International Journal of Applied Mathematical Sciences.
ISSN 0973-0176 Vol.3 No.1 (2006), pp. 11-19

http://www.gbspublisher.com/ijams.htm

A Fixed Point Theorem and its Application in Dynamic Programming

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Abstract

A fixed point theorem for a new contractive type mapping involving a countable family of pseudometrics is established. As its application, the existence, uniqueness and iterative approximation of solution for a functional equation arising in dynamic programming of multistage decision processes are given.

AMS subject classification: 19L20, 49L99, 90C39.
Keywords and Phrases: Fixed point, iterative approximation, solution, functional equation, dynamic programming.

1. Introduction and Preliminaries

Throughout this paper, we assume that $\mathbb{N}$ denotes the set of all positive integers and

$$\omega = \mathbb{N} \cup \{0\}, \quad \mathbb{R} = (-\infty, +\infty), \quad \mathbb{R}_+ = [0, +\infty),$$
$$\Phi = \{\varphi : \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is nondecreasing, right continuous and } \varphi(t) < t \text{ for } t > 0\}.$$

Let X be a nonempty set and $\{d_n\}_{n\in\mathbb{N}}$ be a countable family of pseudometrics on X such that for any distinct $x, y \in X$ there exists some $k \in \mathbb{N}$ with $d_k(x, y) \neq 0$. Define

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(x,y)}{1 + d_k(x,y)}, \ \forall x, y \in X.$$

It is easy to see that d is a metric on X. A sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ converges to a point $x \in X$ if and only if $d_k(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $d_k(x_m, x_n) \to 0$ as $m, n \to \infty$ for any $k \in \mathbb{N}$. Let $f : (X, d) \to (X, d)$ be a mapping. For any $A \subseteq X,\ x, y \in X$ and $k \in \mathbb{N}$, put

$$O_f(x) = \{f^i x : i \in \omega\}, \quad O_f(x,y) = O_f(x) \cup O_f(y),$$
$$\delta_k(A) = \sup\{d_k(a,b) : a, b \in A\}.$$

X is said to be *f-orbitally complete* if every Cauchy sequence which is contained in $O_f(x)$ for $x \in X$ converges in X.

Several classes of functional equations and systems of functional equations in dynamic programming have been studied by some researchers by using various methods and techniques, for example, see [1–19] and the references therein. Bellman and Lee [3] suggested the basic form of the functional equations of dynamic programming as follows:

$$f(x) = opt_{y\in D} H(x, y, f(T(x,y))), \quad \forall x \in S, \tag{1.1}$$

where x and y represent the state and decision vectors, respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return function with initial state x. Bellman and Roosta [4] established approximation solutions for a kind of infinite-stage equations arising in dynamic programming. Bhakta and Choudhury [5] investigated the existence and uniqueness of fixed point for the following contractive type mapping:

$$d_k(fx, fy) \le \varphi(d_k(x,y)), \quad \forall (k, x, y) \in \mathbb{N} \times X \times X, \tag{1.2}$$

where $\varphi \in \Phi$, and using the fixed point theorem, they gave the existence and uniqueness of solution for the functional equation:

$$f(x) = inf_{y\in D} H(x, y, f(T(x,y))), \quad \forall x \in S. \tag{1.3}$$

Liu [9] considered the following more general contractive type mapping and functional equation, respectively:

$$d_k(fx, fy) \le \varphi(\delta_k(O_f(x, y))), \quad \forall (k, x, y) \in \mathbb{N} \times X \times X, \tag{1.4}$$

where $\varphi \in \Phi$, and

$$f(x) = opt_{y \in D}\{u(x, y) + H(x, y, f(T(x, y)))\}, \quad \forall x \in S, \tag{1.5}$$

where opt denotes $\sup$ or $\inf$, and established a few sufficient conditions which ensure the existence, uniqueness and iterative approximation of fixed point and solution for (1.4) and (1.5), respectively.

