International Journal of Applied Mathematical Sciences.
ISSN 0973-0176 Vol.3 No.1 (2006), pp. 69-81

http://www.gbspublisher.com/ijams.htm

Common Stationary Point Theorems for a Pair of Multi-Valued Mappings Involving Caristi and Contractive Type Conditions

Zeqing Liu
Department of Mathematics, Liaoning Normal University,
P. O. Box 200, Dalian, Liaoning 116029, People's Republic of China
E-mail: zeqingliu@dl.cn

Zhengsheng Chen
Department of Mathematics, Liaoning Normal University,
P. O. Box 200, Dalian, Liaoning 116029, People's Republic of China
E-mail: zhengshengchen@126.com

Shin Min Kang*
Department of Mathematics and Research Institute of Natural Science,
Gyeongsang National University, Jinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr
**Corresponding author*

Hyung Gook Kim
Department of Mathematics, Gyeongsang National University,
Jinju 660-701, Korea
E-mail: emath@nongae.gsnu.ac.kr

Abstract

In this paper, some common stationary point theorems for a pair of multi-valued mappings dealing with combinations of Caristi and contractive type conditions are given. A few stationary point theorems for three classes of multi-valued mappings are established. The results presented in this paper extend, improve and unify some results in the literatures.

AMS subject classification: 54H25.
Keywords and Phrases: Multi-valued mappings, Caristi's fixed point theorem, Contractive type mapping, Stationary point, Common stationary point.

1. Introduction

In 1976, Caristi [4] established the following nice result:

Caristi's Fixed Point Theorem: Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists a lower semi-continuous function $\phi : X \to [0, +\infty)$ satisfying

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \forall x \in X,$$

then T has a fixed point in X.

Since then various extensions of Caristi' fixed point theorem have been studied by Aubin-Sigel [1], Basu [2], Bhaka-Basu [3], Chikkala-Baisnab [5], Dien [6], Kirk [7], Liu-Kim [8], Liu-Xu-Cho [9, 10] and others under different conditions. Recently, Chikkala-Baisnab [5] have investigated multi-valued version of Caristi's fixed point theorem and Liu-Kim [8] have obtained a few common fixed point theorems for a pair of multi-valued mappings which are combinations of Caristi and Banach type conditions.

The purpose of this paper is to establish sufficient conditions which guarantee or suggest the existence, uniqueness and iterative approximation of common stationary point for a pair of multi-valued mappings, which deal with combinations of Caristi and contractive type conditions. Several stationary point theorems for three classes of multi-valued mappings are proved. The results presented in this paper generalize, improve and unify the corresponding results of Bhaka-Basu [3], Chikkala-Baisnab [5], Dien [6] and Liu-Kim [8].

2. Preliminaries

Let (X, d) be a metric space and $\mathbb{R}^+ = [0, +\infty)$. $B(X)$ denotes the family of all nonempty bounded subsets of X. For $A, B \in B(X)$ and $x \in X$, define

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},$$
$$\delta(x, B) = \delta(\{x\}, B),\ d(x, B) = \inf_{y \in B} d(x, y).$$

It is clear that

$$\delta(A, B) = \delta(B, A) \geq 0, \quad \delta(A, B) \leq \delta(A, C) + \delta(C, B), \quad \forall A, B, C \in B(X).$$

Let $F, G : X \to B(X)$ be multi-valued mapping. A point $w \in X$ is said to be a stationary point of F if $Fw = \{w\}$. $S(F)$ denotes the set of all stationary points of F in X. A point $w \in X$ is called a common stationary point of F and G if $Fw = \{w\} = Gw$.

Definition 2.1 ([1]): Let $F : X \to B(X)$ be a multi-valued mapping. A sequence $\{x_n\}_{n\geq 0}$ of the elements in X is called a trajectory starting at x if $x_0 = x$ and $x_{n+1} \in Fx_n$ for all $n \geq 0$. $\Im(F, x)$ denote the set of all such trajectories.

Definition 2.2 ([5]): A multi-valued mapping $F : X \to B(X)$ is said to be orbitally continuous at $x_0 \in X$ if, for any $\{x_n\}_{n\geq 0} \in \Im(F, x_0)$ and $w \in X$, $\lim_{n\to\infty} d(x_n, w) = 0$ implies that $\lim_{n\to\infty} \delta(Fx_n, Fw) = 0$. F is called orbitally continuous in X if it is orbitally continuous at every point of X.

3. Stationary Point Theorems

Now we establish the existence, uniqueness and iterative approximation of common stationary point for a pair of multi-valued mappings, which is a combination of Caristi and contractive type conditions.

