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New Resolvent Operator Technique for A Class of General (A, η)-Accretive Equations in Banach Spaces[1]

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Abstract

In this paper, by using the concept of (A, η)-accretive mappings and the new resolvent operators associated with (A, η)-accretive mappings due to Lan et al., we construct some new iterative algorithms to approximate the solutions of a new class of relaxed cocoercive variational inclusion problems involving (A, η)-accretive mappings with non-accretive set-valued mappings. we also prove the existence of solutions and the convergence of the sequences generated by the algorithms in q-uniformly smooth Banach spaces.

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1. Introduction

In this paper, we always assume that $\mathcal{B}$ is a real Banach space with dual space $\mathcal{B}^*$, $\langle \cdot, \cdot \rangle$ is the dual pair between $\mathcal{B}$ and $\mathcal{B}^*$, $CB(\mathcal{B})$ denotes the family of all nonempty closed bounded subsets of $\mathcal{B}$ and $2^{\mathcal{B}}$ denotes the family of all the nonempty subsets of $\mathcal{B}$. For any given constant $\lambda > 0$, single-valued mappings $A, p, g : \mathcal{B} \to \mathcal{B}$, $N : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, set-valued mappings $S, T, G : \mathcal{B} \to CB(\mathcal{B})$, and an any nonlinear mapping $M : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ satisfying for all $t \in \mathcal{B}$, $g(\mathcal{B}) \bigcap M(\cdot, t) \neq \emptyset$ and $M(\cdot, t)$ is an (A, η)-accretive mapping, we consider the following problem of finding $x \in \mathcal{B}$, $u \in S(x)$, $v \in T(x)$, $\omega \in G(x)$ such that

$$0 \in N(p(x), u, \omega) + \lambda M(g(x), v). \tag{1.1}$$

Some examples of the problem (1.1) include the following.

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(1) If $N(x,y,z) = f(y) + T(x,z)$ for all $x, y, z \in \mathcal{B}$, then the problem (1.1) reduces to the following nonlinear variational inclusion:

Find $x \in \mathcal{B}$, $u \in S(x)$, $v \in T(x)$, $\omega \in G(x)$ such that

$$0 \in f(u) + T(p(x), \omega) + \lambda M(g(x), v). \tag{1.2}$$

The problem (1.2) is considered by Jin [10] when $\lambda = 1$, which includes a number of known and new classes of variational inclusions, variational inequalities, and corresponding optimization problems (see, for example, [2]-[4], [8], [12] and the references therein). Furthermore, these types of variational inclusion problems enable us to study many important problems arising in the mathematical, physical, and engineering sciences in a general and unified framework.

(2) If $M(x,t) = M(x)$ for all $x, t \in \mathcal{B}$ and there exists a element $f \in \mathcal{B}$ such that $N(x,y,z) = F(y,z) - f$ for all $x, y, z \in \mathcal{B}$, then the problem (1.1) reduces to finding $x \in \mathcal{B}$, $u \in S(x)$, $\omega \in G(x)$ such that

$$f \in F(u, \omega) + \lambda M(g(x)). \tag{1.3}$$

In 2004, motivated and inspired by the results in [6], Peng [16] introduced and studied the set-valued variational inclusion problem (1.3) involving T-accretive operators (in fact, H-accretive operators due to Fang and Huang [6]) in Banach spaces, when the set-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is a T-accretive operator. As Peng pointed out "the results in [16] can be further generalized to the case of A-accretivity based on the references [17, 18]". Moreover, we note that the author proved the convergence result for iterative sequences generated by the algorithm under the assumption that V, G are $\hat{\mathbf{H}}$-Lipschitz continuous and strongly accretive with respect to g_1 in the first and second arguments, respectively, where $g_1 : \mathcal{B} \to \mathcal{B}$ is defined by $g_1(x) = T \cdot g(x) = T(g(x))$ for all $x \in \mathcal{B}$. By Liu and Li [13], it is easy to know that S, G in Theorem 4.1 of [16] are single-valued mappings.

(3) If $\lambda = 1$, $f = 0$ and $S, G : \mathcal{B} \to \mathcal{B}$ is a single-valued, then the problem (1.3) reduces to the variational inclusion problem of finding $x \in \mathcal{B}$ such that

$$0 \in F(S(x), G(x)) + M(g(x)). \tag{1.4}$$

The problems (1.3)–(1.4) include a number of variational inclusions and nonlinear variational inequalities as special cases, for more details, see [1]-[5], [8]-[14], [16]-[18] and the references therein.

On the other hand, Lan et al. [11] introduced a new concept of (A, η)-accretive mappings, which generalizes the existing monotone or accretive operators, and studied some properties of (A, η)-accretive mappings and defined resolvent operators associated with (A, η)-accretive mappings. By using the resolvent operator technique, the authors constructed some perturbed iterative algorithms for a class of nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings and study applications of (A,η)-accretive mappings to the approximation-solvability of this class of nonlinear relaxed cocoercive variational inclusions in q-uniformly smooth Banach spaces, which are providing mathematical models to some problems arising in economics, mechanics, and engineering science. Very recently, by using the resolvent operator technique, a very important method to find solutions of variational inequality and variational inclusion problems, a number of of nonlinear variational inclusions and many systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years. See, for example, [7], [8], [17], [19] and the references therein.

