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New Inequalities Related to the Jensen-type Inequalities with Repetitive Sample

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Abstract

This paper is devoted to analyzing the monotonicity of some weighted differences between any two terms in a chain of Jensen-type inequalities with repetitive sample. The problem is reduced to the solvability of some weight equations and/or weight inequalities. Complete characterization of the structure of solutions to the weight equations are also given.

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1. Introduction and Main Results

Throughout this paper, we fix a non-empty convex subset I of a real linear space X and a real-valued function f defined on I. For any $n, k \in \mathbb{N}$, $x_i \in I$ and $t_i > 0$

($i = 1, 2, \dots, n$), set

$$\begin{aligned} M(n,k,f) &\equiv M(n,k,f,t_1,\dots,t_n,x_1,\dots,x_n) \\ &\quad \triangleq \sum_{1\le i_1\le\cdots\le i_k\le n} \left(\sum_{j=1}^{k} t_{i_j}\right) f\left(\sum_{j=1}^{k} t_{i_j}x_{i_j} \Big/ \sum_{j=1}^{k} t_{i_j}\right), \\ J(n,k,f) &\equiv J(n,k,f,t_1,\dots,t_n,x_1,\dots,x_n) \triangleq \tfrac{1}{C_{n+k-1}^{k-1}} M(n,k,f), \\ J(n,\infty,f) &\equiv J(n,\infty,f,t_1,\dots,t_n,x_1,\dots,x_n) \triangleq T_n f\left(\sum_{i=1}^{n} t_i x_i \Big/ T_n\right), \\ \bar{f}_{k,n} &\triangleq J(n,k,f,1,\dots,1,x_1,\dots,x_n) = \tfrac{n}{C_{n+k-1}^{k}} \sum_{1\le i_1\le\cdots\le i_k\le n} f\left(\tfrac{1}{k}\sum_{j=1}^{k} x_{i_j}\right), \end{aligned} \tag{1.1}$$

where $T_j = \sum_{i=1}^{j} t_i$ ($j = 1, 2, \dots, n$). It is easy to see that $J(n,k,f)/T_n$ represents a weighted mean of f with repetitive samples.

In [3], when f is a mid-convex function, Pečarić and Svrtan obtained the monotonicity of $\bar{f}_{k,n}$ with respect to k, i.e.,

$$nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \cdots \le \bar{f}_{k+1,n} \le \bar{f}_{k,n} \le \cdots \le \bar{f}_{1,n} = \sum_{i=1}^{n} f(x_i), \qquad \forall\, k \in \mathbb{N}. \tag{1.2}$$

It is easy to show that (1.2) can be strengthened as follows:

Theorem 1.1: Let f be a convex function on I. Then for any positive integers k and r with $k < r$, it holds

$$\begin{aligned} T_n f\left(\sum_{i=1}^{n} t_i x_i \Big/ T_n\right) &= J(n,\infty,f) \le \cdots \le J(n,r,f) \le \cdots \le J(n,k+1,f) \\ &\le J(n,k,f) \le \cdots \le J(n,1,f) = \sum_{i=1}^{n} t_i f(x_i). \end{aligned} \tag{1.3}$$

Clearly, (1.3) is a refinement of the following classical Jensen inequality by inserting infinitely many terms:

$$T_n f\left(\sum_{i=1}^{n} t_i x_i \Big/ T_n\right) = J(n,\infty,f) \le J(n,1,f) = \sum_{i=1}^{n} t_i f(x_i). \tag{1.4}$$

The main purpose of this paper is to analyze the monotonicity of weighted differences between $J(n,k,f)$ and $J(n,r,f)$, or equivalently between $M(n,k,f)$ and $M(n,r,f)$.

For this purpose, we fix two functions $u, v : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$. For any positive integers k, r, set

$$\begin{aligned} G(n,k,r,f) &= u(n,k,r)M(n,k,f) - v(n,k,r)M(n,r,f), \\ A(n,k,r) &= \Big(u(n,k,r) - u(n-1,k,r)\Big)C_{r-1}^{k-1}C_{n+k-1}^{k} - v(n,k,r)C_{r-1}^{k}C_{n+r-2}^{r-1} \\ &\quad + \Big(v(n-1,k,r) - v(n,k,r)\Big)(n+k-1)C_r^k C_{n+r-2}^{r}\Big/ (n-1), \\ B(n,k,r) &= u(n,k,r)C_{n+k-1}^{k-1} - v(n,k,r)C_{n+r-1}^{r-1}. \end{aligned} \tag{1.5}$$

Obviously, $G(n,k,r,f)$ is a weighted difference between $M(n,k,f)$ and $M(n,r,f)$ (with weight functions u and v). We have the following monotonicity result on $G(n,k,r,f)$ with respect to n:

Theorem 1.2: Let f be a convex function on $I, n \geq 2$, k and r ($k < r$) be positive integers. Assume v is non-negative and

$$v(n,k,r) \geq v(n-1,k,r). \tag{1.6}$$

Then

$$G(n,k,r,f) \geq G(n-1,k,r,f) \tag{1.7}$$

provided one of the following three classes of conditions holds

(1)

$$A(n,k,r) = B(n,k,r) = 0; \tag{1.8}$$

(2)

$$A(n,k,r) \geq 0, \quad B(n,k,r) \geq 0 \tag{1.9}$$

and f is non-negative; or

(3)

$$A(n,k,r) \leq 0, \quad B(n,k,r) \leq 0 \tag{1.10}$$

and f is non-positive.

