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\gamma.d(t)y+\tfrac{\alpha_1}{\alpha_4}\gamma.d(t)z+\tfrac{1}{\alpha_4}\gamma.d(t)w\right]^2 \\ & +2\varepsilon\left(\tfrac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz+k+\sum_{i=1}^{4}V_i, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} V_1 :=\ & 2\delta e(t)\int_0^x f_5(\xi)d\xi-\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5).\gamma.d(t)}e^2(t)f_5^2(x), \\ V_2 :=\ & \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{\alpha_1\alpha_4-\alpha_5}\left[2\int_0^y f_4(x,\eta)d\eta-yf_4(x,y)\right] \\ & +\left[\delta\alpha_3-\alpha_1\alpha_5-\tfrac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)^2}-\delta^2\right]y^2, \\ V_3 :=\ & \tfrac{\varepsilon}{\alpha_1}w^2+2b(t)\int_0^w f_2(x,y,z,\rho)d\rho-\alpha_2 w^2, \\ V_4 :=\ & 2\alpha_1 c(t)\int_0^z f_3(x,y,\zeta)d\zeta-\alpha_1\alpha_3 z^2. \end{aligned}$$

Now, the function V_1 here can be estimated as in [15]. In fact the estimates there show that

$$V_1 \geq 2\varepsilon e_0\int_0^x f_5(\xi)d\xi.$$

Since

$$yf_4(x,y)\equiv\int_0^y f_4(x,\eta)d\eta+\int_0^y \eta\frac{\partial}{\partial\eta}f_4(x,\eta)d\eta,$$

then, by using (i) and (vi), we obtain

$$V_2 = \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{\alpha_1\alpha_4-\alpha_5}\left[2\int_0^y f_4(x,\eta)d\eta - yf_4(x,y)\right] + \left[\frac{\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - \varepsilon\left(\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\right)\right] y^2$$

$$= \int_0^y \left[\frac{2\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)d(t)}{\alpha_1\alpha_4-\alpha_5}\left\{\frac{\partial}{\partial\eta}f_4(x,\eta) - \frac{f_4(x,\eta)}{\eta}\right\} - 2\varepsilon\left\{\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\right\}\right]\eta d\eta$$

$$\geq \int_0^y \left[\frac{\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4-\alpha_5)} - 2\varepsilon\left\{\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\right\}\right]\eta d\eta \geq \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2$$

provided that

$$\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)} \geq \varepsilon\left[\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \alpha_3\right]$$

which we now assume

In view of assumptions(i) and (iv) of the theorem, it follows that

$$V_3 \geq \frac{\varepsilon}{\alpha_1}w^2 + 2\int_0^w \left[\frac{f_2(x,y,z,\rho)}{\rho} - \alpha_3\right]\rho d\rho \geq \frac{\varepsilon}{\alpha_1}w^2.$$

Similarly, from (v) $V_4 = 0$ when $z = 0$, and again from (i) and (v) if $z \neq 0$

$$V_4 \geq 2\alpha_1\int_0^z \left[\frac{f_3(x,y,\zeta)}{\zeta} - \alpha_3\right]\zeta d\zeta \geq 0.$$

Therefore V_4 satisfies $V_4 \geq 0$ for all x, y and z.

Now, making the use of the estimates for V_1, V_2, V_3 and V_4 with (34), we have that

$$2V_0 \geq \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z + \delta y\right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4}y\right)^2 + \Delta_2(w+\alpha_1 z)^2$$

$$+2\varepsilon e_0\int_0^x f_5(\xi)d\xi + \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4-\alpha_5)}y^2 + \frac{\varepsilon}{\alpha_1}w^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz + k.$$

The remainder of the proof exists in [15] and hence is omitted.