Motivated and inspired by the above works, the purpose of this paper is to introduce the notion of (A,η)-accretive mappings and the resolvent operators associated with (A,η)-accretive mappings due to Lan et al. [11], and to construct some new iterative algorithms to approximate the solutions of the relaxed cocoercive variational inclusion problem (1.1) involving (A,η)-accretive mappings with non-accretive set-valued mappings. we also prove the existence of solutions and the convergence of the sequences generated by the algorithms in q-uniformly smooth Banach spaces. Our results improve and extend the corresponding results of recent works.

2. Preliminaries

It is well known that the generalized duality mapping $J_q : \mathcal{B} \to 2^{\mathcal{B}^*}$ is defined by

$$J_q(x) = \{f^* \in \mathcal{B}^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in \mathcal{B},$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$, and J_q is single-valued if $\mathcal{B}^*$ is strictly convex, and if $\mathcal{B} = \mathcal{H}$, the Hilbert space, then J_2 becomes the identity mapping on $\mathcal{H}$.

The modulus of smoothness of $\mathcal{B}$ is the function $\rho_{\mathcal{B}} : [0,\infty) \to [0,\infty)$ defined by

$$\rho_{\mathcal{B}}(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le t\}.$$

A Banach space $\mathcal{B}$ is called uniformly smooth if $\lim_{t\to 0} \frac{\rho_{\mathcal{B}}(t)}{t} = 0$. $\mathcal{B}$ is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_{\mathcal{B}}(t) \le ct^q, \quad q > 1$.

Remark that J_q is single-valued if $\mathcal{B}$ is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [20] proved the following result:

Lemma 2.1: Let $\mathcal{B}$ be a real uniformly smooth Banach space. Then $\mathcal{B}$ is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in \mathcal{B}$,

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

In the sequel, we give some concept and lemma needed later.

Definition 2.2: Let $\mathcal{B}$ be a q-uniformly smooth Banach space, $T, A : \mathcal{B} \to \mathcal{B}$ be two single-valued operators. Then T is said to be

(i) accretive if
$$\langle T(x) - T(y), J_q(x-y)\rangle \geq 0, \quad \forall x, y \in \mathcal{B};$$

(ii) strictly accretive, if T is accretive and
$$\langle T(x) - T(y), J_q(x-y)\rangle = 0$$
if and only if $x = y$;

(iii) r-strongly accretive, if there exists a constant $r > 0$ such that
$$\langle T(x) - T(y), J_q(x-y)\rangle \geq r\|x-y\|^q, \quad \forall x, y \in \mathcal{B};$$

(iv) γ-strongly accretive with respect to A, if there exists a constant $\gamma > 0$ such that
$$\langle T(x) - T(y), J_q(A(x)-A(y))\rangle \geq \gamma\|x-y\|^q, \quad \forall x, y \in \mathcal{B};$$

(v) m-relaxed cocoercive with respect to A, if there exists a constant $m > 0$ such that
$$\langle T(x) - T(y), J_q(A(x)-A(y))\rangle \geq -m\|T(x)-T(y)\|^q, \quad \forall x, y \in \mathcal{B};$$

(vi) (α, ξ)-relaxed cocoercive with respect to A, if there exist constants $\alpha, \xi > 0$ such that
$$\langle T(x)-T(y), J_q(A(x)-A(y))\rangle \geq -\alpha\|T(x)-T(y)\|^q + \xi\|x-y\|^q, \quad \forall x, y \in \mathcal{B};$$

(vii) s-Lipschitz continuous, if there exists a constant $s > 0$ such that
$$\|T(x) - T(y)\| \leq s\|x-y\|, \quad \forall x, y \in \mathcal{B}.$$

Remark 2.3: When $\mathcal{B} = \mathcal{H}$, (i)-(iv) of Definition 2.1 reduce to the definitions of monotonicity, strict monotonicity, strong monotonicity, and strong monotonicity with respect to A, respectively (see [5], [6]).

