Int. J. Appl. Math. Sci. Vol. **1**(2004), pp. 97-110

Dynamical Systems Method for solving nonlinear operator equations

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
E:mail: ramm@math.ksu.edu

Abstract Consider an operator equation (*) $B(u)+\epsilon u=0$ in a real Hilbert space, where $\epsilon>0$ is a small constant. The DSM (Dynamical Systems Method) for solving equation (*) consists of finding and solving a Cauchy problem:

$$\dot{u}=\Phi(t,u),\ u(0)=u_0,\ t\geq 0,$$

which has the following properties: 1) it has a global solution $u(t)$,
2) this solution tends to a limit as time tends to infinity, i.e., $u(\infty)$ exists,
3) this limit solves the equation $B(u)=0$, i.e., $B(u(\infty))=0$.
Existence of the unique solution is proved by the DSM for equation (*) with operators B defined on all of H and satisfying a spectral assumption: $||[B'(u)+\epsilon I]^{-1}||\leq c/\epsilon$ for any $u\in H$, where $c>0$ is a constant independent of u and $\epsilon\in(0,\epsilon_0)$. If $\epsilon=0$ and equation (**) $B(u)=0$ is solvable, the DSM yields a solution to (**). The case when B is a monotone, hemicontinuous, defined on all of H operator is also studied, and DSM is justified for this case, that is, above properties 1),2), and 3) are proved. A sufficient condition for surjectivity of a nonlinear map is given. Meyer's generalization of the Hadamard theorem about global homeomorphisms is proved by the DSM. The DSM method is justified for non-differentiable, hemicontinuous, monotone, defined on all of H operators.

AMS (MOS) Subject Classification Codes: 34R30, 35R25, 35R30, 37C35, 37L05, 37N30, 47A52, 47J06, 65M30, 65N21.

Key Words: dynamical systems method, ill-posed problems, monotone operators, iterative methods, fixed-point maps.

1 Introduction

In this paper a version of the DSM, *Dynamical Systems Method*, is proposed for solving nonlinear operator equation of the form:

$$B(v) + \epsilon v = 0, \quad \epsilon = const > 0, \tag{1}$$

where the operator $B : H \to H$ is a nonlinear map in a Hilbert space H, which satisfies some spectral assumption SA). In our earlier work [2], this operator was assumed twice Fréchet differentiable. In Theorems 2 and 3 of this paper the operator may be a non-differentiable, hemicontinuous, defined on all of H, monotone operator. This paper can be considered as further development of the theory presented in [2]. Some of the results presented in [2] and [4] are included in the monograph [5].

The notions related to monotone operators, used in this paper, one finds, e.g., in [1]. The DSM is applied to solving operator equations in [2], where the nonlinear mapping B was assumed locally twice Fréchet differentiable. In this paper the above assumption is kept in the case when B is not monotone, but is dropped if $B(u)$ is a hemicontinuous, defined on all of H, monotone operator.

The Dynamical Systems Method (DSM) for solving nonlinear and linear operator equations, introduced in [2], consists of finding a nonlinearity $\Phi(t, u)$ such that the Cauchy problem:

$$\dot{u} = \Phi(t, u), \; u(0) = u_0, \; t \geq 0, \tag{DSM}$$

has a unique global solution, this solution has a limit at infinity $u(\infty) := \lim_{t\to\infty} u(t)$, and this limit solves the original equation $F(u(\infty)) = 0$. The original equation may have many solutions. A solution $u(\infty)$ depends on the initial approximation u_0.

The DSM is justified in [2] for large classes of operator equations:

a) for any solvable equation $F(u) = 0$ in a Hilbert space for which the assumptions (F) and (W) of Theorem S hold. Moreover, the choices of $\Phi(t, u)$ in [2] under these assumptions guarantee exponential rate of convergence of $u(t)$ to the solution $u(\infty)$. Convergent iterative schemes for solving the above equation are constructed.

b) for any solvable linear operator equation $F(u) := Au - f = 0$ with an arbitrary bounded linear operator A. Convergence is guaranteed to the unique minimal-norm solution to the above equation. Convergent iterative schemes for solving the above equation are constructed.

c) for any solvable operator equation $F(u) = 0$ with any monotone operator F which satisfies conditions (F) of Theorem S. Convergence is guaranteed to the unique minimal-norm solution to the above equation. Convergent iterative schemes for solving the above equation are constructed.

d) for any solvable operator equation $F(u) = 0$ with any operator F which satisfies conditions (F) of Theorem S and the following assumption: if $F(y) = 0$, and $T := [F'(y)]^* F(y)$, then for any $r \in (0, r_0)$, where $r_0 > 0$ is a fixed small number, the operator T maps the ball $B(0, r) := \{u : ||u - y|| \leq r\}$ onto a set which has a non-empty intersection with a punctured ball $B(0, a) \setminus \{0\}$, where $a > 0$ is a fixed number. This assumption implies that there is an element $z \in H$, such that for any $r > 0$ there is an element v, $0 < ||v|| \leq r$ such that $Tv = z - y$.

Specific choices of $\Phi(t, u)$ are proposed in [2] for all of the above four cases. Below Theorem S stands for a surjectivity theorem, formula (F) stands for Fréchet differentiability, and (W) stands for well-posedness: assumption (W) implies that F is a local homeomorphism.