Note, however, that in Theorem 1.2, we assume $r < \infty$, which exclude the important case of $r = \infty$.

In order to include the case of $r = \infty$, we fix two functions $y, z : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$. For any positive integer k, put

$$\begin{aligned} G(n,k,\infty,f) &= y(n,k)M(n,k,f) - z(n,k)J(n,\infty,f), \\ C(n,k) &= y(n,k) - y(n-1,k), \\ D(n,k) &= \frac{y(n,k)C_{n+k-1}^{k-1} - z(n,k)}{C_{n+k-1}^{k-1}}, \\ E(n,k) &= \frac{z(n-1,k)C_{n+k-1}^{k-1} - z(n,k)C_{n+k-2}^{k-1}}{C_{n+k-1}^{k-1}}. \end{aligned} \tag{1.11}$$

Clearly, $G(n,k,\infty,f)$ is a weighted difference between $M(n,k,f)$ and $J(n,\infty,f)$ (with weight functions y, z). We have the following monotonicity result on $G(n,k,\infty,f)$ with respect to n:

Theorem 1.3: Let f be a convex function defined on I, $z \geq 0$ and $n \geq 2$. Then for any positive integer k,

$$G(n,k,\infty,f) \geq G(n-1,k,\infty,f). \tag{1.12}$$

provided one of the following three classes of conditions holds

(1)

$$C(n,k) = D(n,k) = E(n,k) = 0; \tag{1.13}$$

(2)

$$C(n,k) \geq 0, \quad D(n,k) \geq 0, \quad E(n,k) \geq 0 \tag{1.14}$$

and f is non-negative; or

(3)

$$C(n,k) \leq 0, \quad D(n,k) \leq 0, \quad E(n,k) \leq 0 \tag{1.15}$$

and f is non-positive.

Remark 1.4: Theorems 1.2 and 1.3 extend our previous results ([8]) on the Jensen-type inequalities with non-repetitive sample to the present case of repetitive sample. More precisely, in [8] the summation $\sum\limits_{1\leq i_1\leq\cdots\leq i_k\leq n}$ in the definition of $M(n,k,f)$ is replaced by $\sum\limits_{1\leq i_1<\cdots<i_k\leq n}$. As we shall see later, the analysis for the case of repetitive sample is much more complicated. As far as we know, there are no monotonicity results in the references on the weighted difference related to the Jensen-type inequalities with repetitive sample.

Remark 1.5: We refer to [4], [5] and [6] for a few earlier literatures on the monotonicity related to Jensen-type inequalities with non-repetitive sample. Also, we refer to [1], [2] and [7] for updated results on Jensen-type inequality for convex functions.

Remark 1.6: In order to ensure the monotonicity in Theorems 1.2 and 1.3, we introduce some (sufficient) assumptions on the weight functions. It is easy to see that those assumptions are not necessary. It would be quite interesting to give a sufficient and necessary conditions on the weight functions. But this is a by now open problem.

Note also that, if without suitable restrictions on the weight functions, there are no monotonicity for the weighted difference $G(n,k,r,f)$ with respect to n. Indeed, we have the following simple counterexample.

Example 1.7: Take $I = [0,3]$. It is clear that the function

$$f(x) = \begin{cases} 1-4x, & 0 \leq x < 1/4, \\ 0, & 1/4 \leq x \leq 3 \end{cases}$$

is convex in I. Take $t_1 = t_2 = t_3 = 1$, $x_1 = 3$, $x_2 = 2$ and $x_3 = 0$. Then, noting that $f(x) = 0$ whenever $x \in [1/4, 3]$, we get

$$\begin{aligned}
J(3,2,f) - J(3,8,f) &= \frac{1}{C_{3+2-1}^{2-1}} \sum_{1\le i\le j\le 3} (t_i + t_j) f\left(\frac{t_i x_i + t_j x_j}{t_i + t_j}\right) \\
&\quad - \frac{1}{C_{3+8-1}^{8-1}} \sum_{1\le i_1\le\cdots\le i_8\le 3} \left(\sum_{j=1}^{8} t_{i_j}\right) f\left(\sum_{j=1}^{8} t_{i_j} x_{i_j} \Big/ \sum_{j=1}^{8} t_{i_j}\right) \\
&= \frac{2}{4} f(x_3) - \frac{8}{120} f(x_3) = \frac{13}{30}, \\
J(2,2,f) - J(2,8,f) &= \frac{1}{C_{2+2-1}^{2-1}} \sum_{1\le i\le j\le 2} (t_i + t_j) f\left(\frac{t_i x_i + t_j x_j}{t_i + t_j}\right) \\
&\quad - \frac{1}{C_{2+8-1}^{8-1}} \sum_{1\le i_1\le\cdots\le i_8\le 2} \left(\sum_{j=1}^{8} t_{i_j}\right) f\left(\sum_{j=1}^{8} t_{i_j} x_{i_j} \Big/ \sum_{j=1}^{8} t_{i_j}\right) \\
&= 0.
\end{aligned}$$

Therefore

$$J(3,2,f) - J(3,8,f) > J(2,2,f) - J(2,8,f).$$

On the other hand, taking $x_1 = 0$, $x_2 = 3$ and $x_3 = 2$, then similar to the above computation, we get the following converse inequality:

$$J(3,2,f) - J(3,8,f) = 13/30 < J(2,2,f) - J(2,8,f) = 4/9.$$

Hence, $J(n,2,f) - J(n,8,f)$ does not have monotonicity with respect to n.