Lemma 2: *Suppose that all the conditions of the theorem are satisfied. Then there exist positive constants D_i $(i = 11, 12, 13)$ such that*

$$\dot{V_0} \leq -D_{13}\left[y^2+z^2+w^2+u^2\right] + 2D_{12}(y^2+z^2+w^2+u^2)^{\frac{1}{2}}\left[p_1(t)+p_2(t)\right]$$
$$+2D_{12}p_2(t)\left[F_5(x)+y^2+z^2+w^2+u^2\right] + D_{11}\beta_0 V_0. \quad (3.5)$$

Proof: Along any solution (x, y, z, w, u) of (1.2) it follows from (3.1) and (1.2) that (for $y, z, w \neq 0$)

$$\begin{aligned}
\dot{V}_0 = \; & -u^2[f_1(t,x,y,z,w,u)-\alpha_1] - w^2\left[\alpha_1 \tfrac{b(t)f_2(x,y,z,w)}{w} - \left\{\alpha_3 + \tfrac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right\}\right] \\
& -z^2\left[\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\tfrac{c(t)f_3(x,y,z)}{z} - \left\{\delta\alpha_2 + \alpha_1 d(t)\tfrac{\partial}{\partial y}f_4(x,y) - \alpha_5\right\}\right] \\
& -y^2\left[\delta d(t)\tfrac{f_4(x,y)}{y} - \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)f_5'(x)\right] - \alpha_1 uw[f_1(t,x,y,z,w,u)-\alpha_1] \\
& -uzc(t)\left[\tfrac{f_3(x,y,z)}{z} - \alpha_3\right] - \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uz[f_1(t,x,y,z,w,u)-\alpha_1] \\
& -\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}wzb(t)\left[\tfrac{f_2(x,y,z,w)}{w} - \alpha_2\right] - \delta uy[f_1(t,x,y,z,w,u)-\alpha_1] \\
& -wye(t)\left[\alpha_5 - f_5'(x)\right] - \delta wyb(t)\left[\tfrac{f_2(x,y,z,w)}{w} - \alpha_2\right] - \alpha_1 zye(t)\left[\alpha_5 - f_5'(x)\right] \\
& -\delta zyc(t)\left[\tfrac{f_3(x,y,z)}{z} - \alpha_3\right] - wzd(t)\left[\alpha_4 - \tfrac{\partial}{\partial y}f_4(x,y)\right] + d(t)wy\tfrac{\partial}{\partial x}f_4(x,y) \\
& +b(t)y\int_0^w \tfrac{\partial}{\partial x}f_2(x,y,z,\rho)d\rho + b(t)z\int_0^w \tfrac{\partial}{\partial y}f_2(x,y,z,\rho)d\rho + b(t)w\int_0^w \tfrac{\partial}{\partial z}f_2(x,y,z,\rho)d\rho \\
& +\alpha_1 c(t)y\int_0^z \tfrac{\partial}{\partial x}f_3(x,y,\zeta)d\zeta + \alpha_1 c(t)z\int_0^z \tfrac{\partial}{\partial y}f_3(x,y,\zeta)d\zeta + \alpha_1 d(t)zy\tfrac{\partial}{\partial x}f_4(x,y) \\
& +\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}d(t)y\int_0^y \tfrac{\partial}{\partial x}f_4(x,\eta)d\eta + \left[\tfrac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}zw + \delta\alpha_2 yw\right][1-b(t)] \\
& +[\alpha_3 zu + \delta\alpha_3 yz][1-c(t)] - \alpha_4 wz[1-d(t)] - [\alpha_5 yw + \alpha_1\alpha_5 yz][1-e(t)] \\
& +\left[u + \alpha_1 w + \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z + \delta y\right].p(t,x,y,z,w,u) + \tfrac{\partial V_0}{\partial t}.
\end{aligned} \tag{3.6}$$

The assumption (iii) of the theorem shows that

$$[f_1(t,x,y,z,w,u)-\alpha_1] \geq \varepsilon_0. \tag{3.7}$$

It follows from (i), (ii), (iv) and (3.2) that (for $w \neq 0$)

$$\begin{aligned}
& \alpha_1 \tfrac{b(t)f_2(x,y,z,w)}{w} - \left\{\alpha_3 + \tfrac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right\} \\
& \geq \alpha_1\left[\tfrac{f_2(x,y,z,w)}{w} - \alpha_2\right] + \left[\alpha_1\alpha_2 - \alpha_3 + \delta - \tfrac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right] \geq \varepsilon.
\end{aligned} \tag{3.8}$$