Definition 2.4: A single-valued operator $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is said to be Lipschitz continuous if there exists a constant $\tau > 0$ such that
$$\|\eta(x, y)\| \leq \tau\|x-y\|, \quad \forall x, y \in \mathcal{B}.$$

Definition 2.5: Let $\mathcal{B}$ be a q-uniformly smooth Banach space, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $A, H : \mathcal{B} \to \mathcal{B}$ be single-valued mappings. Then multi-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is said to be

(i) accretive if

$$\langle u-v, J_q(x-y)\rangle \geq 0, \quad \forall x,y\in\mathcal{B}, u\in M(x), v\in M(y);$$

(ii) η-accretive if

$$\langle u-v, J_q(\eta(x,y))\rangle \geq 0, \quad \forall x,y\in\mathcal{B}, u\in M(x), v\in M(y);$$

(iii) strictly η-accretive if M is η-accretive and equality holds if and only if $x=y$;

(iv) r-strongly η-accretive if there exists a constant $r>0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \geq r\|x-y\|^q, \quad \forall x,y\in\mathcal{B}, u\in M(x), v\in M(y);$$

(v) α-relaxed η-accretive if there exists a constant $\alpha>0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \geq -\alpha\|x-y\|^q, \quad \forall x,y\in\mathcal{B}, u\in M(x), v\in M(y);$$

(vi) m-accretive if M is accretive and $(I+\rho M)(\mathcal{B})=\mathcal{B}$ for all $\rho>0$, where I denotes the identity operator on $\mathcal{B}$;

(vii) generalized m-accretive if M is η-accretive and $(I+\rho M)(\mathcal{B})=\mathcal{B}$ for all $\rho>0$;

(viii) H-accretive if M is accretive and $(H+\rho M)(\mathcal{B})=\mathcal{B}$ for all $\rho>0$;

(ix) (H,η)-accretive if M is η-accretive and $(H+\rho M)(\mathcal{B})=\mathcal{B}$ for every $\rho>0$.

Remark 2.6: If $\mathcal{B}=\mathcal{H}$, then (i)-(ix) of Definition 2.4 reduce to the definitions of monotone operators, η-monotone operators, strictly η-monotone operators, strongly η-monotone operators, relaxed η-monotone operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators and (H,η)-monotone operators, respectively.

Definition 2.7: Let $T:\mathcal{B}\to 2^{\mathcal{B}}$ be a set-valued mapping. For all $x,y\in\mathcal{B}$, the mapping $N(\cdot,\cdot,\cdot):\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ is called to be

(i) ϵ-Lipschitz continuous with respect to the first argument, if there exists a constant $\epsilon>0$ such that

$$\|N(x,\cdot,\cdot)-N(y,\cdot,\cdot)\| \leq \epsilon\|x-y\| \quad \forall x,y\in\mathcal{B};$$

(ii) T is said to be ζ-$\hat{\mathbf{H}}$-Lipschitz continuous, if there exists a constant $\zeta>0$ such that

$$\hat{\mathbf{H}}(T(x),T(y)) \leq \zeta\|x-y\|, \quad \forall x,y\in\mathcal{B},$$

where $\hat{\mathbf{H}}: 2^{\mathcal{B}}\times 2^{\mathcal{B}}\to(-\infty,+\infty)\cup\{+\infty\}$ is the Hausdorff pseudo-metric, i.e.,

$$\hat{\mathbf{H}}(E,D)=\max\{\sup_{x\in E}\inf_{y\in D}\|x-y\|, \sup_{x\in D}\inf_{y\in E}\|x-y\|\}, \quad \forall E,D\in 2^{\mathcal{B}}.$$

Note that if the domain of $\hat{\mathbf{H}}$ is restricted to closed bounded subsets $CB(\mathcal{B})$, then $\hat{\mathbf{H}}$ is the Hausdorff metric.

In a similar way, we can define Lipschitz continuity of the mapping $N(\cdot,\cdot,\cdot)$ with respect to the second argument.

Definition 2.8: Let $A : \mathcal{B} \to \mathcal{B}$, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be two single-valued operators. Then a multi-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is called (A, η)-accretive if

(i) M is m-relaxed η-accretive,

(ii) $(A + \rho M)(\mathcal{B}) = \mathcal{B}$ for every $\rho > 0$.

Remark 2.9 ([11]):

(i) If $m = 0$, then Definition 2.5 reduces to the definition of (H, η)-accretive operators, which is generalized m-accretive operators and H-accretive operators if $A = I$, $\eta(x, y) = x - y$ for all $x, y \in \mathcal{B}$, respectively. Furthermore, if $m = 0$, $A = I$ and $\eta(x, y) = x - y$ for all $x, y \in \mathcal{B}$, then Definition 2.5 becomes and is classical m-accretive operators.

(ii) When $\mathcal{B} = \mathcal{H}$, Definition 2.5 reduces to the definition of (A, η)-monotone operators, which is a new concept and includes A-monotone operators as special case.

(iii) When $m = 0$ and $\mathcal{B} = \mathcal{H}$, Definition 2.5 is replaced to the definition of (H, η)-monotone operators, which becomes maximal η-monotone operators and classical maximal monotone operators if $A = I$, $\eta(x, y) = x - y$ for all $x, y \in \mathcal{H}$, respectively.