Theorem 3.1: Let (X, d) be a complete metric space and $F, G : X \to B(X)$ be orbitally continuous and satisfy

$$\begin{aligned}\delta(Fx, Gy) &\leq r_1 d(x, y) + r_2\delta(x, Fx) + r_3\delta(y, Gy) + r_4\delta(x, Gy) \\ &\quad + r_5\delta(y, Fx) + \phi(x) - \phi(u) + \psi(y) - \psi(v) \qquad (3.1)\end{aligned}$$

for any $x, y \in X, u \in Fx, v \in Gy$, where r_1, r_2, r_3, r_4, r_5 is nonnegative constants satisfying

$$c = \max\{r_1 + 2r_2 + 2r_3 + 3r_4 + r_5, r_1 + 2r_2 + 2r_3 + r_4 + 3r_5\} < 1 \qquad (3.2)$$

and $\phi, \varphi : X \to \mathbb{R}^+$ are two functions. Assume that there exist a positive constant A and a sequence $\{x_n\}_{n\geq 0} \subset X$ such that $x_{2n+1} \in Fx_{2n}$, $\forall n \geq 0$, and $x_{2n} \in Gx_{2n-1}$, $\forall n \geq 1$ and

$$\sup\left\{\sum_{k=1}^{n}[\phi(x_{2k}) - \phi(x_{2k+1}) + \psi(x_{2k-1}) - \psi(x_{2k})] : n \geq 1\right\} \leq A. \qquad (3.3)$$

Then F and G have a unique common stationary point $w \in X$ and $\lim_{n\to\infty} x_n = w$.

Proof. In light of (3.1) and (3.2), we conclude that

$$\begin{aligned}&\delta(Fx_{2k}, Gx_{2k-1}) \\ &\leq r_1 d(x_{2k}, x_{2k-1}) + r_2\delta(x_{2k}, Fx_{2k}) + r_3\delta(x_{2k-1}, Gx_{2k-1}) \\ &\quad + r_4\delta(x_{2k}, Gx_{2k-1}) + r_5\delta(x_{2k-1}, Fx_{2k}) \\ &\quad + \phi(x_{2k}) - \phi(x_{2k+1}) + \psi(x_{2k-1}) - \psi(x_{2k}) \\ &\leq r_1 d(x_{2k}, x_{2k-1}) + r_2\delta(Fx_{2k}, Gx_{2k-1}) + r_3[d(x_{2k-1}, x_{2k}) \\ &\quad + d(x_{2k}, x_{2k+1}) + \delta(x_{2k+1}, Gx_{2k-1})] + r_4[d(x_{2k}, x_{2k+1}) \\ &\quad + \delta(x_{2k+1}, Gx_{2k-1})] + r_5[d(x_{2k-1}, x_{2k}) + \delta(x_{2k}, Fx_{2k})] \\ &\quad + \phi(x_{2k}) - \phi(x_{2k+1}) + \psi(x_{2k-1}) - \psi(x_{2k}) \\ &\leq (r_1 + r_3 + r_5)d(x_{2k-1}, x_{2k}) + (r_3 + r_4)d(x_{2k}, x_{2k+1}) \\ &\quad + (r_2 + r_3 + r_4 + r_5)\delta(Fx_{2k}, Gx_{2k-1}) \\ &\quad + \phi(x_{2k}) - \phi(x_{2k+1}) + \psi(x_{2k-1}) - \psi(x_{2k}), \quad \forall k \geq 1,\end{aligned}$$

that is,

$$\begin{aligned}
&[1-(r_2+r_3+r_4+r_5)]d(x_{2k},x_{2k+1})\\
&\quad\le[1-(r_2+r_3+r_4+r_5)]\delta(Fx_{2k},Gx_{2k-1})\\
&\quad\le(r_1+r_3+r_5)d(x_{2k-1},x_{2k})+(r_3+r_4)d(x_{2k},x_{2k+1})\\
&\qquad+\phi(x_{2k})-\phi(x_{2k+1})+\psi(x_{2k-1})-\psi(x_{2k}),\quad\forall k\ge 1,
\end{aligned}$$

which and (3.3) imply that

$$\begin{aligned}
&[1-(r_2+2r_3+2r_4+r_5)]\sum_{k=1}^{n}d(x_{2k},x_{2k+1})\\
&\qquad-(r_1+r_3+r_5)\sum_{k=1}^{n}d(x_{2k-1},x_{2k})\\
&\quad\le\sum_{k=1}^{n}[\phi(x_{2k})-\phi(x_{2k+1})+\psi(x_{2k-1})-\psi(x_{2k})]\\
&\quad\le A,\quad\forall n\ge 1.
\end{aligned}\tag{3.4}$$

