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On Convergence of Three Classes of Implicit Iteration Methods

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Abstract

In this paper, a few convergence results of three classes of implicit iteration methods for a finite family of Lipschitz mappings in Banach spaces are established.

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1. Introduction

Let K be a nonempty closed convex subset of a real Banach space X and $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of K. In [3], Xu and Ori constructed the following implicit iteration process:

For each $x_0 \in K$, compute $\{x_n\}_{n\geq 0}$ by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}_{n\geq 1} \subseteq [0,1]$ and $T_n = T_{n \ mod \ N}$, and proved that the implicit iteration process (1.1) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$. Zhou and Chang [4] studied the weak and strong convergence of the implicit iteration process (1.1) to a common fixed point for a family of nonexpansive mappings. Chidume and Shahzade [1] established the convergence theorems of the implicit iteration process (1.1) for a family of asymptotically nonexpansive mappings. Recently, Osilike [2] extended the results of Xu and Ori to a finite family of strictly pseudocontractive mappings.

Motivated and inspired by the results in [1–4], in this paper, we prove several convergence theorems of three kinds of implicit iteration methods involving a finite family of Lipschitz mappings in Banach spaces. Our results unify, improve and generalize the corresponding results in [1–3].

2. Preliminaries

Throughout this paper, let K be a nonempty closed convex subset of a real Banach space X, and $\{T_i\}_{i=1}^N$ be a finite family of Lipschitz self-mappings on K with common Lipschitz constant $L \geq 1$ and the set $F = \{x \in K : T_i x = x, 1 \leq i \leq N\}$ be nonempty.

Let $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be real sequences in $[0,1]$ and $T_n = T_{n \ mod \ N}$ for $n \geq 1$. Based on the implicit iteration process (1.1), we give the following three classes of implicit iteration methods:

(A) Let $x_0 \in K$, compute $\{x_n\}_{n\geq 0}$ by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n, \\ y_n &= \beta_n x_{n-1} + (1-\beta_n)T_n z_n, \\ z_n &= \gamma_n x_{n-1} + (1-\gamma_n)T_n x_n, \quad n \geq 1. \end{aligned}$$

(B) Let $x_0 \in K$, compute $\{x_n\}_{n\geq 0}$ by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n, \\ y_n &= \beta_n x_n + (1-\beta_n)T_n z_n, \\ z_n &= \gamma_n x_n + (1-\gamma_n)T_n x_n, \quad n \geq 1. \end{aligned}$$

(C) Let $x_0 \in K$, compute $\{x_n\}_{n\geq 0}$ by

$$\begin{aligned}
x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n,\\
y_n &= \beta_n x_{n-1} + (1-\beta_n)T_n z_n,\\
z_n &= \gamma_n x_n + (1-\gamma_n)T_n x_n, \quad n \geq 1.
\end{aligned}$$

We now claim that the implicit iteration method (A) is well defined if the following condition

$$L^3(1-\alpha_n)(1-\beta_n)(1-\gamma_n) < 1, \quad n \geq 1 \tag{2.1}$$

is satisfied. In fact, let $T : K \to K$ be a Lipschitz mapping with Lipschitz constant L. For any given $u \in K$, α, β and $\gamma \in [0,1]$, define a mapping $S : K \to K$ by

$$Sx = \alpha u + (1-\alpha)T\{\beta u + (1-\beta)T[\gamma u + (1-\gamma)Tx]\}, \quad x \in K.$$

It is easy to verify that

$$\|Sx - Sy\| \leq L^3(1-\alpha)(1-\beta)(1-\gamma)\|x-y\|, \quad x, y \in K.$$

Since $L^3(1-\alpha)(1-\beta)(1-\gamma) < 1$, S is a contraction mapping and it has a unique fixed point $u_1 \in K$. This implies that if (2.1) holds, the implicit iteration method (A) is well defined.