Proposition 2.10 ([11]): Let $A : \mathcal{B} \to \mathcal{B}$ be a r-strongly η-accretive mapping, $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Definition 2.11: Let $A : \mathcal{B} \to \mathcal{B}$ be a strictly η-accretive operator and $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. The resolvent operator $J_{\eta,M}^{\rho,A} : \mathcal{B} \to \mathcal{B}$ is defined by:

$$J_{\eta,M}^{\rho,A}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in \mathcal{B}.$$

Remark 2.12: Resolvent operators associated with (A, η)-accretive mappings include the corresponding resolvent operators associated with (H, η)-accretive mappings, (H, η)-monotone operators, H-accretive operators, generalized m-accretive mappings, maximal η-monotone operators, H-monotone operators, A-monotone operators, η-subdifferential operators, the classical m-accretive and maximal monotone operators (see, [3]-[7], [10]-[12], [14], [16], [17] and the references therein).

Proposition 2.13 ([11]): Let $\mathcal{B}$ be a q-uniformly smooth Banach space, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be τ-Lipschitz continuous, $A : \mathcal{B} \to \mathcal{B}$ be a r-strongly η-accretive mapping and M :

$\mathcal{B} \to 2^{\mathcal{B}}$ be an (A,η)-accretive mapping. Then the resolvent operator $J_{\eta,M}^{\rho,A} : \mathcal{B} \to \mathcal{B}$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,

$$\|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\| \leq \frac{\tau^{q-1}}{r-\rho m}\|x-y\|, \quad \forall x,y \in \mathcal{B},$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

3. Iterative Algorithms and Convergence

In this section, by using the new resolvent operator technique associated with (A,η)-accretive mappings duo to section 2, we shall construct a new iterative algorithm for solving the relaxed cocoercive variational inclusion problems (1.1) in Banach spaces.

Lemma 3.1: Let $A : \mathcal{B} \to \mathcal{B}$ be r-strongly η-accretive, $M(\cdot,t) : \mathcal{B} \to 2^{\mathcal{B}}$ is (A,η)-accretive in the first argument for all $t \in \mathcal{B}$, and $p,g : \mathcal{B} \to \mathcal{B}$, $N : \mathcal{B}\times\mathcal{B}\times\mathcal{B} \to \mathcal{B}$ and $S,T,G : \mathcal{B} \to CB(\mathcal{B})$ be any nonlinear mappings. Then a given element (x,u,v,ω) is a solution to the problem (1.1) if and only if (x,u,v,ω) satisfies

$$g(x) = J_{\eta,M(\cdot,v)}^{\rho\lambda,A}(A(g(x)) - \rho N(p(x),u,\omega)), \tag{3.1}$$

where $J_{\eta,M(\cdot,v)}^{\rho\lambda,A} = (A + \rho\lambda M(\cdot,v))^{-1}$ and $\rho > 0$ is a constant.

Proof. The conclusion follows directly from Definition 2.7. ■

Remark 3.2: The equality (3.1) can be written as

$$x = x - g(x) + J_{\eta,M(\cdot,v)}^{\rho\lambda,A}(A(g(x)) - \rho N(p(x),u,\omega)),$$

where $\rho,\lambda > 0$ are constants. This fixed point formulation enables us to suggest the following iterative algorithm.

Algorithm 3.1: For any given $x_0 \in \mathcal{B}$, take $u_0 \in S(x_0)$, $v_0 \in T(x_0)$ and $\omega_0 \in G(x_0)$. It follows from Lemma 3.1 that there exists $x_1 \in \mathcal{B}$ such that

$$x_1 = x_0 - g(x_0) + J_{\eta,M(\cdot,v_0)}^{\rho\lambda,A}((A(g(x_0)) - \rho N(p(x_0),u_0,\omega_0)) + e_0).$$

Since $u_0 \in S(x_0)$, $v_0 \in T(x_0)\, \omega_0 \in G(x_0)$, by Nadler's result [15], there exist $u_1 \in S(x_1)$, $v_1 \in T(x_1)$ and $\omega_1 \in G(x_1)$ such that

$$\|u_0 - u_1\| \leq (1+1)\hat{\mathbf{H}}(S(x_0),S(x_1)),$$
$$\|v_0 - v_1\| \leq (1+1)\hat{\mathbf{H}}(T(x_0),T(x_1)),$$
$$\|\omega_0 - \omega_1\| \leq (1+1)\hat{\mathbf{H}}(G(x_0),G(x_1)).$$

From Lemma 3.1, we know that there exists $x_2 \in \mathcal{B}$ such that

$$x_2 = x_1 - g(x_1) + J^{\rho\lambda,A}_{\eta,M(\cdot,v_1)}((A(g(x_1)) - \rho N(p(x_1), u_1, \omega_1)) + e_1).$$

By Nadler's result [15], there exist $u_2 \in S(x_2)$, $v_2 \in T(x_2)$ and $\omega_2 \in G(x_2)$ such that

$$\begin{aligned} \|u_1 - u_2\| &\leq (1 + 2^{-1})\hat{\mathbf{H}}(S(x_1), S(x_2)), \\ \|v_1 - v_2\| &\leq (1 + 2^{-1})\hat{\mathbf{H}}(T(x_1), T(x_2)), \\ \|\omega_0 - \omega_1\| &\leq (1 + 2^{-1})\hat{\mathbf{H}}(G(x_1), G(x_2)). \end{aligned}$$