In view of (3.1), we infer that

$$\begin{aligned}
&\delta(Fx_{2k-2},Gx_{2k-1})\\
&\quad\le r_1d(x_{2k-2},x_{2k-1})+r_2\delta(x_{2k-2},Fx_{2k-2})+r_3\delta(x_{2k-1},Gx_{2k-1})\\
&\qquad+r_4\delta(x_{2k-2},Gx_{2k-1})+r_5\delta(x_{2k-1},Fx_{2k-2})\\
&\qquad+\phi(x_{2k-2})-\phi(x_{2k-1})+\psi(x_{2k-1})-\psi(x_{2k})\\
&\quad\le r_1d(x_{2k-2},x_{2k-1})+r_2[d(x_{2k-2},x_{2k-1})+\delta(x_{2k-1},Gx_{2k-1})\\
&\qquad+\delta(Fx_{2k-2},Gx_{2k-1})]+r_3\delta(Fx_{2k-2},Gx_{2k-1})+r_4[d(x_{2k-2},x_{2k-1})\\
&\qquad+\delta(x_{2k-1},Gx_{2k-1})]+r_5[\delta(x_{2k-1},Gx_{2k-1})+\delta(Gx_{2k-1},Fx_{2k-2})]\\
&\qquad+\phi(x_{2k-2})-\phi(x_{2k-1})+\psi(x_{2k-1})-\psi(x_{2k})\\
&\quad\le(r_1+r_2+r_4)d(x_{2k-2},x_{2k-1})\\
&\qquad+(2r_2+r_3+r_4+2r_5)\delta(Fx_{2k-2},Gx_{2k-1})\\
&\qquad+\phi(x_{2k-2})-\phi(x_{2k-1})+\psi(x_{2k-1})-\psi(x_{2k}),\qquad\forall k\ge 1,
\end{aligned}$$

that is,

$$\begin{aligned}
&[1-(2r_2+r_3+r_4+2r_5)]\sum_{k=1}^{n} d(x_{2k-1},x_{2k})\\
&\quad -(r_1+r_2+r_4)\sum_{k=1}^{n-1} d(x_{2k},x_{2k+1})\\
&\le \sum_{k=1}^{n-1}[\phi(x_{2k})-\phi(x_{2k+1})+\psi(x_{2k-1})-\psi(x_{2k})]\\
&\quad + d(x_0,x_1)+\phi(x_0)-\phi(x_1)+\psi(x_1)-\psi(x_2)\\
&\le A+d(x_0,x_1)+\phi(x_0)-\phi(x_1)+\psi(x_1)-\psi(x_2), \quad \forall n\ge 2.
\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we deduce that

$$\begin{aligned}
&(1-c)\sum_{k=1}^{n}[d(x_{2k-1},x_{2k})+d(x_{2k},x_{2k+1})]\\
&\le [1-(r_1+2r_2+2r_3+3r_4+r_5)]\sum_{k=1}^{n} d(x_{2k},x_{2k+1})\\
&\quad + [1-(r_1+2r_2+2r_3+r_4+3r_5)]\sum_{k=1}^{n} d(x_{2k-1},x_{2k})\\
&\le 2A+\phi(x_0)+\psi(x_1)+d(x_0,x_1), \quad \forall n\ge 2,
\end{aligned}$$

which gives that,

$$\begin{aligned}
&\sum_{k=1}^{n}[d(x_{2k-1},x_{2k})+d(x_{2k},x_{2k+1})]\\
&\le \frac{1}{1-c}[2A+\phi(x_0)+\psi(x_1)+d(x_0,x_1)], \quad \forall n\ge 2,
\end{aligned}$$

which ensures that

$$\begin{aligned}
\sum_{k=1}^{n} d(x_k,x_{k+1}) &\le \sum_{k=1}^{n}[d(x_{2k-1},x_{2k})+d(x_{2k},x_{2k+1})]\\
&\le \frac{1}{1-c}[2A+\phi(x_0)+\psi(x_1)+d(x_0,x_1)], \quad \forall n\ge 2,
\end{aligned}$$

which yields that the series $\sum_{k=1}^{\infty} d(x_n,x_{n+1})$ is convergent. Consequently, the sequence $\{x_n\}_{n\ge 0}$ is a Cauchy sequence and converges to some point $w\in X$ by completeness of X. Since F is orbitally continuous, it follows that

$$\delta(Fx_n,Fw)\to 0 \quad \text{and} \quad \delta(Gx_n,Gw)\to 0 \quad \text{as } n\to\infty. \tag{3.6}$$

Note that

$$\begin{aligned}\delta(Gw, w) &\le \delta(Gw, x_{2n+1}) + d(x_{2n+1}, w)\\ &\le \delta(Gw, Gx_{2n}) + d(x_{2n+1}, w), \quad \forall n \ge 1.\end{aligned}$$

