

Comparisons of two integral inequalities with Hermite-Hadamard-Jensen's integral inequality

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1. Introduction

In [4, 6], motivated by Problem 121 in [7, p. 62] (See also [8]), K. S. K. Iyengar proposed proving the following inequality by means of geometrical considerations.

Theorem 1. *Let $f(x)$ be continuous and not identically a constant on $[a, b]$. If M is the upper bound of $|f'(x)|$ in (a, b) , then*

$$\begin{aligned} \frac{1}{4M}[f(b) - f(a)]^2 - \frac{M(b-a)^2}{4} &< \int_a^b f(t) dt - \frac{1}{2}(b-a)[f(a) + f(b)] \\ &< \frac{M(b-a)^2}{4} - \frac{1}{4M}[f(b) - f(a)]^2. \end{aligned} \quad (1.1)$$

Inequality (1.1) is sharp in the sense that it can not be improved.

In [6], among other things, an alternative geometrical proof of inequality (1.1) was provided by G. S. Mahajani. So, we would like to call inequality (1.1) Iyengar-Mahajani's integral inequality.

In [5], the following integral inequalities are established by analytic approach.

Theorem 2. *If $f'(b)$ is not equal to zero, then*

$$\int_a^b f(x) dx \geq (b-a)f(a) + \frac{1}{2} \frac{[f(b) - f(a)]^2}{f'(b)} \quad (1.2)$$

according as $f''(x) \geq 0$ in (a, b) .

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Inequalities in (1.2) are called Kevasa Menon's integral inequalities.

The following integral inequality about convex functions is well known.

Theorem 3. For f being convex on $[a, b]$,

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x) dx < (b-a)\left[\frac{f(a)+f(b)}{2}\right]. \quad (1.3)$$

Inequality (1.3) is often called Hadamard's inequality [1], Hermite-Hadamard's inequality [10], or Jensen-Hadamard's inequality [11], since ones are not aware that this inequality appeared first in a letter by Hermite on November 22, 1881 to the journal *Mathesis* and an extract from that letter was published in *Mathesis* **3** (1883), page 82. For a historical consideration of (1.3), please refer to [3, Chapter 1] and the reference therein. Here, we would like to call inequality (1.3) Hermite-Hadamard-Jensen's integral inequality.

Some comparisons between inequalities (1.1), (1.2) and (3) had been studied in [11]. However, because the inequality for the case of $f''(x) > 0$ in (1.2) was reversedly referenced, all of results related to (1.2) in [11] must be wrong. For example, the inequality (12) in [11], $e^{2b} - e^{2a} < 2(b-a)e^{a+b}$, does not hold for $a = 0$ and $b = 1$.

In this short note, we shall reconsider certain comparisons of Iyengar-Mahajani's integral inequality and Kesava Menon's integral inequality in the case of $f''(x) > 0$ with Hermite-Hadamard-Jensen's integral inequality and then correct some errors in [11].

Our main results are as follows.

Theorem 4. Denote

$$S_0(a, b) = \frac{f(b) - f(a)}{b - a}. \quad (1.4)$$

(1) Let $f'(x)$ is increasing and $|f'(x)| \leq M$ on $[a, b]$.

(a) The right hand side of inequality (1.3) is better than the second inequality in (1.1).

(b) If

$$f'(a)f'(b) > [S_0(a, b)]^2, \quad (1.5)$$

then the left hand side of inequality (1.3) is better than the first inequality in (1.1).

(2) Let $f'(b) \neq 0$ and $f''(x) > 0$ on $[a, b]$.

(a) If

$$\frac{[S_0(a, b)]^2}{f'(b)} < f'(a), \quad (1.6)$$

then the left hand side of (1.3) is better than inequality (1.2).

(b) If

$$\frac{[S_0(a, b)]^2}{f'(b)} > f'\left(\frac{a+b}{2}\right), \quad (1.7)$$

then the left hand side of (1.3) is weaker than inequality (1.2).

Moreover, applications of these theorems to elementary functions are carried out and several inequalities involving mean values are obtained.

2. Proofs of main results

2.1. Comparison between (1.1) and (1.3)

In this subsection, assume that $f'(x)$ is increasing and $|f'(x)| \leq M$ on $[a, b]$. It is easy to see that $M = f'(b)$ or $M = -f'(a)$.

It is clear that the right hand side of inequality (1.3) is better than the second inequality in (1.1), since $\frac{M(b-a)^2}{4} - \frac{1}{4M}[f(b) - f(a)]^2$ is positive.

The first inequality in (1.1) is better (weaker) than the left hand side of (1.3) if and only if

$$\frac{f(a) + f(b)}{2} \geq f\left(\frac{a+b}{2}\right) + \frac{(b-a)M}{4} - \frac{b-a}{4M} \left[\frac{f(b) - f(a)}{b-a} \right]^2. \quad (2.1)$$

By Lagrange's mean value theorem on $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ respectively, we obtain

$$\frac{f(a) + f(b)}{2} = f\left(\frac{a+b}{2}\right) + \frac{b-a}{4} [f'(\xi_2) - f'(\xi_1)], \quad (2.2)$$

where $\xi \in (a, \frac{a+b}{2})$ and $\xi_2 \in (\frac{a+b}{2}, b)$. So, inequality (2.1) is satisfied if

$$f'(x) - f'(y) \geq M - \frac{1}{M} \left[\frac{f(b) - f(a)}{b-a} \right]^2 \quad (2.3)$$

for all $y \in (a, \frac{a+b}{2})$ and $x \in (\frac{a+b}{2}, b)$. Therefore, if

$$f'(b) - f'(a) < M - \frac{1}{M} \left[\frac{f(b) - f(a)}{b-a} \right]^2 \quad (2.4)$$

holds, then the first inequality in (1.1) is weaker than the left hand side of (1.3).