Continuing this way, we can obtain sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{\omega_n\}$ satisfying

$$\begin{cases} x_{n+1} = x_n - g(x_n) \\ \qquad\quad + J^{\rho\lambda,A}_{\eta,M(\cdot,v_n)}((A(g(x_n)) - \rho N(p(x_n), u_n, \omega_n)) + e_n), \\ u_n \in S(x_n),\ \|u_n - u_{n+1}\| \leq (1 + \frac{1}{n+1})\hat{\mathbf{H}}(S(x_n), S(x_{n+1})), \\ v_n \in T(x_n),\ \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})\hat{\mathbf{H}}(T(x_n), T(x_{n+1})), \\ \omega_n \in G(x_n),\ \|\omega_n - \omega_{n+1}\| \leq (1 + \frac{1}{n+1})\hat{\mathbf{H}}(G(x_n), G(x_{n+1})), \end{cases} \tag{3.2}$$

where $\rho, \lambda > 0$ are constants, $e_n \in \mathcal{B}(n \geq 0)$ is an error to take into account a possible inexact computation of the resolvent operator point, and $\hat{\mathbf{H}}(\cdot, \cdot)$ is the Hausdorff pseudo-metric on $2^{\mathcal{B}}$.

From Algorithm 3.1, we have the following algorithm for solving the problem (1.3).

Algorithm 3.2: For any given $x_0 \in \mathcal{B}$, we can obtain sequences $\{x_n\}$, $\{u_n\}$, $\{\omega_n\}$ satisfying

$$\begin{cases} x_{n+1} = x_n - g(x_n) + J^{\rho\lambda,A}_{\eta,M}((A(g(x_n)) - \rho(F(u_n, \omega_n) - f)) + e_n), \\ u_n \in S(x_n),\ \|u_n - u_{n+1}\| \leq (1 + \frac{1}{n+1})\hat{\mathbf{H}}(S(x_n), S(x_{n+1})), \\ \omega_n \in G(x_n),\ \|\omega_n - \omega_{n+1}\| \leq (1 + \frac{1}{n+1})\hat{\mathbf{H}}(G(x_n), G(x_{n+1})), \end{cases}$$

where $\rho, \lambda > 0$ and e_n are the same as in Algorithm 3.1.

Remark 3.3: If we choose suitable e_n, A, η, M, N, S, T, g, G and $\mathcal{B}$, then Algorithms 3.1-3.2 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities, complementarity problems, and variational inclusions (see, for example, [1]-[6], [9]-[13]).

Now we prove the existence of a solution of problem (1.1) and the convergence of Algorithm 3.1.

Theorem 3.4: Let $\mathcal{B}$ be a q-uniformly smooth Banach space and $A : \mathcal{B} \to \mathcal{B}$ be r-strongly η-accretive and σ-Lipschitz continuous, respectively. Let $S, T, G : \mathcal{B} \to CB(\mathcal{B})$ be ξ-$\hat{\mathbf{H}}$-Lipschitz continuous, γ-$\hat{\mathbf{H}}$-Lipschitz continuous and ζ-$\hat{\mathbf{H}}$-Lipschitz continuous, respectively. Suppose that $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is τ-Lipschitz continuous and for each fixed $t \in \mathcal{B}$, $M(\cdot, t) : \mathcal{B} \to 2^{\mathcal{B}}$ is a (A,η)-accretive. Let $g : \mathcal{B} \to \mathcal{B}$ be (d,α)-relaxed cocoercive and β-Lipschitz continuous, $p : \mathcal{B} \to \mathcal{B}$ be π-strongly accretive and ϱ-Lipschitz continuous, and $N : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be (s,ι)-relaxed cocoercive with respect to g_1 in the first argument and Lipschitz continuous with respect to three arguments with constants $\nu > 0$, $\delta > 0$ and $\epsilon > 0$, respectively, where $g_1 : \mathcal{B} \to \mathcal{B}$ is defined by $g_1(x) = A \circ g(x) = A(g(x))$ for all $x \in \mathcal{B}$. If there exist constants $\rho > 0$ and $\mu > 0$ such that for each $x, y, t \in \mathcal{B}$,

$$\|J^{\rho\lambda,A}_{\eta,M(\cdot,x)}(t) - J^{\rho\lambda,A}_{\eta,M(\cdot,y)}(t)\| \le \mu\|x-y\|, \tag{3.3}$$

$$\begin{cases} h = \mu\gamma + \sqrt[q]{1 - q\alpha + (c_q + dq)\beta^q} < 1 - \frac{\rho\tau^{q-1}(\delta\xi+\epsilon\zeta)}{r-\rho\lambda m}, \quad r > \rho\lambda m, \\ \sigma^q\beta^q - q\rho\iota + q\rho s\nu^q\pi^q + c_q\rho^q\nu^q\pi^q \\ \qquad < [(1-h)(r-\rho\lambda m)\tau^{1-q} - \rho(\delta\xi + \epsilon\zeta)]^q, \\ \sum\limits_{i=1}^{\infty} \|e_i - e_{i-1}\|\, k^{-i} < \infty,\ \forall k \in (0,1), \quad \lim\limits_{n\to\infty} e_n = 0, \end{cases} \tag{3.4}$$

where c_q is the constant as in Lemma 2.1, then the iterative sequences $\{x_n\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to x^*, u^*, v^* and w^*, respectively, and (x^*, u^*, v^*, w^*) is a solution of problem (1.1).