Let n tend to infinity, we deduce that $\delta(Gw, w) \le 0$ by (3.6). This is, $Gw = \{w\}$. Similarly, we have $Fw = \{w\}$. Suppose that $v \in X \setminus \{w\}$ is also a common stationary point of F and G. In view of (3.1) we infer immediately that

$$\begin{aligned}d(v, w) &= \delta(Fv, Gw)\\ &\le r_1 d(v, w) + r_2\delta(v, Fv) + r_3\delta(w, Gw) + r_4\delta(v, Gw)\\ &\quad + r_5\delta(w, Fv) + \phi(v) - \phi(v) + \psi(w) - \psi(w)\\ &= (r_1 + r_4 + r_5)d(v, w)\\ &< d(v, w),\end{aligned}$$

which is a contradiction. Hence $v = w$, that is, w is a unique common stationary point of F and G. This completes the proof. ■

Theorem 3.2: Let (X, d) be a complete metric space and $F : X \to B(X)$ be orbitally continuous and satisfy

$$\begin{aligned}\delta(Fx, Fy) &\le r_1 d(x, y) + r_2\delta(x, Fx) + r_3\delta(y, Fy) + r_4\delta(x, Fy)\\ &\quad + r_5\delta(y, Fx) + \phi(x) - \phi(u) + \phi(y) - \phi(v)\end{aligned} \tag{3.7}$$

for all $x, y \in X$, $u \in Fx, v \in Fy$, where r_1, r_2, r_3, r_4, r_5 are nonnegative constants with $r_1 + r_2 + 3r_3 + 2r_4 + 2r_5 < 1$ and $\phi : X \to \mathbb{R}^+$ is a function. Then F has a unique stationary point $w \in X$ and each trajectory in $\Im(F, x)$ for any $x \in X$ converges to w.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n\ge 0} \in \Im(F, x_0)$. Note that $\sum_{k=1}^{n}[\phi(x_{2k}) - \phi(x_{2k+1}) + \phi(x_{2k-1}) - \phi(x_{2k})] \le \phi(x_1)$. The rest of the proof is similar that of Theorem 3.1, and is omitted. This completes the proof. ■

Theorem 3.3: Let (X, d) be a complete metric space and $F, G : X \to B(X)$ be orbitally continuous and satisfy

$$\begin{aligned}\delta(Fx, Gy) &\le r_1 d(x, y) + r_2 d(x, Fx) + r_3 d(y, Gy) + r_4 d(x, Gy)\\ &\quad + r_5 d(y, Fx) + \phi(x) - \phi(u) + \psi(y) - \psi(v)\end{aligned} \tag{3.8}$$

for all $x, y \in X$, $u \in Fx, v \in Gy$, where r_1, r_2, r_3, r_4, r_5 are nonnegative constants satisfying (3.2) and $\phi, \varphi : X \to \mathbb{R}^+$ are two functions. Then F and G have a unique common stationary point $w \in X$ and each trajectory in $\Im(F, x)$ and $\Im(G, y)$ for any $x, y \in X$ converges to w.

Proof. Let x_0, y_0 be arbitrary points in X and $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ be trajectories starting at x_0 and y_0, respectively. From (3.8) we have

$$\begin{aligned} d(x_k, y_k) &\leq \delta(Fx_{k-1}, Gy_{k-1}) \\ &\leq r_1 d(x_{k-1}, y_{k-1}) + r_2 d(x_{k-1}, Fx_{k-1}) + r_3 d(y_{k-1}, Gy_{k-1}) \\ &\quad + r_4 d(x_{k-1}, Gy_{k-1}) + r_5 d(y_{k-1}, Fx_{k-1}) \\ &\quad + \phi(x_{k-1}) - \phi(x_k) + \psi(y_{k-1}) - \psi(y_k) \\ &\leq r_1 d(x_{k-1}, y_{k-1}) + r_2 d(x_{k-1}, x_k) + r_3 d(y_{k-1}, y_k) \\ &\quad + r_4 d(x_{k-1}, y_k) + r_5 d(y_{k-1}, x_k) \\ &\quad + \phi(x_{k-1}) - \phi(x_k) + \psi(y_{k-1}) - \psi(y_k) \\ &\leq (r_1 + r_2 + r_4) d(x_{k-1}, y_{k-1}) + (r_2 + r_3 + r_4 + r_5) d(x_k, y_{k-1}) \\ &\quad + (r_3 + r_4) d(x_k, y_k) \\ &\quad + \phi(x_{k-1}) - \phi(x_k) + \psi(y_{k-1}) - \psi(y_k), \quad \forall k \geq 1, \end{aligned}$$

which yields that

$$\begin{aligned} &[1 - (r_3 + r_4)] d(x_k, y_k) \\ &\leq (r_1 + r_2 + r_4) d(x_{k-1}, y_{k-1}) + (r_2 + r_3 + r_4 + r_5) d(x_k, y_{k-1}) \\ &\quad + \phi(x_{k-1}) - \phi(x_k) + \psi(y_{k-1}) - \psi(y_k), \quad \forall k \geq 1. \end{aligned} \tag{3.9}$$