Remark 1.8: Theorems 1.2–1.3 reduce the monotonicity of weighted differences to the solvability of suitable weight equations and/or weight inequalities. To the best of our knowledge, the following two weight inequalities (with unknowns (u,v) and (y,z), respectively)

$$\begin{cases} A(n,k,r) \ge 0, \\ B(n,k,r) \ge 0, \end{cases}$$

and

$$\begin{cases} C(n,k) \ge 0, \\ D(n,k) \ge 0, \\ E(n,k) \ge 0, \end{cases}$$

are new, and very little is known about their solutions. (The other weight inequalities appeared in Theorems 1.2–1.3 can be easily reduced to the above ones). It would be interesting to analyze the structure of their solutions. But this is by now an open problem.

Finally, we have the following two results, which characterize the structure of solutions to the weight equations appeared in Theorems 1.2–1.3.

Theorem 1.9: Let $n \geq 2$, k and r ($k < r$) be given positive integers. Then $u(n,k,r)$ and $v(n,k,r)$ satisfy the weight equations

$$A(n,k,r) = B(n,k,r) = 0 \tag{1.16}$$

if and only if

$$u(n,k,r) = \frac{C_{n+r-1}^{r-1}}{C_{n+k-1}^{k-1}} v(n,k,r). \tag{1.17}$$

Theorem 1.10: Let $n \geq 2$ and k be given positive integers. Then $y(n,k)$ and $z(n,k)$ satisfy satisfy the weight equations

$$C(n,k) = D(n,k) = E(n,k) = 0 \tag{1.18}$$

if and only if

$$\begin{cases} z(n,k) = C_{n+k-1}^{k-1} y(n,k), \\ y(n,k) = y(n-1,k). \end{cases} \tag{1.19}$$

The rest of this paper is organized as follows. In Section 2, we show some preliminary results. The proof of Theorems 1.1–1.3 will be given in Sections 3–5 respectively. In Section 6, we shall prove Theorems 1.9 and 1.10.

2. Several preliminaries

In the sequel, when $1 \leq i_1 \leq \cdots \leq i_q \leq n$ and $\{i'_1, \ldots, i'_p\}$ is a subsequence of $\{i_1, \ldots, i_q\}$ ($p \leq q$), we shall denote it by $\{i'_1, \ldots, i'_p\} \subset \{i_1, \ldots, i_q\}$. Clearly, there are C_q^p such subsequences in $\{i_1, \ldots, i_q\}$.

First of all, we need the following known result (*see* Lemma 3.2 in [8]).

Lemma 2.1: Let f be a convex function defined on $\mathbb{I}$. Then for any $m, \ell \in \mathbb{N}$ with $1 \leq \ell \leq m \leq n$, it holds

$$f\left(\sum_{i=1}^{m} t_i x_i \Big/ T_m\right) \leq \frac{1}{C_{m-1}^{\ell-1} T_m} \sum_{\{i_1,\ldots,i_\ell\} \subset \{1,\ldots,m\}} \left(\sum_{j=1}^{\ell} t_{i_j}\right) f\left(\sum_{j=1}^{\ell} t_{i_j} x_{i_j} \Big/ \sum_{j=1}^{\ell} t_{i_j}\right).$$

Next, we show the following two lemmas, which are repetitive sample counterparts of Lemmas 3.3–3.4 in [8].

Lemma 2.2: Let $k, r, m \in \mathbb{N}$ satisfy $1 \leq k \leq r \leq m$. Then for any function $g: \mathbb{N}^k \to \mathbb{R}$, it holds

$$\sum_{1 \leq i_1 \leq \cdots \leq i_r \leq m} \sum_{\{i'_1,\ldots,i'_k\} \subset \{i_1,\ldots,i_r\}} g(r'_1, \ldots, i'_k) = \frac{C_{m+r-1}^{r} C_r^k}{C_{m+k-1}^{k}} \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq m} g(j_1, \ldots, j_k). \tag{2.1}$$

Proof. By symmetry, it is easy to see that

$$\sum_{1\le i_1\le\cdots\le i_r\le m}\ \sum_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_r\}} g(i'_1,\ldots,i'_k) \tag{2.2}$$

is equal to some integer times of the following summation

$$\sum_{1\le j_1\le\cdots\le j_k\le m} g(j_1,\ldots,j_k). \tag{2.3}$$

Let us compute this integer. From the elementary combinational theory, it is easy to see that there are $C_{m+r-1}^{r}C_r^k$ terms (including repeated terms) in (2.2); while in (2.3) there are C_{m+k-1}^{k} terms. Therefore, the desired integer is equal to

$$\frac{C_{m+r-1}^{r}C_r^k}{C_{m+k-1}^{k}}.$$

This completes the proof of Lemma 2.2. ■

Lemma 2.3: Let f be a convex function defined on I. Then for any positive integers k and r with $k<r$, it holds

$$M(n,r,f)\le\frac{C_{n+r-1}^{r}C_r^k}{C_{r-1}^{k-1}C_{n+k-1}^{k}}M(n,k,f). \tag{2.4}$$