Motivated and inspired by the work in [1–19], in this paper we introduce a class of new and more general contractive type mapping as follows:

There exist $p, q \in \mathbb{N}$ and $\varphi \in \Phi$ satisfying

$$d_k(f^p x, f^q y) \le \varphi(\delta_k(O_f(x, y))), \quad \forall (k, x, y) \in \mathbb{N} \times X \times X. \tag{1.6}$$

Under certain conditions we establish the existence, uniqueness and iterative approximation of fixed point for the mapping f, which satisfies (1.6). As an application, we provide a sufficient condition which guarantees the existence, uniqueness and iterative approximation of solution for the functional equation (1.5). The results presented in this paper are generalizations of the corresponding results in [4–6, 9].

2. A Fixed Point Theorem

Now we prove the following fixed point theorem.

Theorem 2.1: Let (X, d) be a f-orbitally complete metric space and $f : (X, d) \to (X, d)$ satisfy (1.6) for some $(p, q, \varphi) \in \{1\} \times \{1, 2\} \times \Phi$. If for any $k \in \mathbb{N}$ and $x \in X$, there exists $h(k, x) > 0$ such that $\delta_k(O_f(x)) \le h(k, x)$, then f has a unique fixed point $w \in X$ and the iterative sequence $\{f^n x\}_{n \in \mathbb{N}}$ converges to w for each $x \in X$.

Proof. Let $(k, x_0) \in \mathbb{N} \times X$. Put $x_n = f^n x_0$ and $d_{ki} = d_{ki}(x_0) = \delta_k(O_f(x_i))$ for $n \in \mathbb{N}$ and $i \in \omega$. Since $\{d_{ki}\}_{i \in \omega}$ is nonincreasing and bounded below by 0, it follows that $\{d_{ki}\}_{i \in \omega}$ converges to $r \ge 0$. We now assert that $r = 0$. Otherwise $r > 0$. For any given $\varepsilon > 0$, there exists $j \in \mathbb{N}$ such that $r \le d_{ki} < r + \varepsilon$ for all $i \ge j$. It follows from (1.6) that

$$\begin{aligned} d_k(f^{n+q}x_0, f^{m+q}x_0) &= d_k(f f^{n+q-1}x_0, f^q f^m x_0) \\ &\le \varphi(\delta_k(O_f(f^{n+q-1}x_0, f^m x_0))) \\ &\le \varphi(\delta_k(O_f(x_{i+j}))) \\ &= \varphi(d_{ki+j}), \quad \forall n, m \ge i + j, \end{aligned}$$

which implies that $d_{ki+j+q} \le \varphi(d_{ki+j})$ for all $i \ge j$. Letting $i \to \infty$, by the right continuity of φ, we deduce that $0 < r \le \varphi(r) < r$, which is a contradiction. Consequently, we have

$$\lim_{i\to\infty} d_{ki}(x_0) = 0, \tag{2.1}$$

which yields that $\{f^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Because (X, d) is f-orbital complete, it follows that $\{f^n x_0\}_{n\in\mathbb{N}}$ converges to a point $w \in X$. Put

$$c_{kn} = \sup\{d_k(f^m x_0, f^m w) : m \ge n\}, \quad \forall n \in \mathbb{N}.$$

It is clear that $\{c_{kn}\}_{n\in\mathbb{N}}$ is nonincreasing and

$$\lim_{n\to\infty} c_{kn} = s \quad \text{for some } s \ge 0. \tag{2.2}$$

Suppose that $s > 0$. (2.1) and (2,2) ensure that for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$s \le c_{kn} < s + \varepsilon, \quad d_{kn}(x_0) < \varepsilon \quad \text{and} \quad d_{kn}(w) < \varepsilon, \forall n \ge m. \tag{2.3}$$