Using (3.8), we get that

$$\begin{aligned} d(x_{k+1}, y_k) &\leq \delta(Fx_k, Gy_{k-1}) \\ &\leq r_1 d(x_k, y_{k-1}) + r_2 d(x_k, Fx_k) + r_3 d(y_{k-1}, Gy_{k-1}) \\ &\quad + r_4 d(x_k, Gy_{k-1}) + r_5 d(y_{k-1}, Fx_k) \\ &\quad + \phi(x_k) - \phi(x_{k+1}) + \psi(y_{k-1}) - \psi(y_k) \\ &\leq r_1 d(x_k, y_{k-1}) + r_2 d(x_k, x_{k+1}) + r_3 d(y_{k-1}, y_k) \\ &\quad + r_4 d(x_k, y_k) + r_5 d(y_{k-1}, x_{k+1}) \\ &\quad + \phi(x_k) - \phi(x_{k+1}) + \psi(y_{k-1}) - \psi(y_k) \\ &\leq (r_1 + r_3 + r_5) d(x_k, y_{k-1}) + (r_2 + r_5) d(x_{k+1}, y_k) \\ &\quad + (r_2 + r_3 + r_4 + r_5) d(x_k, y_k) \\ &\quad + \phi(x_k) - \phi(x_{k+1}) + \psi(y_{k-1}) - \psi(y_k), \quad \forall k \geq 1, \end{aligned}$$

which gives that

$$\begin{aligned} &[1 - (r_2 + r_5)] d(x_{k+1}, y_k) \\ &\leq (r_1 + r_3 + r_5) d(x_k, y_{k-1}) + (r_2 + r_3 + r_4 + r_5) d(x_k, y_k) \\ &\quad + \phi(x_k) - \phi(x_{k+1}) + \psi(y_{k-1}) - \psi(y_k), \quad \forall k \geq 1. \end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10), we arrive at

$$\begin{aligned}
&[1-(r_2+2r_3+2r_4+r_5)]d(x_k,y_k)+[1-(r_2+r_5)]d(x_{k+1},y_k)\\
&\le (r_1+r_2+r_4)d(x_{k-1},y_{k-1})+(r_1+r_2+2r_3+r_4+2r_5)d(x_k,y_{k-1})\\
&\quad+[\phi(x_{k-1})-\phi(x_k)+\psi(y_{k-1})-\psi(y_k)]\\
&\quad+[\phi(x_k)-\phi(x_{k+1})+\psi(y_{k-1})-\psi(y_k)],\quad \forall k\ge 1.
\end{aligned}$$

This means that

$$\begin{aligned}
&[1-(r_2+2r_3+2r_4+r_5)]\sum_{k=1}^{n}d(x_k,y_k)+[1-(r_2+r_5)]\sum_{k=1}^{n}d(x_{k+1},y_k)\\
&\le (r_1+r_2+r_4)\sum_{k=1}^{n}d(x_{k-1},y_{k-1})\\
&\quad+(r_1+r_2+2r_3+r_4+2r_5)\sum_{k=1}^{n}d(x_k,y_{k-1})\\
&\quad+\sum_{k=1}^{n}[\phi(x_{k-1})-\phi(x_k)+\psi(y_{k-1})-\psi(y_k)]\\
&\quad+\sum_{k=1}^{n}[\phi(x_k)-\phi(x_{k+1})+\psi(y_{k-1})-\psi(y_k)]\\
&\le (r_1+r_2+r_4)\sum_{k=1}^{n}d(x_k,y_k)+d(x_0,y_0)\\
&\quad+(r_1+r_2+2r_3+r_4+2r_5)\sum_{k=1}^{n}d(x_{k+1},y_k)+d(x_1,y_0)\\
&\quad+\phi(x_0)+\phi(x_1)+2\psi(y_2),\quad \forall n\ge 1,
\end{aligned}$$

which leads to

$$\sum_{k=1}^{n}[d(x_k,y_k)+d(x_{k+1},y_k)]\le B,\quad \forall n\ge 1, \tag{3.11}$$