Proof. By Lemmas 2.1–2.2, we have

$$\begin{aligned}
&M(n,r,f)\\
&=\sum_{1\le i_1\le\cdots\le i_r\le n}\left(\sum_{j=1}^{r}t_{i_j}\right)f\left(\sum_{j=1}^{r}t_{i_j}x_{i_j}\Big/\sum_{j=1}^{r}t_{i_j}\right)\\
&\le\sum_{1\le i_1\le\cdots\le i_r\le n}\frac{1}{C_{r-1}^{k-1}}\sum_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_r\}}\left(\sum_{j=1}^{k}t_{i'_j}\right)f\left(\sum_{j=1}^{k}t_{i'_j}x_{i'_j}\Big/\sum_{j=1}^{k}t_{i'_j}\right)\\
&=\frac{C_{n+r-1}^{r}C_r^k}{C_{r-1}^{k-1}C_{n+k-1}^{k}}\sum_{1\le j_1\le\cdots\le j_k\le n}\left(\sum_{q=1}^{k}t_{j_q}\right)f\left(\sum_{q=1}^{k}t_{j_q}x_{j_q}\Big/\sum_{q=1}^{k}t_{j_q}\right)\\
&=\frac{C_{n+r-1}^{r}C_r^k}{C_{r-1}^{k-1}C_{n+k-1}^{k}}M(n,k,f).
\end{aligned}$$

This completes the proof of Lemma 2.3. ■

Finally, we need the following simple combinational identity, which is possibly known but we do not find an exact reference.

Lemma 2.4: For any positive integers n, r, k with $k < r$, it holds

$$\sum_{p=k}^{r-1} C_{n+p-2}^{p} C_{p}^{k} = \frac{n-1}{n+k-1} C_{n+r-2}^{r-1} C_{r-1}^{k}. \tag{2.5}$$

Proof. The left of (2.5) can be rewritten as

$$\begin{aligned} &\sum_{p=k}^{r-1} C_{n+p-2}^{p} C_{p}^{k} = \sum_{p=k}^{r-1} \frac{(n+p-2)!}{(n-2)!k!(p-k)!} \\ &= (n-1) C_{n+r-2}^{r-1} C_{r-1}^{k} \left[\sum_{p=k}^{r-2} \frac{(r-k-1)\cdots(p-k+1)}{(n+r-2)\cdots(n+p-1)} + \frac{1}{n+r-2} \right]. \end{aligned} \tag{2.6}$$

However

$$\begin{aligned} &\sum_{p=k}^{r-2} \frac{(r-k-1)\cdots(p-k+1)}{(n+r-2)\cdots(n+p-1)} + \frac{1}{n+r-2} \\ &= \frac{(r-k-1)\cdots 3\cdot 2}{(n+r-2)\cdots(n+k)} \left(\frac{1}{n+k-1} + 1 \right) + \frac{(r-k-1)\cdots 4\cdot 3}{(n+r-2)\cdots(n+k+1)} + \cdots \\ &\quad + \frac{r-k-1}{(n+r-2)(n+r-3)} + \frac{1}{n+r-2} \\ &= \frac{(r-k-1)\cdots 4\cdot 3}{(n+r-2)\cdots(n+k+1)} \left(\frac{2}{n+k-1} + 1 \right) + \cdots \\ &\quad + \frac{r-k-1}{(n+r-2)(n+r-3)} + \frac{1}{n+r-2} \\ &= \cdots = \frac{r-k-1}{(n+r-2)(n+r-3)} \left(\frac{r-k-2}{n+k-1} + 1 \right) + \frac{1}{n+r-2} \\ &= \frac{1}{n+r-2} \left(\frac{r-k-1}{n+k-1} + 1 \right) = \frac{1}{n+k-1}. \end{aligned} \tag{2.7}$$

Now, combining (2.6) and (2.7), we obtain the desired identity (2.5). This completes the proof of Lemma 2.4. ∎

3. Proof of Theorem 1.1

For $k = 1, 2, \cdots$, using Lemma 2.3, we obtain

$$\begin{aligned} &C_{n+k}^{k} J(n, k+1, f) = M(n, k+1, f) \le \frac{C_{n+k}^{k+1} C_{k+1}^{k}}{C_{k}^{k-1} C_{n+k-1}^{k}} M(n, k, f) \\ &= \frac{n+k}{k} C_{n+k-1}^{k-1} J(n, k, f) = C_{n+k}^{k} J(n, k, f), \end{aligned}$$

i.e.,

$$J(n,k+1,f) \leq J(n,k,f), \qquad k = 1,2,\cdots .$$

It remains to show that $J(n,\infty,f) \leq J(n,k,f)$ for $k = 1,2,\cdots$. For this, noting the following simple combinatorial identities

$$\sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \sum_{j=1}^{k} t_{i_j} = C_{n+k-1}^{k-1} T_n,$$
$$\sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \sum_{j=1}^{k} t_{i_j} x_{i_j} = C_{n+k-1}^{k-1} \sum_{i=1}^{n} x_i t_i,$$

and using (1.4), we see that

$$\begin{aligned} J(n,\infty,f) &= T_n f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \\ &= T_n f\left(\frac{1}{C_{n+k-1}^{k-1} T_n} \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \left(\sum_{j=1}^{k} t_{i_j}\right)\left(\sum_{j=1}^{k} t_{i_j} x_{i_j} \Big/ \sum_{j=1}^{k} t_{i_j}\right)\right) \\ &\leq \frac{1}{C_{n+k-1}^{k-1}} \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \left(\sum_{j=1}^{k} t_{i_j}\right) f\left(\sum_{j=1}^{k} t_{i_j} x_{i_j} \Big/ \sum_{j=1}^{k} t_{i_j}\right) = J(n,k,f). \end{aligned}$$

This completes the proof of Theorem 1.1. ■

4. Proof of Theorem 1.2

The proof is divided into several steps.