Proof. From (3.2), condition (3.3) and Proposition 2.13, we have

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\le \|x_n - x_{n-1} - [g(x_n) - g(x_{n-1})]\| \\ &\quad + \|J^{\rho\lambda,A}_{\eta,M(\cdot,v_n)}((A(g(x_n)) - \rho N(p(x_n),u_n,\omega_n)) + e_n) \\ &\quad - J^{\rho\lambda,A}_{\eta,M(\cdot,v_{n-1})}((A(g(x_n)) - \rho N(p(x_n),u_n,\omega_n)) + e_n)\| \\ &\quad + \|J^{\rho\lambda,A}_{\eta,M(\cdot,v_{n-1})}((A(g(x_n)) - \rho N(p(x_n),u_n,\omega_n)) + e_n) \\ &\quad - J^{\rho\lambda,A}_{\eta,M(\cdot,v_{n-1})}((A(g(x_{n-1})) - \rho N(p(x_{n-1}),u_{n-1},\omega_{n-1})) + e_{n-1})\| \end{aligned}$$

$$\begin{aligned} &\le \|x_n - x_{n-1} - [g(x_n) - g(x_{n-1})]\| \\ &\quad + \frac{\tau^{q-1}}{r-\rho\lambda m}\|A(g(x_n) - A(g(x_{n-1}))) \\ &\qquad - \rho[N(p(x_n),u_n,\omega_n) - N(p(x_{n-1}),u_n,\omega_n)]\| \\ &\quad + \frac{\rho\tau^{q-1}}{r-\rho\lambda m}\|N(p(x_{n-1}),u_n,\omega_n) - N(p(x_{n-1}),u_{n-1},\omega_{n-1})\| \\ &\quad + \mu\|v_n - v_{n-1}\| + \frac{\tau^{q-1}}{r-\rho\lambda m}\|e_n - e_{n-1})\| \end{aligned} \tag{3.5}$$

By the assumptions and Lemma 2.1, we know that

$$\begin{aligned}
&\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^q \\
&\le \|x_n - x_{n-1}\|^q + c_q\|g(x_n) - g(x_{n-1})\|^q \\
&\quad -q\langle g(x_n) - g(x_{n-1}), J_q(x_n - x_{n-1})\rangle \\
&\le (1 - q\alpha + (c_q + dq)\beta^q)\|x_n - x_{n-1}\|^q, \qquad (3.6)\\
&\|v_n - v_{n-1}\| \le (1 + n^{-1})\hat{\mathbf{H}}(T(x_n), T(x_{n-1})) \le \gamma(1 + n^{-1})\|x_n - x_{n-1}\|, \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
&\|A(g(x_n) - A(g(x_{n-1}))) - \rho[N(p(x_n), u_n, \omega_n) - N(p(x_{n-1}), u_n, \omega_n)]\|^q \\
&\le \|A(g(x_n)) - A(g(x_{n-1}))\|^q - q\rho\langle N(p(x_n), u_n, \omega_n) \\
&\qquad -N(p(x_{n-1}), u_n, \omega_n), J_q(A(g(x_n)) - A(g(x_{n-1})))\rangle \\
&\quad +c_q\rho^q\|N(p(x_n), u_n, \omega_n) - N(p(x_{n-1}), u_n, \omega_n)\|^q \\
&\le \sigma^q\beta^q\|x_n - x_{n-1}\|^q + c_q\rho^q\nu^q\|p(x_n) - p(x_{n-1})\|^q \\
&\quad -q\rho(-s\|N(p(x_n), u_n, \omega_n) - N(p(x_{n-1}), u_n, \omega_n)\|^q + \iota\|x_n - x_{n-1}\|^q) \\
&\le (\sigma^q\beta^q - q\rho\iota + q\rho s\nu^q\pi^q + c_q\rho^q\nu^q\pi^q)\|x_n - x_{n-1}\|^q \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
&\|N(p(x_{n-1}), u_n, \omega_n) - N(p(x_{n-1}), u_{n-1}, \omega_{n-1})\| \\
&\le \|N(p(x_{n-1}), u_n, \omega_n) - N(p(x_{n-1}), u_{n-1}, \omega_n)\| \\
&\quad +\|N(p(x_{n-1}), u_{n-1}, \omega_n) - N(p(x_{n-1}), u_{n-1}, \omega_{n-1})\| \\
&\le \delta\|u_n - u_{n-1}\| + \epsilon\|\omega_n - \omega_{n-1}\| \\
&\le (1 + n^{-1})(\delta\xi + \epsilon\zeta)\|x_n - x_{n-1}\|. \qquad (3.9)
\end{aligned}$$