where

$$B=\frac{1}{1-c}[d(x_0,y_0)+d(x_1,y_0)+\phi(x_0)+\phi(x_1)+2\psi(y_2)].$$

Note that

$$\sum_{k=1}^{n}d(x_k,x_{k+1})\le\sum_{k=1}^{n}[d(x_k,y_k)+d(x_{k+1},y_k)]\le B,\quad \forall n\ge 1,$$

which yields that the series $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent and hence the sequence $\{x_n\}_{n\geq 0}$ is a Cauchy sequence and converges to some point $w \in X$ by completeness of X. It follows from (3.11) that $\sum_{n=1}^{\infty} d(x_n, y_n) < \infty$, which means that $\lim_{n\to\infty} y_n = w$. Since F and G are orbitally continuous, it follows that

$$\delta(Fx_n, Fw) \to 0 \quad \text{and} \quad \delta(Gx_n, Gw) \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Note that

$$\begin{aligned} \delta(Fw, w) &\leq \delta(Fw, x_{n+1}) + d(x_{n+1}, w) \\ &\leq \delta(Fw, Fx_n) + d(x_{n+1}, w), \quad \forall n \geq 0. \end{aligned} \tag{3.13}$$

Let n tend to infinity in (3.13), we infer that $\delta(Fw, w) \leq 0$ by (3.12). That is, $Fw = \{w\}$. Similarly, we have $Gw = \{w\}$. Suppose that $x \in X$ is also a stationary point of F different from w. From (3.8) and (3.2), we get immediately that

$$\begin{aligned} d(x, w) &= \delta(Fx, Gw) \\ &\leq r_1 d(x, w) + r_2 d(x, Fx) + r_3 d(w, Gw) + r_4 d(x, Gw) \\ &\quad + r_5 d(w, Fx) + \phi(x) - \phi(x) + \psi(w) - \psi(w) \\ &= (r_1 + r_4 + r_5) d(x, w) \\ &< d(x, w), \end{aligned}$$

which is impossible and hence $x = w$. Consequently, w is a unique stationary point of F. Similarly, we could prove that w is also the unique stationary point of G. This completes the proof. ■

Remark 3.1: (a) If $r_1 = r_2 = r_3 = r_4 = r_5 = 0$, then Theorem 3.3 reduces to Theorem 2.2 of Chikkala and Baisnab [5] and the corresponding theorem of Bhakta and Basu [3].

(b) In case $r_2 = r_3 = r_4 = r_5 = 0$, then Theorem 3.3 reduces to Theorem 2.1 of Liu and Kim [8] and the corresponding theorem of Dien [6].

Example 3.1: Let $X = \{2, 3, 4\}$ with the usual metric. Define $F, G : X \to B(X)$ and $\phi, \psi : X \to \mathbb{R}^+$ by $F2 = F3 = F4 = \{2\}, G2 = G3 = \{2\}, G4 = \{2, 3\}$ and $\phi(t) = 2t, \psi(t) = t, \forall t \in X$. Let r_1, r_2, r_3, r_4 and r_5 be positive constants satisfying (3.2). It is easy to see that the conditions of Theorem 3.3 are fulfilled and F and G have a unique stationary point $2 \in X$.

Theorem 3.4: Let (X, d) be a complete metric space and $F : X \to B(X)$ be orbitally continuous. If there exist $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi : X \to \mathbb{R}^+$ satisfy the following conditions:

(a) f is nondecreasing in $\mathbb{R}^+$;

(b) $f(s + t) \leq f(s) + f(t)$, $\forall s, t \in \mathbb{R}^+$;

(c) $\lim_{m,n\to\infty} f(d(x_m, x_n)) = 0$ implies that $\{x_n\}_{n\geq 0} \subset X$ is a Cauchy sequence;

(d) for any $x \in X$ there exists $y \in Fx$ satisfying

$$f(d(x,y)) \le \phi(x) - \phi(y) + f(\phi(x)) - f(\phi(y)), \tag{3.14}$$

then $S(F) \neq \emptyset$ and for any $x_0 \in X$, each trajectory in $\Im(F, x_0)$ converges to some stationary point of F.