Step 1. For any positive integer s, we define a function L by

$$L(n,s,f) = \begin{cases} \sum\limits_{1\leq j_1\leq\cdots\leq j_{s-1}\leq n} \left(t_n + \sum\limits_{p=1}^{s-1} t_{j_p}\right) f\left(\left(t_n x_n + \sum\limits_{p=1}^{s-1} t_{j_p} x_{j_p}\right) \Big/ \left(t_n + \sum\limits_{p=1}^{s-1} t_{j_p}\right)\right), & s > 1, \\ t_n f(x_n), & s = 1. \end{cases} \tag{4.1}$$

Recall the definition of $M(n,k,f)$ in (1.1), we see that

$$\begin{aligned} &M(n,k,f)\\ &=\sum_{1\le j_1\le\cdots\le j_k\le n-1}\left(\sum_{p=1}^{k}t_{j_p}\right)f\left(\sum_{p=1}^{k}t_{j_p}x_{j_p}\Big/\sum_{p=1}^{k}t_{j_p}\right)\\ &\quad+\sum_{1\le j_1\le\cdots\le j_{k-1}\le n}\left(t_n+\sum_{p=1}^{k-1}t_{j_p}\right)f\left(\left(t_nx_n+\sum_{p=1}^{k-1}t_{j_p}x_{j_p}\right)\Big/\left(t_n+\sum_{p=1}^{k-1}t_{j_p}\right)\right)\\ &=M(n-1,k,f)+L(n,k,f). \end{aligned} \tag{4.2}$$

By (4.2) and noting the definition of $G(n,k,r,f)$ in (1.5), we see that

$$\begin{aligned} &G(n,k,r,f)-G(n-1,k,r,f)\\ &=[u(n,k,r)-u(n-1,k,r)]M(n-1,k,f)+u(n,k,r)L(n,k,f)\\ &\quad+[v(n-1,k,r)-v(n,k,r)]M(n-1,r,f)-v(n,k,r)L(n,r,f). \end{aligned} \tag{4.3}$$

Step 2. We now use $M(n-1,k,f)$ and/or $L(n,k,f)$ to bound $M(n-1,r,f)$ and $L(n,r,f)$ (Recall that $k<r$).

To bound $M(n-1,r,f)$ by $M(n-1,k,f)$ is quite easy. Indeed, by Lemma 2.3, one obtains that

$$M(n-1,r,f)\le\frac{C_{n+r-2}^{r}C_r^k}{C_{r-1}^{k-1}C_{n+k-2}^{k}}M(n-1,k,f). \tag{4.4}$$

However, it is much more complicated to bound $L(n,r,f)$ by $M(n-1,k,f)$ and $L(n,k,f)$. For this purpose, by (4.1), we rewrite $L(n,r,f)$ as follows:

$$\begin{aligned} &L(n,r,f)\\ &=\sum_{w=0}^{k-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\\ &\qquad\times f\left(\left((r-w)t_nx_n+\sum_{q=1}^{w}t_{i_q}x_{i_q}\right)\Big/\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\right)\\ &\quad+\sum_{w=k}^{r-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\\ &\qquad\times f\left(\left((r-w)t_nx_n+\sum_{q=1}^{w}t_{i_q}x_{i_q}\right)\Big/\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\right), \end{aligned} \tag{4.5}$$

where and henceforth, when $w = 0$, we agree that

$$\sum_{1\le i_1\le\cdots\le i_w\le n-1}\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right) f\left(\left((r-w)t_nx_n+\sum_{q=1}^{w}t_{i_q}x_{i_q}\right)\Big/\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\right) = rt_nf(x_n)$$

Using Lemma 2.1, we conclude that

$$\begin{aligned}
&\sum_{w=0}^{k-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\\
&\qquad\times f\left(\left((r-w)t_nx_n+\sum_{q=1}^{w}t_{i_q}x_{i_q}\right)\Big/\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\right)\\
&\le\sum_{w=0}^{k-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\frac{1}{C_{r-1}^{k-1}}\sum_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_w,\underbrace{n,\ldots,n}_{r-w\ \text{times}}\}}\left(\sum_{q=1}^{k}t_{i'_q}\right)\\
&\qquad\times f\left(\sum_{q=1}^{k}t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{k}t_{i'_q}\right)\\
&\overset{def}{=}\frac{1}{C_{r-1}^{k-1}}\sum_{(1)}\left(t_n+\sum_{q=1}^{k-1}t_{i'_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{i'_q}x_{i'_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{i'_q}\right)\right),
\end{aligned}\tag{4.6}$$