Similarly, we get that if the following condition

$$L(1-\alpha_n)[\beta_n + L(1-\beta_n)\gamma_n + L^2(1-\beta_n)(1-\gamma_n)] < 1, \quad n \geq 1 \tag{2.2}$$

is fulfilled, then the implicit iteration method (B) is well defined and if the following condition

$$L^2(1-\alpha_n)(1-\beta_n)[\gamma_n + L(1-\gamma_n)] < 1, \quad n \geq 1 \tag{2.3}$$

is satisfied, then the implicit iteration method (C) is well defined.

3. Main Results

Our main results are as follows.

Theorem 3.1: Let K be a nonempty closed convex subset of a real Banach space X, and $\{T_i\}_{i=1}^N$ be a finite family of Lipschitz self-mappings on K with common Lipschitz constant L and the set $F = \{x \in K : T_i x = x, 1 \leq i \leq N\} \neq \emptyset$. Assume that $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ are real sequences in $[0,1]$ satisfying (2.1) and

$$\sum_{n=1}^{\infty}(1-\alpha_n) < +\infty. \tag{3.1}$$

Then for any given $x_0 \in K$, there exists some $q \in K$ such that the sequence $\{x_n\}_{n\geq 0}$ defined by the implicit iteration method (A) converges to q. Moreover, if the following condition

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 1. \tag{3.2}$$

is satisfied, then the sequences $\{y_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ defined by the implicit iteration method (A) converge to q, respectively.

Proof. It follows from (3.1) that there exists a positive integer m satisfying $1 - L^3(1 - \alpha_n) > \frac{1}{2}$ for $n \geq m.$ Put

$$A = 1 + L + L^2 + L^3 + 6L^6, \quad C = A\|x_m - p\| \exp\left\{6L^3 \sum_{i=m+1}^{\infty} (1 - \alpha_n)\right\}.$$

Note that (2.1) ensures that the sequence $\{x_n\}_{n\geq 0}$ defined by (A) is well defined. For each $p \in F$, we conclude that

$$\begin{aligned}
\|x_n - p\| &\leq \alpha_n\|x_{n-1} - p\| + L(1 - \alpha_n)\|y_n - p\| \\
&\leq [\alpha_n + L(1 - \alpha_n)\beta_n]\|x_{n-1} - p\| + L^2(1 - \alpha_n)(1 - \beta_n)\|z_n - p\| \\
&\leq [\alpha_n + L(1 - \alpha_n)\beta_n + L^2(1 - \alpha_n)(1 - \beta_n)\gamma_n]\|x_{n-1} - p\| \\
&\quad + L^3(1 - \alpha_n)(1 - \beta_n)(1 - \gamma_n)\|x_n - p\|, \quad n \geq 1,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_n - p\| &\leq \frac{\alpha_n + L(1 - \alpha_n)\beta_n + L^2(1 - \alpha_n)(1 - \beta_n)\gamma_n}{1 - L^3(1 - \alpha_n)(1 - \beta_n)(1 - \gamma_n)}\|x_{n-1} - p\| \\
&\leq \left(1 + \frac{3L^3(1 - \alpha_n)}{1 - L^3(1 - \alpha_n)}\right)\|x_{n-1} - p\| \\
&\leq (1 + 6L^3(1 - \alpha_n))\|x_{n-1} - p\| \\
&\leq \exp\{6L^3(1 - \alpha_n)\}\|x_{n-1} - p\| \\
&\leq \|x_m - p\| \exp\left\{6L^3 \sum_{i=m+1}^{n} (1 - \alpha_i)\right\}, \quad n \geq m + 1.
\end{aligned} \tag{3.3}$$

In view of (A) and (3.3), we infer that

$$\begin{aligned}
\|x_n - x_{n-1}\| &= (1 - \alpha_n)\|x_{n-1} - T_n y_n\| \\
&\leq (1 - \alpha_n)(\|x_{n-1} - p\| + \|T_n y_n - p\|) \\
&\leq (1 - \alpha_n)(\|x_{n-1} - p\| + L\|y_n - p\|) \\
&\leq (1 - \alpha_n)[(1 + L\beta_n + L^2(1 - \beta_n)\gamma_n)\|x_{n-1} - p\| \\
&\quad + L^3(1 - \beta_n)(1 - \gamma_n)\|x_n - p\|] \\
&\leq A(1 - \alpha_n)\|x_{n-1} - p\| \\
&\leq A(1 - \alpha_n)\|x_m - p\| \exp\left\{6L^3 \sum_{i=m+1}^{n-1} (1 - \alpha_i)\right\} \\
&\leq C(1 - \alpha_n), \quad n \geq m + 1,
\end{aligned}$$