By using (i), (ii), (v) and (3.2) we have that (for $z \neq 0$)

$$\begin{aligned} &\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\tfrac{c(t)f_3(x,y,z)}{z} - \left[\delta\alpha_2 + \alpha_1 d(t)\tfrac{\partial}{\partial y}f_4(x,y) - \alpha_5\right] \\ &\geq \tfrac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \left[\delta\alpha_2 + \alpha_1 d(t)\tfrac{\partial}{\partial y}f_4(x,y) - \alpha_5\right] \\ &= \tfrac{(\alpha_1\alpha_2-\alpha_3)(\alpha_3\alpha_4-\alpha_2\alpha_5)}{\alpha_1\alpha_4-\alpha_5} - \left[\alpha_1 d(t)\tfrac{\partial}{\partial y}f_4(x,y) - \alpha_5\right] - \varepsilon\alpha_2 > (\varepsilon\alpha_2). \end{aligned} \tag{3.9}$$

By noting the assumptions (i), (vi), (vii) of the theorem,(2.10) and (3.2) we obtain that (for $y \neq 0$)

$$\begin{aligned} \delta d(t)\tfrac{f_4(x,y)}{y} - \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)f_5'(x) &\geq \delta E\alpha_4 - \tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}Ef_5'(x) \\ &= \varepsilon E\alpha_4 + \tfrac{\alpha_4 E(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\alpha_5 - f_5'(x)\right] \\ &\geq \varepsilon\alpha_4 E. \end{aligned} \tag{3.10}$$

Hence the first four terms, involving u^2, w^2, z^2 and y^2 in (3.6), are majorizable by

$$-(\varepsilon_0 u^2 + \varepsilon w^2 + \varepsilon\alpha_2 z^2 + \varepsilon\alpha_4 E y^2).$$

It is also clear that

$$\begin{aligned} &y\int_0^z \tfrac{\partial}{\partial x}f_3(x,y,\zeta)d\zeta \leq 0, z\int_0^z \tfrac{\partial}{\partial y}f_3(x,y,\zeta)d\zeta \leq 0, y\int_0^w \tfrac{\partial}{\partial x}f_2(x,y,z,\rho)d\rho \leq 0, \\ &z\int_0^w \tfrac{\partial}{\partial y}f_2(x,y,z,\rho)d\rho \leq 0, w\int_0^w \tfrac{\partial}{\partial z}f_2(x,y,z,\rho)d\rho \leq 0 \text{ by (2.7) and (2.9).} \end{aligned} \tag{3.11}$$

Now consider the terms

$$V_6 = d(t)\left[wy\frac{\partial}{\partial x}f_4(x,y) + \alpha_1 zy\frac{\partial}{\partial x}f_4(x,y)\right] + \frac{d(t)\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}y^2\left[\frac{1}{y}\int_0^y \frac{\partial}{\partial x}f_4(x,\eta)dn\right],$$

which are contained in (3.6).