and

$$\sum_{w=k}^{r-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)$$
$$\times f\left(\left((r-w)t_nx_n+\sum_{q=1}^{w}t_{i_q}x_{i_q}\right)\Big/\left((r-w)t_n+\sum_{q=1}^{w}t_{i_q}\right)\right)$$
$$\le\sum_{w=k}^{r-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\frac{1}{C_{r-1}^{k-1}}\sum_{\{i'_1,\dots,i'_k\}\subset\{i_1,\dots,i_w,\underbrace{n,\dots,n}_{r-w\ \text{times}}\}}\left(\sum_{q=1}^{k}t_{i'_q}\right)$$
$$\times f\left(\sum_{q=1}^{k}t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{k}t_{i'_q}\right)$$
$$\overset{def}{=}\frac{1}{C_{r-1}^{k-1}}\sum_{w=k}^{r-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\sum_{\{i'_1,\dots,i'_k\}\subset\{i_1,\dots,i_w\}}\left(\sum_{q=1}^{k}t_{i'_q}\right)f\left(\sum_{q=1}^{k}t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{k}t_{i'_q}\right)$$
$$+\frac{1}{C_{r-1}^{k-1}}\sum_{(2)}\left(t_n+\sum_{q=1}^{k-1}t_{i'_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{i'_q}x_{i'_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{i'_q}\right)\right).\tag{4.7}$$

Combining (4.5)–(4.7), we arrive at

$$L(n,r,f)$$
$$\le\frac{1}{C_{r-1}^{k-1}}\sum_{w=k}^{r-1}\sum_{1\le i_1\le\cdots\le i_w\le n-1}\sum_{\{i'_1,\dots,i'_k\}\subset\{i_1,\dots,i_w\}}\left(\sum_{q=1}^{k}t_{i'_q}\right)f\left(\sum_{q=1}^{k}t_{i'_q}x_{i'_q}\Big/\sum_{q=1}^{k}t_{i'_q}\right)$$
$$+\frac{1}{C_{r-1}^{k-1}}\left[\sum_{(1)}+\sum_{(2)}\right]\left(t_n+\sum_{q=1}^{k-1}t_{i'_q}\right)f\left(\left(t_nx_n+\sum_{q=1}^{k-1}t_{i'_q}x_{i'_q}\right)\Big/\left(t_n+\sum_{q=1}^{k-1}t_{i'_q}\right)\right).\tag{4.8}$$

In the rest of this step, we will rewrite the right hand side of (4.8) as a linear combination of $M(n-1,k,f)$ and $L(n,k,f)$. Thanks to Lemmas 2.2 and 2.4, the first line in the right hand side of (4.8) can be written as follows:

$$\begin{aligned}
&\sum_{w=k}^{r-1} \sum_{1\le i_1\le\cdots\le i_w\le n-1} \sum_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_w\}} \left(\sum_{q=1}^{k} t_{i'_q}\right) f\left(\sum_{q=1}^{k} t_{i'_q}x_{i'_q} \Big/ \sum_{q=1}^{k} t_{i'_q}\right)\\
&= \sum_{w=k}^{r-1} \frac{C_{n+w-2}^{w} C_{w}^{k}}{C_{n+k-2}^{k}} \sum_{1\le j_1\le\cdots\le j_k\le n-1} \left(\sum_{q=1}^{k} t_{j_q}\right) f\left(\sum_{q=1}^{k} t_{j_q}x_{j_q} \Big/ \sum_{q=1}^{k} t_{j_q}\right) \qquad (4.9)\\
&= \frac{(n-1)C_{n+r-2}^{r-1}C_{r-1}^{k}}{(n+k-1)C_{n+k-2}^{k}} M(n-1,k,f) = \frac{C_{n+r-2}^{r-1}C_{r-1}^{k}}{C_{n+k-1}^{k}} M(n-1,k,f).
\end{aligned}$$

We need to be more careful to treat the second line in the right hand side of (4.8). The difficulty is to analyze how many terms there are in the last two summations $\sum\limits_{(1)} + \sum\limits_{(2)}$ in (4.8).

To overcome this difficulty, we note that, from (4.1), using Lemma 2.1 again, one obtains

$$\begin{aligned}
&L(n,r,f)\\
&= \sum_{1\le i_1\le\cdots\le i_{r-1}\le n} \left(t_n + \sum_{q=1}^{r-1} t_{i_q}\right) f\left(\left(t_n x_n + \sum_{q=1}^{r-1} t_{i_q}x_{i_q}\right) \Big/ \left(t_n + \sum_{q=1}^{r-1} t_{i_q}\right)\right)\\
&\le \frac{1}{C_{r-1}^{k-1}} \sum_{1\le i_1\le\cdots\le i_{r-1}\le n} \sum_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_{r-1},n\}} \left(\sum_{q=1}^{k} t_{i'_q}\right) f\left(\sum_{q=1}^{k} t_{i'_q}x_{i'_q} \Big/ \sum_{q=1}^{k} t_{i'_q}\right).
\end{aligned} \qquad (4.10)$$

Obviously, the number of all terms in the first summation $\sum\limits_{w=k}^{r-1} \sum\limits_{1\le i_1\le\cdots\le i_w\le n-1} \sum\limits_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_w\}}$ and the second one $\sum\limits_{(1)} + \sum\limits_{(2)}$ of the right side of (4.8) is equal to that of all terms in the summation $\sum\limits_{1\le i_1\le\cdots\le i_{r-1}\le n} \sum\limits_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_{r-1},n\}}$ of the right side of (4.10) (including repeated terms).