which means that

$$\|x_{n+r}-x_n\| \le \sum_{i=n}^{n+r}\|x_{i+1}-x_i\| \le C\sum_{i=n}^{n+r}(1-\alpha_n), \quad n \ge m,\ r \ge 1. \tag{3.4}$$

It follows from (3.1) and (3.4) that $\{x_n\}_{n\ge 0}$ is a Cauchy sequence in K and it converges to some $q \in K$.

Assume that (3.2) holds. Since $\{T_i\}_{i=1}^N$ are a finite family of Lipschitz mappings on K and $\{x_n\}_{n\ge 1}$ converges to q, it follows that $\{T_n x_n\}_{n\ge 1}$ is bounded. Using (A) and (3.2), we know that

$$\|z_n-q\| \le \gamma_n\|x_{n-1}-q\| + (1-\gamma_n)\|T_n x_n - q\| \to 0 \quad \text{as } n \to \infty,$$

which gives that $\lim_{n\to\infty} z_n = q$ and $\{T_n z_n\}_{n\ge 1}$ is bounded. By virtue of (A) and (3.2), we get that

$$\|y_n-q\| \le \beta_n\|x_{n-1}-q\| + (1-\beta_n)\|T_n z_n - q\| \to 0 \quad \text{as } n \to \infty,$$

which implies that $\lim_{n\to\infty} y_n = q$. This completes the proof. ■

Theorem 3.2: Let K, $\{T_i\}_{i=1}^N$ and F be as in Theorem 3.1. Assume that $\{\alpha_n\}_{n\ge 1}$, $\{\beta_n\}_{n\ge 1}$ and $\{\gamma_n\}_{n\ge 1}$ are real sequences in $[0,1]$ satisfying (2.2) and (3.1). Then for any given $x_0 \in K$, there exists some $q \in K$ such that the sequence $\{x_n\}_{n\ge 0}$ defined by the implicit iteration method (B) converges to q. In addition, if (3.3) holds, then the sequences $\{y_n\}_{n\ge 1}$ and $\{z_n\}_{n\ge 1}$ defined by the implicit iteration method (B) converge to q, respectively.

Proof. It is clear that (3.1) implies that there exists a positive integer m satisfying $(1-\alpha_n)L^6 > \frac{1}{9}$ for $n \ge m$. Put

$$A = (1+L+L^3+L^4)(1+3L^3), \quad C = A\|x_m-p\|\exp\left\{3L^3\sum_{i=m+1}^{\infty}(1-\alpha_n)\right\}.$$

It follows from (2.2) that the sequence $\{x_n\}_{n\ge 0}$ defined by (B) is well defined. In view of (B) we deduce that for each $p \in F$,

$$\begin{aligned}\|x_n-p\| &\le \alpha_n\|x_{n-1}-p\| + L(1-\alpha_n)\|y_n-p\| \\ &\le \alpha_n\|x_{n-1}-p\| + L(1-\alpha_n)[\beta_n\|x_n-p\| + L(1-\beta_n)\|z_n-p\|] \\ &\le \alpha_n\|x_{n-1}-p\| + L(1-\alpha_n)[\beta_n \\ &\quad + L(1-\beta_n)(\gamma_n + L(1-\gamma_n))]\|x_n-p\|, \quad n \ge 1,\end{aligned}$$

which yields that

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n \|x_{n-1} - p\|}{1 - L(1-\alpha_n)[\beta_n + L(1-\beta_n)\gamma_n + L^2(1-\beta_n)(1-\gamma_n)]}\\ &\le \frac{\alpha_n}{1 - 3L^3(1-\alpha_n)}\|x_{n-1} - p\|\\ &\le (1 + 3L^3(1-\alpha_n))\|x_{n-1} - p\|,\\ &\le \|x_m - p\| \exp\left\{3L^3 \sum_{i=m+1}^{n} (1-\alpha_i)\right\}, \quad n \ge m+1.\end{aligned} \tag{3.5}$$