Combining (3.5)–(3.9), we have

$$\|x_{n+1} - x_n\| \le \theta_n\|x_n - x_{n-1}\| + \frac{\tau^{q-1}}{r - \rho\lambda m}\|e_n - e_{n-1})\|, \qquad (3.10)$$

where

$$\begin{aligned}
\theta_n \quad &= \sqrt[q]{1 - q\alpha + (c_q + dq)\beta^q} + (1 + n^{-1})[\mu\gamma + \frac{\rho\tau^{q-1}}{r - \rho\lambda m}(\delta\xi + \epsilon\zeta)] \\
&\quad + \frac{\tau^{q-1}}{r - \rho\lambda m}\sqrt[q]{\sigma^q\beta^q - q\rho\iota + q\rho s\nu^q\pi^q + c_q\rho^q\nu^q\pi^q}.
\end{aligned}$$

Let $\theta = \mu\gamma + \sqrt[q]{1 - q\alpha + (c_q + dq)\beta^q} + \frac{\tau^{q-1}[\rho(\delta\xi+\epsilon\zeta) + \sqrt[q]{\sigma^q\beta^q - q\rho\iota + q\rho s\nu^q\pi^q + c_q\rho^q\nu^q\pi^q}]}{r - \rho\lambda m}$. Then we know that

$$\theta_n \downarrow \theta \quad \text{as} \quad n \to \infty.$$

From condition (3.4), we know that $0 < \theta < 1$ and hence there exist $n_0 > 0$ and $\theta_0 \in (\theta, 1)$ such that $\theta_n \leq \theta_0$ for all $n \geq n_0$. Therefore, by (3.10), we have

$$\|x_{n+1} - x_n\| \leq \theta_0\|x_n - x_{n-1}\| + \frac{\tau^{q-1}}{r - \rho\lambda m}\|e_n - e_{n-1}\|, \quad \forall n \geq n_0. \tag{3.11}$$

(3.11) implies that

$$\|x_{n+1} - x_n\| \leq \theta_0^{n-n_0}\|x_{n_0+1} - x_{n_0}\| + \sum_{j=1}^{n-n_0} \theta_0^{j-1} t_{n-(j-1)}, \quad \forall n \geq n_0,$$

where $t_n = \frac{\tau^{q-1}}{r-\rho\lambda m}\|e_n - e_{n-1}\|$ for all $n > n_0$. Hence, for any $m \geq n > n_0$, we have

$$\begin{aligned}
\|x_m - x_n\| &\leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \\
&\leq \sum_{i=n}^{m-1} \theta_0^{i-n_0}\|x_{n_0+1} - x_{n_0}\| + \sum_{i=n}^{m-1}[\sum_{j=1}^{i-n_0} \theta_0^{j-1} t_{i-(j-1)}] \\
&\leq \sum_{i=n}^{m-1} \theta_0^{i-n_0}\|x_{n_0+1} - x_{n_0}\| + \sum_{i=n}^{m-1} \theta_0^{i}[\sum_{j=1}^{i-n_0} \frac{t_{i-(j-1)}}{\theta_0^{i-(j-1)}}].
\end{aligned} \tag{3.12}$$

Since $\sum_{i=1}^{\infty} t_i\, k^{-i} < \infty,\ \forall k \in (0, 1)$ and $\theta_0 < 1$, it follows from (3.12) that $\|x_m - x_n\| \to 0$, as $n \to \infty$, and so $\{x_n\}$ is a Cauchy sequence in $\mathcal{B}$. Thus, there exists $x^* \in \mathcal{B}$ such that $x_n \to x^*$ as $n \to \infty$.

Now we prove that $u_n \to u^* \in S(x^*)$, $v_n \to v^* \in T(x^*)$ and $\omega_n \to \omega^* \in G(x^*)$. In fact, it follows from (3.7) and (3.9) that $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ are also Cauchy sequences in $\mathcal{B}$. Let $u_n \to u^*$, $v_n \to v^*$ and $\omega_n \to \omega^*$, respectively. In the sequel, we will show that $u^* \in S(x^*)$, $v^* \in T(x^*)$ and $\omega^* \in G(x^*)$. Noting $u_n \in S(x_n)$, we have

$$\begin{aligned}
d(u^*, S(x^*)) \quad &= \inf\{\|u_n - y\| : y \in S(x^*)\} \leq \|u^* - u_n\| + d(u_n, S(x^*)) \\
&\leq \|u^* - u_n\| + \hat{\mathbf{H}}(S(x_n), S(x^*)) \\
&\leq \|u^* - u_n\| + \xi\|x_n - x^*\| \to 0.
\end{aligned}$$

Hence $d(u^*, S(x^*)) = 0$ and therefore $u^* \in S(x^*)$. Similarly, we can prove that $v^* \in T(x^*)$ and $\omega^* \in G(x^*)$.