It is easy see that there are $C_{n+(r-1)-1}^{r-1}C_r^k$ terms (including repeated terms) in the summation $\sum\limits_{1\le i_1\le\cdots\le i_{r-1}\le n} \sum\limits_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_{r-1},n\}}$ of the right side in (4.10). However, by means of Lemma 2.4, we see that there are

$$\sum_{w=k}^{r-1} C_{n+w-2}^{w} C_{w}^{k} = \frac{n-1}{n+k-1} C_{n+r-2}^{r-1} C_{r-1}^{k}$$

terms (including repeated terms) in the first summation $\sum\limits_{w=k}^{r-1} \sum\limits_{1\le i_1\le\cdots\le i_w\le n-1} \sum\limits_{\{i'_1,\ldots,i'_k\}\subset\{i_1,\ldots,i_w\}}$ of the right side of (4.8). Therefore, we conclude that there are

$$C_{n+r-2}^{r-1}C_r^k - \frac{n-1}{n+k-1}C_{n+r-2}^{r-1}C_{r-1}^k = \frac{n}{n+k-1}C_{n+r-1}^{r-1}C_{r-1}^{k-1}$$

terms (including repeated terms) in the last two summations $\sum_{(1)} + \sum_{(2)}$ in (4.8). However, there are at most C_{n+k-2}^{k-1} different terms in $\sum_{(1)} + \sum_{(2)}$. Hence, we end up with

$$\begin{aligned}
&\left[\sum_{(1)} + \sum_{(2)}\right]\left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right) f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{i'_q} x_{i'_q}\right)\Big/\left(t_n + \sum_{q=1}^{k-1} t_{i'_q}\right)\right)\\
&= \frac{nC_{n+r-1}^{r-1}C_{r-1}^{k-1}}{(n+k-1)C_{n+k-2}^{k-1}} \sum_{1\le j_1\le\cdots\le j_{k-1}\le n}\left(t_n + \sum_{q=1}^{k-1} t_{j_q}\right)\\
&\qquad \times f\left(\left(t_n x_n + \sum_{q=1}^{k-1} t_{j_q} x_{j_q}\right)\Big/\left(t_n + \sum_{q=1}^{k-1} t_{j_q}\right)\right)\\
&= \frac{C_{n+r-1}^{r-1}C_{r-1}^{k-1}}{C_{n+k-1}^{k-1}} L(n,k,f).
\end{aligned} \tag{4.11}$$

Now, combining (4.8), (4.9) and (4.11), we conclude that

$$L(n,r,f) \le \frac{C_{n+r-2}^{r-1}C_{r-1}^k}{C_{r-1}^{k-1}C_{n+k-1}^k} M(n-1,k,f) + \frac{C_{n+r-1}^{r-1}}{C_{n+k-1}^{k-1}} L(n,k,f). \tag{4.12}$$

Step 3. Recalling that v is assumed to be non-negative and $v(n,k,r) \ge v(n-1,k,r)$, and using (4.3), (4.4) and (4.12), we see that

$$\begin{aligned}
&G(n,k,r,f) - G(n-1,k,r,f) \ge \frac{A(n,k,r)}{C_{r-1}^{k-1}C_{n+k-1}^k} M(n-1,k,f)\\
&\quad + \frac{B(n,k,r)}{C_{n+k-1}^{k-1}} L(n,k,f).
\end{aligned} \tag{4.13}$$

Finally, noting our assumption on $A(n,k,r), B(n,k,r)$ and f, the desired result follows from (4.13) immediately. This completes the proof of Theorem 1.2. ■

5. Proof of Theorem 1.3

It is easy to see that

$$\begin{aligned}&\sum_{1\le i_1\le\cdots\le i_{k-1}\le n}\left(t_n+\sum_{j=1}^{k-1}t_{i_j}\right)+C_{n+k-2}^{k-1}T_{n-1}\\&=C_{n+k-2}^{k-1}(T_{n-1}+t_n)+\sum_{1\le i_1\le\cdots\le i_{k-1}\le n}\sum_{j=1}^{k-1}t_{i_j}\\&=C_{n+k-2}^{k-1}T_n+\frac{k-1}{n}C_{n+k-2}^{k-1}T_n=C_{n+k-1}^{k-1}T_n.\end{aligned}$$

Similarly,

$$\sum_{1\le i_1\le\cdots\le i_{k-1}\le n}\left(t_nx_n+\sum_{j=1}^{k-1}t_{i_j}x_{i_j}\right)+C_{n+k-2}^{k-1}\sum_{i=1}^{n-1}t_ix_i=C_{n+k-1}^{k-1}\sum_{i=1}^{n}t_ix_i. \tag{5.1}$$