Proof. Let $x_0 \in X$. From (3.14), we may choose a sequence $\{x_n\}_{n\ge 0} \in \Im(F, x_0)$ satisfying

$$f(d(x_n, x_{n+1})) \le \phi(x_n) - \phi(x_{n+1}) + f(\phi(x_n)) - f(\phi(x_{n+1})), \quad \forall n \ge 0. \tag{3.15}$$

Notice that (3.15) means that

$$\begin{aligned} \phi(x_{n+1}) + f(\phi(x_{n+1})) &\le \phi(x_n) + f(\phi(x_n)) - f(d(x_n, x_{n+1})) \\ &\le \phi(x_n) + f(\phi(x_n)), \quad \forall n \ge 0, \end{aligned}$$

which guarantees that the sequence $\{\phi(x_n) + f(\phi(x_n))\}_{n\ge 0}$ is nonincreasing and bounded below by 0. Hence it converges to some $c \ge 0$. By virtue of (a), (b) and (d), we deduce that

$$\begin{aligned} f(d(x_n, x_m)) &\le f\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \\ &\le \sum_{k=n}^{m-1} f(d(x_k, x_{k+1})) \\ &\le \sum_{k=n}^{m-1} [\phi(x_k) - \phi(x_{k+1}) + f(\phi(x_k)) - f(\phi(x_{k+1}))] \\ &= \phi(x_n) + f(\phi(x_n)) - [\phi(x_m) + f(\phi(x_m))], \quad \forall m > n \ge 0, \end{aligned}$$

which implies that $\lim_{m,n\to\infty} f(d(x_m, x_n)) = 0$. It follows from (iii) that $\{x_n\}_{n\ge 0}$ is a Cauchy sequence in X. Thus there exists some $w \in X$ such that $\lim_{n\to\infty} x_n = w$ by completeness of X. From the orbital continuity of F, we infer that

$$\begin{aligned} \delta(w, Fw) &\le d(w, x_{n+1}) + \delta(x_{n+1}, Fw) \\ &\le d(w, x_{n+1}) + \delta(Fx_n, Fw) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $Fw = \{w\}$. This completes the proof. ■

Remark 3.2 ([8]): The following functions satisfy (a)-(c) in Theorem 3.4:

$$t^{\frac{1}{2}}, \quad \ln(1+t), \quad \frac{t}{1+t}, \quad rt, \quad \forall t \in \mathbb{R}^+,$$

where $r > 0$ is a constant.

Theorem 3.5: Let (X, d) be a complete metric space and $F : X \to B(X)$ be orbitally continuous. If there exist $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi : X \to \mathbb{R}^+$ satisfy (a)-(c) and

$$f(\delta(x, Fx)) \le \phi(x) - \phi(y) + f(\phi(x)) - f(\phi(y)), \quad \forall x \in X, y \in Fx, \qquad (3.16)$$

then $S(F) \neq \emptyset$ and for any $x_0 \in X$, each trajectory in $\Im(F, x_0)$ converges to some stationary point of F.

Proof. Note that $d(x, y) \le \delta(x, Fx), \forall x \in X, y \in Fx$. That is, (3.16) implies (3.14). Thus Theorem 3.5 follows from Theorem 3.4. This completes the proof. ■

The examples 3.2 and 3.3 below reveal that the stationary points of the mapping F in Theorems 3.4 and 3.5, respectively, may not be unique.

Example 3.2: Let $X = [0, 1]$ with the usual metric. Define $F : X \to B(X)$, $\phi : X \to \mathbb{R}^+$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\begin{aligned} Fx &= \Big[\frac{x^2+x}{2}, \frac{x^2+2x}{3}\Big], \quad \phi(x) = x, \quad \forall x \in X, \\ f(t) &= t, \quad \forall t \in \mathbb{R}^+. \end{aligned}$$

It is clear that for each $x \in X$ there exists $y = \frac{x^2+x}{2} \in Fx$ satisfying

$$\begin{aligned} f(d(x, y)) &= x - \frac{x^2+x}{2} \le 2\Big(x - \frac{x^2+x}{2}\Big) \\ &= \phi(x) - \phi(y) + f(\phi(x)) - f(\phi(y)). \end{aligned}$$

Now we verify that F is orbitally continuous in X. Let $x_0 \in [0, 1)$. For each sequence $\{x_n\}_{n\ge 0} \in \Im(F, x_0)$, it follows that

$$\begin{aligned} 0 &\le \frac{x_{n-1}^2 + x_{n-1}}{2} \le x_n \le \frac{x_{n-1}^2 + 2x_{n-1}}{3} \\ &\le x_{n-1} \le \ldots \le x_0 < 1, \quad \forall n \ge 1. \end{aligned} \qquad (3.17)$$

It is easy to see that the sequence $\{x_n\}_{n\ge 0}$ converges to some $t \in [0, 1)$. Letting $n \to \infty$ in (3.17), we have

$$0 \le \frac{t^2+t}{2} \le t \le \frac{t^2+2t}{3} \le t \le x_0 < 1,$$

which implies that $t = 0$ and therefore $x_n \to 0$ as $n \to \infty$. It follows that

$$\begin{aligned} \delta(Fx_n, F0) &= \delta\Big(\Big[\frac{x_n^2+x_n}{2}, \frac{x_n^2+2x_n}{3}\Big], 0\Big) \\ &= \frac{x_n^2+2x_n}{3} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Let $x_0 = 1$. For each sequence $\{x_n\}_{n\geq 0} \in \Im(F, x_0)$, we conclude easily that $x_n = x_0$ and $\delta(Fx_n, Fx_0) = 0, \forall n \geq 1$. Therefore F is orbitally continuous in X. Thus the conditions of Theorem 3.4 are satisfied. However, $S(F) = \{0, 1\}$, that is, F has two stationary points 0 and 1.