In light of (B) and (3.5), we infer that

$$\begin{aligned}\|x_n - x_{n-1}\| &= (1-\alpha_n)\|x_{n-1} - T_n y_n\|\\ &\le (1-\alpha_n)(\|x_{n-1} - p\| + L\|y_n - p\|)\\ &\le (1-\alpha_n)[\|x_{n-1} - p\| + L(\beta_n\|x_n - p\| + L(1-\beta_n)\|z_n - p\|)]\\ &\le (1-\alpha_n)\{\|x_{n-1} - p\|\\ &\quad + L[\beta_n + L(1-\beta_n)(\gamma_n + L(1-\gamma_n))]\|x_n - p\|\}\\ &\le A(1-\alpha_n)\|x_{n-1} - p\|\\ &\le A(1-\alpha_n)\|x_m - p\| \exp\left\{3L^3 \sum_{i=m+1}^{n-1} (1-\alpha_i)\right\}\\ &\le C(1-\alpha_n), \quad n \ge m+1,\end{aligned}$$

which implies that (3.4) holds. The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. ■

Theorem 3.3: Let K, $\{T_i\}_{i=1}^N$ and F be as in Theorem 3.1. Assume that $\{\alpha_n\}_{n\ge1}$, $\{\beta_n\}_{n\ge1}$ and $\{\gamma_n\}_{n\ge1}$ are real sequences in $[0,1]$ satisfying (2.3) and (3.1). Then for any given $x_0 \in K$, there exists some $q \in K$ such that the sequence $\{x_n\}_{n\ge0}$ defined by the implicit iteration method (C) converges to q. In addition, if (3.3) holds, then the sequences $\{y_n\}_{n\ge1}$ and $\{z_n\}_{n\ge1}$ defined by the implicit iteration method (C) converge to q, respectively.

Proof. Obviously (3.1) guarantees that there exits a positive integer m satisfying $(1 - \alpha_n)L^2(1+L) < \frac{1}{2}$ for $n \ge m$. Set

$$\begin{aligned}A &= (1+L)(1 + L^2 + 2L^2(L + L^2 + L^3 - 1)),\\ C &= A\|x_m - p\| \exp\left\{(L + L^2 + L^3 - 1) \sum_{i=m+1}^{\infty} (1-\alpha_n)\right\}.\end{aligned}$$

Let p be in F. It follows from (C) that

$$\begin{aligned}\|x_n - p\| &\le \alpha_n\|x_{n-1} - p\| + L(1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n\|x_{n-1} - p\| + L(1-\alpha_n)[\beta_n\|x_{n-1} - p\| + L(1-\beta_n)\|z_n - p\|] \\ &\le (\alpha_n + L(1-\alpha_n)\beta_n)\|x_{n-1} - p\| + \\ &\quad L^2(1-\alpha_n)(1-\beta_n)(\gamma_n + L(1-\gamma_n))\|x_n - p\|, \quad n \ge 1,\end{aligned}$$

which gives that

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n + L(1-\alpha_n)\beta_n}{1 - L^2(1-\alpha_n)(1-\beta_n)[\gamma_n + L(1-\gamma_n)]}\|x_{n-1} - p\| \\ &\le \frac{\alpha_n + L(1-\alpha_n)\beta_n}{1 - L^2(1+L)(1-\alpha_n)}\|x_{n-1} - p\| \\ &= \left[1 + \frac{(1-\alpha_n)(L^2 + L^3 + L\beta_n - 1)}{1 - L^2(1+L)(1-\alpha_n)}\right]\|x_{n-1} - p\| \\ &\le [1 + 2(L + L^2 + L^3 - 1)(1-\alpha_n)]\|x_{n-1} - p\|, \\ &\le \|x_m - p\| \exp\left\{2(L + L^2 + L^3 - 1)\sum_{i=m+1}^{n-1}(1-\alpha_i)\right\}, \quad n \ge m+1.\end{aligned} \tag{3.6}$$