By (5.1) and (5.1), and using (1.4), we obtain

$$\begin{aligned}&J(n,\infty,f)=T_nf\left(\sum_{i=1}^{n}t_ix_i\Big/T_n\right)\\&=T_nf\left(\left[\sum_{1\le i_1\le\cdots\le i_{k-1}\le n}\left(t_n+\sum_{j=1}^{k-1}t_{i_j}\right)\left(\left(t_nx_n+\sum_{j=1}^{k-1}t_{i_j}x_{i_j}\right)\Big/\left(t_n+\sum_{j=1}^{k-1}t_{i_j}\right)\right)\right.\right.\\&\qquad\left.\left.+C_{n+k-2}^{k-1}T_{n-1}\left(\sum_{i=1}^{n-1}t_ix_i\Big/T_{n-1}\right)\right]\Big/\left(C_{n+k-1}^{k-1}T_n\right)\right)\\&\le\frac{1}{C_{n+k-1}^{k-1}}\left[\sum_{1\le i_1\le\cdots\le i_{k-1}\le n}\left(t_n+\sum_{j=1}^{k-1}t_{i_j}\right)f\left(\left(t_nx_n+\sum_{j=1}^{k-1}t_{i_j}x_{i_j}\right)\Big/\left(t_n+\sum_{j=1}^{k-1}t_{i_j}\right)\right)\right.\\&\qquad\left.+C_{n+k-2}^{k-1}T_{n-1}f\left(\sum_{i=1}^{n-1}t_ix_i\Big/T_{n-1}\right)\right]\\&=\frac{1}{C_{n+k-1}^{k-1}}L(n,k,f)+\frac{C_{n+k-2}^{k-1}}{C_{n+k-1}^{k-1}}J(n-1,\infty,f),\end{aligned} \tag{5.2}$$

where $L(n,k,f)$ is defined by (4.1).

Noting that $z(n,k)$ is assumed to be non-negative, by (1.5) and (5.2), we conclude that

$$\begin{aligned}
&G(n,k,\infty,f)-G(n-1,k,\infty,f)\\
&=y(n,k)[M(n-1,k,f)+L(n,k,f)]-z(n,k)J(n,\infty,f)\\
&\quad-[y(n-1,k)M(n-1,k,f)-z(n-1,k)J(n-1,\infty,f)]\\
&\geq C(n,k)M(n-1,k,f)+D(n,k)L(n,k,f)+E(n,k)J(n-1,\infty,f).
\end{aligned}\tag{5.3}$$

Finally, by (5.3), and noting our assumptions on $C(n,k), D(n,k), E(n,k)$ and f, we obtain the desired result immediately. This completes the proof of Theorem 1.3. ■

6. Proof of Theorems 1.9 and 1.10

This section is devoted to prove Theorems 1.9 and 1.10.

Proof of Theorem 1.9. The "only if" part is a direct consequence of $B(n,k,r)=0$.

The "if" part. It suffices to show $A(n,k,r)=0$. By (1.17), we have

$$u(n,k,r)=\frac{C_{n+r-1}^{r-1}}{C_{n+k-1}^{k-1}}v(n,k,r),\qquad u(n-1,k,r)=\frac{C_{n+r-2}^{r-1}}{C_{n+k-2}^{k-1}}v(n-1,k,r).$$

Therefore, from the definition of $A(n,k,r)$ in (1.5), we have

$$\begin{aligned}
&A(n,k,r)\\
&=\left[\frac{C_{n+r-1}^{r-1}C_{r-1}^{k-1}C_{n+k-1}^{k}}{C_{n+k-1}^{k-1}}-\frac{n+k-1}{n-1}C_{n+r-2}^{r}C_{r}^{k}-C_{n+r-2}^{r-1}C_{r-1}^{k}\right]v(n,k,r)\\
&\quad+\left[\frac{n+k-1}{n-1}C_{n+r-2}^{r}C_{r}^{k}-\frac{C_{n+r-2}^{r-1}C_{r-1}^{k-1}C_{n+k-1}^{k}}{C_{n+k-2}^{k-1}}\right]v(n-1,k,r).
\end{aligned}\tag{6.1}$$

A simple computation shows that

$$\frac{C_{n+r-1}^{r-1}C_{r-1}^{k-1}C_{n+k-1}^{k}}{C_{n+k-1}^{k-1}}-\frac{n+k-1}{n-1}C_{n+r-2}^{r}C_{r}^{k}-C_{n+r-2}^{r-1}C_{r-1}^{k}=0\tag{6.2}$$

and

$$\frac{n+k-1}{n-1}C_{n+r-2}^{r}C_{r}^{k}-\frac{C_{n+r-2}^{r-1}C_{r-1}^{k-1}C_{n+k-1}^{k}}{C_{n+k-2}^{k-1}}=0.\tag{6.3}$$

Now, combining (6.1)-(6.3), we see that $A(n,k,r)=0$. This completes the proof of Theorem 1.9. ■

Proof of Theorem 1.10. The "only if" part is a direct consequence of $D(n,k)=0$ and $C(n,k)=0$.

The “if” part. It suffices to show $E(n,k) = 0$. By (1.19), we have

$$z(n,k) = C_{n+k-1}^{k-1} y(n,k), \qquad z(n-1,k) = C_{n+k-2}^{k-1} y(n-1,k) = C_{n+k-2}^{k-1} y(n,k).$$

Therefore, by the definition of $E(n,k)$ in (1.11), we see that

$$E(n,k) = \frac{y(n,k)\Big(C_{n+k-2}^{k-1} C_{n+k-1}^{k-1} - C_{n+k-1}^{k-1} C_{n+k-2}^{k-1}\Big)}{C_{n+k-1}^{k-1}} = 0.$$

This completes the proof of Theorem 1.10. ■

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