Example 3.3: Let (X, d) and f be as in Example 3.2. Define $F : X \to X$ and $\phi : X \to \mathbb{R}^+$ by

$$Fx = \begin{cases} \{\frac{1}{9}, \frac{1}{3}\}, & x \in \left[0, \frac{1}{9}\right) \\ \{x\}, & x \in \left[\frac{1}{9}, \frac{1}{3}\right) \\ \{\frac{x}{3}\}, & x \in \left[\frac{1}{3}, 1\right] \end{cases} \quad \text{and} \quad \phi(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{9}\right) \\ \frac{1}{2}, & x \in \left[\frac{1}{9}, \frac{1}{3}\right) \\ 2x, & x \in \left[\frac{1}{3}, 1\right]. \end{cases}$$

In order to show that F and ϕ satisfy (3.16), we have to consider the following cases:

Case 1. Suppose that $x \in [0, \frac{1}{9})$ and $y \in Fx$. It follows that

$$\delta(x, Fx) = \frac{1}{3} - x \leq 1 = 2\Big(1 - \frac{1}{2}\Big) = \phi(x) - \phi(y) + f(\phi(x)) - f(\phi(y));$$

Case 2. Suppose that $x \in [\frac{1}{9}, \frac{1}{3})$ and $y \in Fx$. It is clear that

$$\delta(x, Fx) = 0 = \phi(x) - \phi(y) + f(\phi(x)) - f(\phi(y));$$

Case 3. Suppose that $x \in [\frac{1}{3}, 1)$ and $y \in Fx$. It is easy to verify that

$$\delta(x, Fx) = \frac{2x}{3} \leq 2\Big(2x - \frac{1}{2}\Big) = \phi(x) - \phi(y) + f(\phi(x)) - f(\phi(y));$$

Case 4. Suppose that $x = 1$ and $y \in F1 = \{\frac{1}{3}\}$. It follows that

$$\delta(1, F1) = \frac{2}{3} \leq \frac{8}{3} = \phi(1) - \phi\Big(\frac{1}{3}\Big) + f(\phi(1)) - f\Big(\phi\Big(\frac{1}{3}\Big)\Big).$$

Hence all the conditions of Theorem 3.5 are satisfied. However $S(F) = [\frac{1}{9}, \frac{1}{3})$, that is, the stationary points of F is not unique.

Acknowledgement

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2006).

References

[1] J. P. Aubin and J. Siegel, Fixed point and stationary points of dissipate multi-valued maps, *Proc. Amer. Math. Soc.*, **78**, pp. 391–398, 1980.

[2] T. Basu, Some fixed point theorems for multi-valued mappings using Caristi and Kirk type condition, *J. Indian Acad. Math.*, **8**, pp. 58–68, 1986.

[3] P. C. Bhakta and T. Basu, Some fixed point theorems on metric spaces, *J. Indian Math. Soc.*, **45**, pp. 399–404, 1981.

[4] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Tran. Amer. Math. Soc.*, **215**, pp. 241–251, 1976.

[5] R. Chikkala and A. P. Baisnab, A simultaneous fixed point theorems with application in control theory, *Indian J. Pure. Appl. Math.*, **21**, pp. 144–149, 1990.

[6] N. H. Dien, Some remarks on common fixed point theorems, *J. Math. Anal. Appl.*, **36**, pp. 81–86, 1977.

[7] W. A. Kirk, Caristi's fixed point theorem and metric convexity, *Colloq. Math.*, **36**, pp. 81–86, 1977.

[8] Z. Liu and J. K. Kim, Stationary point theorems for multi-valued mappings in metric spaces, *Fixed Point Theory And Applications*, **3**, pp. 7–17, 2001.

[9] Z. Liu, Y. Xu and Y. J. Cho, On characterizations of fixed points, *J. Math. Anal. Appl.*, **222**, pp. 494–504, 1998.

[10] Z. Liu , Y. Xu and Y. J. Cho, Common fixed point theorems for single-valued and multi-valued mappings, *Publ. Math. Debrecen*, **61**, pp. 439–453, 2002.