$$
\begin{aligned}
-V_6 &\geq -d(t)\left[wy\tfrac{\partial}{\partial x}f_4(x,y)+\alpha_1 zy\tfrac{\partial}{\partial x}f_4(x,y)\right]+\tfrac{d(t)\alpha_4^2\varepsilon(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}y^2\\
&=\tfrac{d(t)\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{4(\alpha_1\alpha_4-\alpha_5)}\left[y^2-\tfrac{4(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}yw\tfrac{\partial}{\partial x}f_4(x,y)\right]\\
&+\tfrac{d(t)\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{4(\alpha_1\alpha_4-\alpha_5)}\left[y^2-\tfrac{4\alpha_1(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}yz\tfrac{\partial}{\partial x}f_4(x,y)\right]\\
&=\tfrac{d(t)\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{4(\alpha_1\alpha_4-\alpha_5)}\left[y-\tfrac{2(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}w\tfrac{\partial}{\partial x}f_4(x,y)\right]^2-\tfrac{d(t)(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}w^2\left[\tfrac{\partial}{\partial x}f_4(x,y)\right]^2\\
&+\tfrac{d(t)\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{4(\alpha_1\alpha_4-\alpha_5)}\left[y-\tfrac{2\alpha_1(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}z\tfrac{\partial}{\partial x}f_4(x,y)\right]^2-\tfrac{d(t)\alpha_1^2(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}z^2\left[\tfrac{\partial}{\partial x}f_4(x,y)\right]^2\\
&\geq -\tfrac{D(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}w^2\left[\tfrac{\partial}{\partial x}f_4(x,y)\right]^2-\tfrac{D\alpha_1^2(\alpha_1\alpha_4-\alpha_5)}{\varepsilon\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}z^2\left[\tfrac{\partial}{\partial x}f_4(x,y)\right]^2\\
&\geq -\left(\tfrac{\varepsilon}{16}\right)w^2-\left(\tfrac{\varepsilon\alpha_2}{16}\right)z^2,
\end{aligned}
$$

by (i), (ii), (2.13) and (2.14).

Hence

$$V_6 \leq \left(\frac{\varepsilon}{16}\right)w^2+\left(\frac{\varepsilon\alpha_2}{16}\right)z^2. \tag{3.12}$$

Now, let $R(t,x,y,z,w,u)$ denote the sum of the remaining terms in (3.6). Following the procedure indicated in [15], one can conclude that

$$
\begin{aligned}
|R(t,x,y,z,w,u)| \leq\ & D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)(y^2+z^2+w^2+u^2)\\
&+\left|u+\alpha_1 w+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right|.\,|p(t,x,y,z,w,u)|+\tfrac{\partial V_0}{\partial t}
\end{aligned}
\tag{3.13}
$$

for a constant $D_9>0$.

On substituting the estimates (3.7)-(3.13) in (3.6) we find

$$
\begin{aligned}
\dot{V_0} &\leq -\varepsilon_0 u^2-\left(\tfrac{15\varepsilon}{16}\right)w^2-\left(\tfrac{15\varepsilon\alpha_2}{16}\right)z^2-(\varepsilon\alpha_4 E)y^2\\
&+D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)(y^2+z^2+w^2+u^2)\\
&+\left|u+\alpha_1 w+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right|.\,|p(t,x,y,z,w,u)|+\tfrac{\partial V_0}{\partial t}\\
&\leq -\tfrac{1}{2}\min\left\{\varepsilon_0,\tfrac{15\varepsilon}{16},\tfrac{15\varepsilon\alpha_2}{16},\varepsilon\alpha_4 E\right\}(y^2+z^2+w^2+u^2)\\
&+\left|u+\alpha_1 w+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right|.\,|p(t,x,y,z,w,u)|+\tfrac{\partial V_0}{\partial t}
\end{aligned}
$$

provided that

$$D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5) \leq \frac{1}{2}\min\left\{\varepsilon_0, \frac{15\varepsilon}{16}, \frac{15\varepsilon\alpha_2}{16}, \varepsilon\alpha_4 E\right\}. \qquad (3.14)$$

Now we assume that D_9 and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_5$ are so small that (3.14) holds. The case $y, z, w = 0$ is obvious. From (3.1) we get

$$\begin{aligned}\frac{\partial V_0}{\partial t} = \;& e'(t)\left[wf_5(x)+\alpha_1 z f_5(x)+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}yf_5(x)+\delta\int_0^x f_5(\xi)d\xi\right]\\ &+d'(t)\left[wf_4(x,y)+\alpha_1 z f_4(x,y)+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^y f_4(x,\eta)d\eta\right]\\ &+\alpha_1 c'(t)\int_0^z f_3(x,y,\zeta)d\zeta + b'(t)\int_0^w f_2(x,y,z,\rho)d\rho.\end{aligned}$$

The relation (3.3) and assumptions (iv)-(vii)of the theorem show that

$$\frac{\partial V_0}{\partial t} \leq D_{10}\left[b'_+(t)+c'_+(t)+|d'(t)|+|e'(t)|\right]\left[F_5(x)+y^2+z^2+w^2+u^2\right] \leq D_{11}\beta_0 V_0, \qquad (3.15)$$

where D_{10} is a positive constant and $D_{11} = \frac{D_{10}}{D_7}$.