By continuity, x^*, u^*, v^*, w^* satisfy

$$g(x^*) = J^{\rho\lambda, A}_{\eta, M(\cdot, v^*)}(A(g(x^*)) - \rho N(p(x^*), u^*, \omega^*)).$$

By Lemma 3.1, (x^*, u^*, v^*, w^*) is a solution of problem (1.1). This completes the proof. ■

From Theorem 3.4, we have the following results.

Theorem 3.5: Let $\mathcal{B}$, A, S, G, η, g be the same as in Theorem 3.1. Assume that $M : \mathcal{B} \to 2^{\mathcal{B}}$ is (A, η)-accretive and $F : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be Lipschitz continuous with respect to both arguments with constants $\delta > 0$ and $\epsilon > 0$, respectively. If there exists constant $\rho > 0$

$$\begin{cases} h = \sqrt[q]{1 - q\alpha + (c_q + dq)\beta^q} < 1 - \sigma\beta r^{-1}\tau^{q-1}, \\ \rho < \max\{\frac{r(1-h)-\sigma\beta\tau^{q-1}}{\lambda m(1-h)+\tau^{q-1}(\delta\xi+\epsilon\zeta)}, \frac{r}{\lambda m}\}, \\ \sum_{i=1}^{\infty} \|e_i - e_{i-1}\| \, k^{-i} < \infty, \ \forall k \in (0,1), \quad \lim_{n\to\infty} e_n = 0, \end{cases}$$

where c_q is the constant as in Lemma 2.1, then the iterative sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ generated by Algorithm 3.2 converge strongly to x^*, u^* and w^*, respectively, and (x^*, u^*, w^*) is a solution of problem (1.3).

Proof. From Algorithm 3.2, the assumptions, and Lemma 2.1, we get

$$\begin{aligned} \|x_{n+1} - x_n\| \quad & \leq \|x_n - x_{n-1} - [g(x_n) - g(x_{n-1})]\| \\ & \quad + \|J_{\eta,M}^{\rho\lambda,A}((A(g(x_n)) - \rho(F(u_n, \omega_n) - f) + e_n) \\ & \quad - J_{\eta,M}^{\rho\lambda,A}((A(g(x_{n-1})) - \rho F(u_{n-1}, \omega_{n-1}) - f) + e_{n-1})\| \\ & \leq \|x_n - x_{n-1} - [g(x_n) - g(x_{n-1})]\| \\ & \quad + \frac{\tau^{q-1}}{r - \rho\lambda m}(\|A(g(x_n) - A(g(x_{n-1})))\| \\ & \quad + \rho\|F(u_n, \omega_n) - F(u_{n-1}, \omega_{n-1})\| + \|e_n - e_{n-1}\|) \\ & \leq \vartheta_n \|x_n - x_{n-1}\| + \frac{\tau^{q-1}}{r - \rho\lambda m}\|e_n - e_{n-1}\|, \end{aligned}$$

where

$$\vartheta_n = \sqrt[q]{1 - q\alpha + (c_q + dq)\beta^q} + \frac{\tau^{q-1}}{r - \rho\lambda m}[\sigma\beta + \rho(1 + n^{-1})(\delta\xi + \epsilon\zeta)].$$

Let $\vartheta = \sqrt[q]{1 - q\alpha + (c_q + dq)\beta^q} + \frac{\tau^{q-1}}{r-\rho\lambda m}[\sigma\beta + \rho(\delta\xi + \epsilon\zeta)]$. Then the rest proof can be completes by Theorem 3.4. ■

Theorem 3.6: Assume that $\mathcal{B}$, A, S, G, η, M and F are the same as in Theorem 3.2. Let $g : \mathcal{B} \to \mathcal{B}$ be α-strongly accretive and β-Lipschitz continuous. If there exists constant $\rho > 0$

$$\begin{cases} h = \sqrt[q]{1 - q\alpha + c_q\beta^q} < 1 - \sigma\beta r^{-1}\tau^{q-1}, \\ \rho < \max\{\frac{r(1-h)-\sigma\beta\tau^{q-1}}{\lambda m(1-h)+\tau^{q-1}(\delta\xi+\epsilon\zeta)}, \frac{r}{\lambda m}\}, \\ \sum_{i=1}^{\infty} \|e_i - e_{i-1}\| \, k^{-i} < \infty, \ \forall k \in (0,1), \quad \lim_{n\to\infty} e_n = 0, \end{cases}$$

where c_q is the constant as in Lemma 2.1, then the iterative sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ generated by Algorithm 3.2 converge strongly to x^*, u^* and w^*, respectively, and (x^*, u^*, w^*) is a solution of problem (1.3).

Remark 3.7: Theorem 3.6 is a correction and improvement of Theorem 4.1 in [16].

Remark 3.8: If M is (H, η)-accretive operator, A-accretive operator or other existing accretive operator and monotone operator, then we can obtain the corresponding results of Theorems 3.4–3.6. Our results improve and generalize the corresponding results of recent works.

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