In view of (1.6) and (2.3), we get that for $i, j \in \omega$,

$$\begin{aligned} d_k(f^{m+i+q}x_0, f^{m+j+q}w) &= d_k(ff^{m+i+q-1}x_0, f^q f^{m+j}w) \\ &\le \varphi(\delta_k(O_f(f^{m+i+q-1}x_0, f^{m+j}w))) \\ &\le \varphi(\delta_k(O_f(f^m x_0, f^m w))) \\ &\le \varphi(d_{km}(x_0) + d_{km}(w) + c_{km}), \end{aligned}$$

which implies that $s \le c_{km+q} \le \varphi(3\varepsilon + s)$. That is, $s \le \varphi(3\varepsilon + s)$. Letting $\varepsilon \to 0$, we gain that $0 < s \le \varphi(s) < s$, which is impossible. Therefore $s = 0$ and (2.2) yields that $\lim_{n\to\infty} d_k(f^n x_0, f^n w) = 0$.

We now prove that $\lim_{n\to\infty} d(f^n x_0, f^n w) = 0$. For any given $\varepsilon > 0$, determine $t \in \mathbb{N}$ with $\sum_{k=t+1}^{\infty} \frac{1}{2^k} < \frac{1}{2}\varepsilon$. Since $\lim_{n\to\infty} \left[\sum_{k=1}^{t} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1+d_k(f^n x_0, f^n w)}\right] = 0$, there exists some $M \in \mathbb{N}$ such that $\sum_{k=1}^{t} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1+d_k(f^n x_0, f^n w)} < \frac{1}{2}\varepsilon$ for $n \ge M$. Hence,

$$\begin{aligned} d(f^n x_0, f^n w) &= \sum_{k=1}^{t} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1 + d_k(f^n x_0, f^n w)} + \sum_{k=t+1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f^n x_0, f^n w)}{1 + d_k(f^n x_0, f^n w)} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \ \forall n \ge M. \end{aligned}$$

This means that $\lim_{n\to\infty} d(f^n x_0, f^n w) = 0$. By virtue of $\lim_{n\to\infty} f^n x_0 = w$, we see that $\lim_{n\to\infty} f^n w = w$.

Next we prove that

$$d_{k0}(w) = d_{k1}(w) = \cdots = d_{ki}(w), \quad \forall i \in \mathbb{N}. \tag{2.4}$$

Suppose that $d_{k0}(w) > d_{k1}(w)$. From

$$\begin{aligned} d_{k0}(w) &= \max\{\sup\{d_k(f^n w, w) : n \in \mathbb{N}\}, d_{k1}(w)\} \\ &= \sup\{d_k(f^n w, w) : n \in \mathbb{N}\}, \end{aligned} \tag{2.5}$$

it follows that

$$\begin{aligned} d_k(w, f^m w) &\le d_k(w, f^n w) + d_k(f^n w, f^m w) \\ &\le d_k(w, f^n w) + d_{k1}(w), \quad \forall n, m \in \mathbb{N} \text{ with } n > m. \end{aligned}$$

Letting $n \to \infty$, we have

$$d_k(w, f^m w) \le d_{k1}(w), \quad \forall m \in \mathbb{N}. \tag{2.6}$$

By virtue of (2.5) and (2.6), we conclude that $d_{k0}(w) \le d_{k1}(w)$, which is a contradiction. Hence $d_{k0}(w) \le d_{k1}(w)$. Since $\{d_{ki}(w)\}_{i\in\varpi}$ is nonincreasing, we get that $d_{k0}(w) = d_{k1}(w)$. That is, (2.4) holds for $i = 1$. Suppose that (2.4) holds for some $i = j$. If (2.4) does not hold for for $i = j + 1$, that is,

$$d_{k0}(w) = d_{k1}(w) = \cdots = d_{kj}(w) > d_{kj+1}(w). \tag{2.7}$$

Note that (2.7) means that

$$\begin{aligned} d_{kj}(w) &= \max\{\sup\{d_k(f^n w, f^j w) : n > j\}, d_{kj+1}(w)\} \\ &= \sup\{d_k(f^n w, f^j w) : n > j\}. \end{aligned} \tag{2.8}$$