Let

$$D_{13} = \frac{1}{2}\min\left\{\varepsilon_0, \frac{15\varepsilon}{16}, \frac{15\varepsilon\alpha_2}{16}, \varepsilon\alpha_4 E\right\} \text{ and } D_{12} = \max\left\{1, \alpha_1, \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_4}, \delta\right\}.$$

Using the Cauchy inequality and the assumption (ix) of the theorem we obtain

$$\begin{aligned}\dot{V}_0 &\leq -2D_{13}(y^2+z^2+w^2+u^2)+\left|u+\alpha_1 w+\tfrac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right|.|p(t,x,y,z,w,u)|+D_{11}\beta_0V_0\\ &\leq -2D_{13}(y^2+z^2+w^2+u^2)+2D_{12}(y^2+z^2+w^2+u^2)^{\frac{1}{2}}.|p(t,x,y,z,w,u)|+D_{11}\beta_0V_0\\ &\leq -2D_{13}(y^2+z^2+w^2+u^2)+2D_{12}(y^2+z^2+w^2+u^2)^{\frac{1}{2}}\\ &\times\left[p_1(t)+p_2(t)\left\{F_5(x)+y^2+z^2+w^2+u^2\right\}^{\frac{\sigma}{2}}+\Delta(y^2+z^2+w^2+u^2)^{\frac{1}{2}}\right]\\ &+D_{11}\beta_0V_0.\end{aligned} \qquad (3.16)$$

It is obvious from (3.15) and (3.16) that

$$\begin{aligned}\dot{V}_0 &\leq -2D_{13}(y^2+z^2+w^2+u^2)+2D_{12}(y^2+z^2+w^2+u^2)^{\frac{1}{2}}\\ &\times\left[p_1(t)+p_2(t)\left\{F_5(x)+y^2+z^2+w^2+u^2\right\}^{\frac{\sigma}{2}}+\Delta(y^2+z^2+w^2+u^2)^{\frac{1}{2}}\right]+D_{11}\beta_0V_0.\end{aligned}$$

Let's fix Δ in what follows, to satisfy $\Delta = \frac{D_{13}}{2D_{12}}$. By this limitation on Δ, we obtain

$$\begin{aligned}\dot{V_0} &\leq -D_{13}(y^2+z^2+w^2+u^2) + 2D_{12}(y^2+z^2+w^2+u^2)^{\frac{1}{2}} \\ &\times \left[p_1(t) + p_2(t)\left\{F_5(x)+y^2+z^2+w^2+u^2\right\}^{\frac{\sigma}{2}}\right] + D_{11}\beta_0 V_0.\end{aligned}$$

Using the inequality

$$\left[F_5(x)+y^2+z^2+w^2+u^2\right]^{\frac{\sigma}{2}} \leq 1 + \left[F_5(x)+y^2+z^2+w^2+u^2\right]^{\frac{1}{2}}$$

we find

$$\begin{aligned}\dot{V_0} &\leq -D_{13}(y^2+z^2+w^2+u^2) + 2D_{12}(y^2+z^2+w^2+u^2)^{\frac{1}{2}}\left[p_1(t)+p_2(t)\right] \\ &+2D_{12}p_2(t)\left[F_5(x)+y^2+z^2+w^2+u^2\right] + D_{11}\beta_0 V_0.\end{aligned}$$

This completes the proof of the lemma.

4. Completion of the proof of the theorem

The completion proof of the theorem is similar to that of the theorem established in [15] and hence it is omitted.

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