By (C) and (3.6), we get that

$$\begin{aligned}\|x_n - x_{n-1}\| &= (1-\alpha_n)\|x_{n-1} - T_n y_n\| \\ &\le (1-\alpha_n)(\|x_{n-1} - p\| + L\|y_n - p\|) \\ &\le (1-\alpha_n)[\|x_{n-1} - p\| + L(\beta_n\|x_{n-1} - p\| + L(1-\beta_n)\|z_n - p\|)] \\ &\le (1-\alpha_n)\{(1 + L\beta_n)\|x_{n-1} - p\| \\ &\quad + L^2(1-\beta_n)[\gamma_n + L(1-\gamma_n)]\|x_n - p\|\} \\ &\le (1-\alpha_n)A\|x_{n-1} - p\| \\ &\le (1-\alpha_n)A\|x_m - p\| \exp\left\{2(L + L^2 + L^3 - 1)\sum_{i=m+1}^{n-1}(1-\alpha_i)\right\} \\ &\le C(1-\alpha_n), \quad n \ge m+1,\end{aligned}$$

which implies that (3.4) holds. The rest of the proof is identical with the proof of Theorem 3.1 and is omitted. This completes the proof. ■

Remark 3.1: Theorems 3.1–3.3 unify, improve and generalize the correspondent results in [1–3].

4. Examples

In this section, we construct two examples to explain Theorems 3.1–3.3.

Example 4.1: Let $X = \mathbb{R}$, $K = [0,1]$ and

$$\alpha_n = 1 - \frac{1}{4(n+4)^2}, \quad \beta_n = 1 - \frac{1}{n+1}, \quad \gamma_n = 1 - \frac{1}{\sqrt{2n+1}}, \quad n \geq 1.$$

Define three mappings $T_1, T_2, T_3 : K \to K$ by

$$T_1(x) = \frac{x}{2+\sin x}, \quad T_2(x) = \sqrt{1+x^2} - 1, \quad T_3 = \sin x, \quad x \in K.$$

It is easy to verify that $\{T_i : i = 1,2,3\}$ is a finite family of Lipschitz self-mappings on K with common Lipschitz constant $L = 4$, $0 \in F = \{x \in K : x = T_i x, i = 1,2,3\} \neq \emptyset$ and (2.1)–(2.3), (3.1) and (3.2) hold. That is, the assumptions of Theorems 3.1–3.3 are fulfilled. It follows from any one of Theorems 3.1–3.3 that for each $x_0 \in K$, there exists $q \in K$ such that the sequences $\{x_n\}_{n\geq1}$, $\{y_n\}_{n\geq1}$ and $\{z_n\}_{n\geq1}$ defined by any one of the implicit iteration methods (A)-(C) converge to q.

Example 4.2: Let $X = \mathbb{R}$, $K = [0, +\infty)$.

$$\alpha_n = 1 - \frac{1}{3(n^2+1)}, \quad \beta_n = 1 - \frac{1}{1+\sqrt{n}}, \quad \gamma_n = 1 - \frac{1}{n+n^2}, \quad n \geq 1.$$

Define mappings $T_1, T_2 : K \to K$ by

$$T_1 x = \frac{x^2}{1+x}, \quad T_2 = \frac{1}{3}|\sin^3 x|, \quad x \in K.$$

Clearly, $\{T_i : i = 1,2\}$ is a finite family of Lipschitz self-mappings on K with common Lipschitz constant $L = 1$, $0 \in F = \{x \in K : x = T_i x, i = 1,2\} \neq \emptyset$ and (2.1)–(2.3), (3.1) and (3.2) hold. Therefore, the assumptions of Theorems 3.1–3.3 are satisfied. Consequently, each of Theorems 3.1–3.3 implies that for each $x_0 \in K$, there exists $q \in K$ such that the sequences $\{x_n\}_{n\geq1}$, $\{y_n\}_{n\geq1}$ and $\{z_n\}_{n\geq1}$ defined by each of the implicit iteration methods (A)-(C) converge to q.

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