If $M = f'(b)$, then inequality (2.4) is reduced to (1.5). If $M = -f'(a)$, then inequality (2.4) is also reduced to inequality (1.5).

Now it is seen that inequality (1.5) is a sufficient condition such that the left hand side of inequality (1.3) is better than the first inequality in (1.1).

2.2. Comparison between (1.2) and (1.3)

In this subsection, let us suppose that $f'(b) \neq 0$ and $f''(x) > 0$ on $[a, b]$.

It is clear that the first derivative $f'(x)$ is strictly increasing on $[a, b]$.

The left hand side of (1.3) is better (weaker) than inequality (1.2) if and only if

$$(b-a)f(a) + \frac{1}{2} \frac{[f(b) - f(a)]^2}{f'(b)} \leq (b-a)f\left(\frac{a+b}{2}\right), \quad (2.5)$$

which are equivalent with

$$\frac{1}{2} \frac{[f(b) - f(a)]^2}{f'(b)} \leq (b-a) \left[f\left(\frac{a+b}{2}\right) - f(a) \right] = \frac{(b-a)^2}{2} f'(\xi), \quad (2.6)$$

where $\xi \in (a, \frac{a+b}{2})$. Therefore, inequality (2.5) holds true if for $x \in (a, \frac{a+b}{2})$

$$\left[\frac{f(b) - f(a)}{b-a} \right]^2 \frac{1}{f'(b)} \leq f'(x). \quad (2.7)$$

Hence, if inequality (1.6) holds, then the left hand side of (1.3) is better than inequality (1.2); if inequality (1.6) holds, then the left hand side of (1.3) is weaker than inequality (1.2).

3. Applications of inequalities (1.1), (1.2) and (1.3)

3.1.

Let $f(x) = \pm \ln x$ in (1.2). It is obvious that $f''(x) = \mp \frac{1}{x^2}$ for $0 < a \leq x \leq b$. Then a simple computation gives

$$\frac{1}{b-a} \int_a^b \ln x \, dx < \ln a + \frac{b(b-a)}{2} \left[\frac{\ln b - \ln a}{b-a} \right]^2. \quad (3.1)$$

Recall that the so-called identric or exponential mean and logarithmic mean of two positive numbers a and b are defined (see [2, 9]) respectively by

$$I(a, b) = \exp \left(\frac{1}{b-a} \int_a^b \ln x \, dx \right), \quad (3.2)$$

$$L(a, b) = \frac{b-a}{\ln b - \ln a}. \quad (3.3)$$

By this notation it is easy to see that (3.1) transforms into

$$\ln \frac{I(a, b)}{a} < \frac{b(b-a)}{2[L(a, b)]^2}. \quad (3.4)$$

3.2.

For $0 < a < b$, applying $f(x) = -\ln x$ to the first inequality in (1.1) and the right hand side of (1.3) and remarking that $M = \frac{1}{a}$. Straightforward computation yields

$$\frac{\ln a + \ln b}{2} < \frac{1}{b-a} \int_a^b \ln x \, dx < \frac{\ln a + \ln b}{2} + \frac{b-a}{4a} - \frac{a(b-a)}{4} \left[\frac{\ln b - \ln a}{b-a} \right]^2, \quad (3.5)$$

which can be rewritten as

$$0 < \ln \frac{I(a, b)}{G(a, b)} < \frac{b-a}{4a} \left\{ 1 - \frac{a^2}{[L(a, b)]^2} \right\}, \quad (3.6)$$

where $G(a, b) = \sqrt{ab}$ denotes the geometric mean of two positive numbers a and b .

3.3.

Set $f(x) = e^x$ in (1.2), the first inequality of (1.1) and the right hand side of (1.3).

After certain elementary calculus, inequality (1.2) gives us

$$e^{2b} - e^{2a} > 2(b-a)e^{a+b}, \quad (3.7)$$

which can be rewritten as

$$\frac{e^b - e^a}{b-a} > e^{(a+b)/2}. \quad (3.8)$$

This inequality is equivalent to applying the left hand side of (1.3) to the exponential function e^x . So inequality (1.2) and the left hand side of (1.3) is equivalent each other when $f(x) = e^x$.

From the first inequality in (1.1) and the right hand side of (1.3) one obtains

$$\frac{e^a + e^b}{2} - \frac{e^2(b-a)}{4} < \frac{e^b - e^a}{b-a} \frac{3e^b + e^a}{4e^b} \quad (3.9)$$

and

$$\frac{e^b - e^a}{b-a} < \frac{e^a + e^b}{2}. \quad (3.10)$$

Lagrange's mean value theorem applied to the function e^{2x} gives

$$e^{2b} - e^{2a} = 2(b-a)e^{2\xi} > 2(b-a)e^{2a} \quad (3.11)$$

which is much weaker than (3.7).

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