In light of (1.6), we derive that

$$\begin{aligned} d_k(f^j w, f^{j+r} w) &\le \varphi(\delta_k(O_f(f^{j-1} w, f^{j+r-q} w))) \\ &\le \varphi(d_{kj-1}(w)), \quad \forall r \in \mathbb{N}. \end{aligned} \tag{2.9}$$

From (2.7)–(2.9), we deduce that

$$d_{kj}(w) \le \varphi(d_{kj-1}(w)) = \varphi(d_{kj}(w)),$$

which yields that $d_{kj}(w) = 0$. It follows from (2.7) that

$$0 > d_{k(j+1)}(w) \ge 0,$$

which is a contradiction. Hence (2.4) holds for $i = j + 1$. Thus (2.4) holds. Clearly, (2.1) and (2.4) ensure that $d_{k0}(w) = 0$. Therefore,

$$d(w, fw) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(w, fw)}{1 + d_k(w, fw)} \le \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_{k0}(w)}{1 + d_{k0}(w)} = 0,$$

which means that $w = fw$.

Finally, we claim that w is the unique fixed point of f. In fact, if $v \in X$ is another fixed point of f different from w. It follows from (1.6) that

$$d_k(w,v) = d_k(f^p w, f^q v) \le \varphi(\delta_k(O_f(w,v))) = \varphi(d_k(w,v)),$$

which implies that $d_k(w,v) = 0$. This leads to

$$d(w,v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(w,v)}{1 + d_k(w,v)} = 0,$$

that is, $w = v$, which is a contradiction. This completes the proof. ■

Remark 2.2: Theorem 2.1 extends the corresponding result of Bellman and Roosta [4], Theorem 2.1 of Bhakta and Choudhury [5], Theorem 1.1 of Bhakta and Mitra [6] and Theorem 2.1 of Liu [9].

Remark 2.3: We could only prove Theorem 2.1 for $(p,q) \in \{1\} \times \{1,2\}$. For further research, in our opinion, it is interesting and important to study the following problems:

Problem 2.1: Does Theorem 2.1 hold for $(p,q) \in \{1\} \times (\mathbb{N} \setminus \{1,2\})$?

Problem 2.2: Does Theorem 2.1 hold for $(p,q) \in \{2\} \times (\mathbb{N} \setminus \{1\})$?

Problem 2.3: Does Theorem 2.1 hold for $(p,q) \in (\mathbb{N} \setminus \{1,2\}) \times (\mathbb{N} \setminus \{1,2\})$?

3. Existence and Uniqueness of Solution

Throughout this section, let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real Banach spaces, $S \subset X$ be the state space, and $D \subset Y$ be the decision space. $BB(S)$ denotes the set of all real-value mappings on S which are bounded on bounded subsets of S. It is easy to see that $BB(S)$ is a linear space over $\mathbb{R}$ under usual definitions of addition and multiplication by scalars. For any $k \in \mathbb{N}$ and $a, b \in BB(S)$, let

$$d_k(a,b) = \sup\{\|a(x) - b(x)\| : x \in \overline{B}(0,k)\},$$
$$d(a,b) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(a,b)}{1 + d_k(a,b)},$$

where $\overline{B}(0,k) = \{x : x \in S \text{ and } \|x\| \le k\}$. Clearly, $\{d_k\}_{k\in\mathbb{N}}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space. Let $H : D \times D \times BB(S) \to \mathbb{R}$, $T : S \times D \to S$ and $u : S \times D \to \mathbb{R}$.

Theorem 3.1: Assume that the following conditions hold.

(a) For any $k \in \mathbb{N}$ and $a \in BB(S)$, there exists $h(k,a) > 0$ such that

$$\max\{|u(x,y)| + |H(x,y,a(T(x,y)))|, \delta_k(O_f(a))\} \le h(k,a), \quad \forall (x,y) \in \overline{B}(0,k) \times D,$$

where the mapping f is defined as

$$fa(x) = opt_{y\in D}\{u(x,y) + H(x,y,a(T(x,y)))\}, \quad \forall (x,a) \in S \times BB(S); \qquad (3.1)$$

(b) There exist $\varphi \in \Phi_1$ and $q \in \{1,2\}$ such that

$$\begin{aligned} &|H(x,y,a(t)) - H(x,y,f^{q-1}b(t))| \\ &\le \varphi(\delta_k(O_f(a,b))), \quad \forall (k,x,y,t) \in \mathbb{N} \times \overline{B}(0,k) \times D \times S. \end{aligned}$$

Then the functional equation (1.5) possesses a unique solution $w \in BB(S)$ and $\{f^n a\}_{n\in\mathbb{N}}$ converges to w for each $a \in BB(S)$.

Proof. Let k be in $\mathbb{N}$ and a be in $BB(S)$. Using (a), we know that

$$|u(x,y)| + |H(x,y,a(T(x,y)))| \le h(k,a), \ \forall (x,y) \in \overline{B}(0,k) \times D.$$

(3.1) ensures that f maps $BB(S)$ into itself. For any $k \in \mathbb{N}, a, b \in BB(S)$ and $x \in S$, we have to consider the following possible cases:

Case 1. Suppose that $opt = \inf$. For any $\varepsilon > 0$, there exist $y, z \in D$ such that

$$fa(x) > u(x,y) + H(x,y,a(T(x,y))) - \varepsilon \qquad (3.2)$$

and

$$f^q b(x) > u(x,z) + H(x,z,f^{q-1}b(T(x,z))) - \varepsilon. \qquad (3.3)$$

Clearly,

$$fa(x) \le u(x,z) + H(x,z,a(T(x,z))) \qquad (3.4)$$

and

$$f^q b(x) \le u(x,y) + H(x,y,f^{q-1}b(T(x,y))). \qquad (3.5)$$

It follows from (3.3), (3.4) and (b) that

$$\begin{aligned} fa(x) - f^q b(x) &< H(x,z,a(T(x,z))) - H(x,z,f^{q-1}b(T(x,z))) + \varepsilon \\ &\le |H(x,z,a(T(x,z))) - H(x,z,f^{q-1}b(T(x,z)))| + \varepsilon \\ &\le \varphi(\delta_k(O_f(a,b))) + \varepsilon. \end{aligned} \qquad (3.6)$$

By virtue of (3.2), (3.5) and (b), we infer that

$$\begin{aligned} fa(x) - f^q b(x) &> H(x,y,a(T(x,y))) - H(x,y,f^{q-1}b(T(x,y))) - \varepsilon \\ &\ge -|H(x,y,a(T(x,y))) - H(x,y,f^{q-1}b(T(x,y)))| - \varepsilon \\ &\ge -\varphi(\delta_k(O_f(a,b))) - \varepsilon. \end{aligned} \qquad (3.7)$$

It follows from (3.6) and (3.7) that

$$|fa(x) - f^q b(x)| \le \varphi(\delta_k(O_f(a,b))) + \varepsilon,$$

which implies that

$$d_k(fa, fb) \leq \varphi(\delta_k(O_f(a, b))) + \varepsilon.$$

Letting $\varepsilon \to \infty$, we derive that

$$d_k(fa, f^q b) \leq \varphi(\delta_k(O_f(a, b))). \tag{3.8}$$

Case 2. Suppose that $opt = \sup$. As in the proof of Case 1, we infer that (3.8) holds also. Therefore Theorem 2.1 guarantees that f has a unique fixed point $w \in BB(S)$ and $\{f^n a\}_{n\in\mathbb{N}}$ converges to w for each $a \in BB(S)$. That is, $w(x)$ is a unique solution of the functional equation (1.5). Thus the proof is completed. ■

Remark 3.2: Theorem 3.1 extends Theorem 3.1 of Bhakta and Choudury [5], Theorem 2.1 and Corollary 2.1 of Bhakta and Mitra [6] and Theorem 3.1 of Liu [9].

Acknowledgement

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2006) and the Research Institute of Natural Science Grant, Gyeongsang National University, 2005.

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