The novel results in this paper include:

a) a proof of the existence of a solution to equation (1) under a spectral assumption on B and a justification of the DSM for construction of a solution to (1); singular perturbation problem is investigated: conditions are given for the solution u_ϵ of the equation $B(u_\epsilon) + \epsilon(u_\epsilon - z) = 0$, where z is a suitably chosen element, to converge to the limit y which solves the limiting equation $B(y) = 0$. The limiting equation is assumed to have a solution. This assumption is necesssary as a simple example shows: take $H = \mathbb{R}^1$ and $B(u) = e^u$. Then spectral assumption SA) is satisfied, $c = 1$, $\epsilon_0 = \infty$, equation $e^u + \epsilon u = 0$ is uniquely solvable in $\mathbb{R}^1$ for any $\epsilon > 0$, but the limiting equation $e^u = 0$ has no solutions in $\mathbb{R}^1$ and even in $\mathbb{C}$.

b) a justification of the DSM for monotone, hemicontinuous, defined on all of H operators,

and

c) a novel technique for estimation of the derivative of the solutions to equations (2) and (7).

d) a sufficient condition for surjectivity of a nonlinear map (cf [4]).

e) a novel proof, based on the DSM, of an Hadamard-type theorem on global homeomorphism in a Hilbert space H.

The Hadamard theorem (1906) says that if $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the estimate $[||F'(u)]^{-1}|| \leq c$ for all $u \in \mathbb{R}^n$, then F is a global homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. This result was generalized by Levy (1920) to $F : H \to H$, where H is a Hilbert space, by Meyer(1968) under the assumption $[||F'(u)]^{-1}|| \leq a||u|| + b$ for all $u \in H$ with some positive constants, a, b, independent of u. An Hadamard-type theorem for Banach spaces is discussed in [3].

Our sufficient condition for surjectivity of a nonlinear map $F : H \to H$ is given in the following theorem

Theorem S: *Let $F : H \to H$ be twice Frćhet differentiable,*

$$\sup_{u \in B(u_0,R)} ||F^{(j)}(u)|| \leq M_j(R), \; j = 1, 2, \tag{F}$$

where $u_0 \in H$ is some element and $R > 0$ is a number. Assume that

$$\sup_{u \in B(u_0,R)} ||[F'(u)]^{-1}|| \leq m(R). \tag{W}$$

If

$$\sup_{R>0} \frac{R}{m(R)} = \infty, \tag{S}$$

then F is surjective, that is, equation $F(u) = f$ has a solution for any $f \in H$.

Remark: Condition (S) (surjectivity) is essential. For example, if $F(u) := e^u$, $H = \mathbb{R}^1$, then equation $e^u = 0$ does not have a solution, conditions (F) (Fréchet differentiability) and (W) (well-posedness in the sense of [2]) hold, but (S) does not hold: $m(R) = e^R$.

Proposition: *If $||F'(u)^{-1}|| \leq a||u|| + b$ for all $u \in H$ with some positive constants, a, b, independent of u, then F is a homeomorphism of H onto H.*

Let us make the following assumption:

Spectral assumption SA): *The map B is twice Fréchet differentiable and $||(A+\epsilon)^{-1}|| \leq \frac{c}{\epsilon}$ for all $u \in H$, where $A := B'(u)$, $c > 0$ is a constant independent of u, $\epsilon \in (0, \epsilon_0)$, and ϵ_0 is an arbitrary small fixed positive number.*

Under this assumption we prove

Theorem 1. *Equation (1) has a solution. Each solution of (1) is isolated. If the equation $B(u) = 0$ has a solution, then there is a solution to equation $B(u_\epsilon) + \epsilon(u_\epsilon - z) = 0$, such that $||u_\epsilon - y|| = O(\epsilon)$, as $\epsilon \to 0$, and $B(y) = 0$. Here z is a suitably chosen element independent of ϵ.*

An alternative assumption we use is:

Assumption A) *B is a monotone, possibly nonlinear, hemicontinuous, defined on all of H operator in a real Hilbert space H.*

If assumption A) holds, then the set $N := \{u : B(u) = 0\}$, if it is non-empty, is closed and convex, and therefore it has the unique element y with minimal norm.

Consider the dynamical system (that is, the Cauchy problem):

$$\dot{w} = -B(w) - \epsilon w, \quad w(0) = w_0, \tag{2}$$

where w_0 is arbitrary. A solution to (2) is a strongly differentiable function which satisfies (2) whenever the operators F or B satisfy assumption (F) of Fréchet differentiabilty. But when we use assumption A) in Theorems 2 and 3, then the solution $w(t)$ of (2) or $u(t)$ of (7) are assumed weakly differentiable, because in Theorems 2 and 3 the global existence of the solution is derived by means of Lemma 1, in the proof of which one establishes only weak differentiability of the solution.

The DSM in this paper consists of solving equation (1) by solving (2), and proving that for any initial approximation w_0 the following results (3) and (4) hold:

$$\exists w(t) \forall t > 0, \quad \exists V_\epsilon := w(\infty) := \lim_{t\to\infty} w(t), \quad B(V_\epsilon) + \epsilon V_\epsilon = 0. \tag{3}$$

If assumption A) holds and $y \in N$ is the (unique) minimal-norm solution to the equation $B(u) = 0$, then

$$\lim_{\epsilon\to 0} ||V_\epsilon - y|| = 0. \tag{4}$$

The rate of convergence in (4) can be arbitrarily slow, in general. It can be estimated only under additional assumptions on the operator B.

If B satisfies assumption A), then conclusion (4) is known, but we give in subsection 2.5 a simple proof of it to make the presentation self-contained for convenience of the reader.

It is also known that equation (1) under the assumtions A) has a solution and this solution is unique. This known result is a consequence of Theorem S under the additional smoothness assumptions on F formulated in this Theorem, because the spectral assumption is satisfied by monotone operators with $c = 1$ and $\epsilon_0 = \infty$.

We prove also the existence of the solution to (1) under the assumption A), without assuming Fréchet differentiability of B. This proof is given by the DMS method. If $\epsilon = 0$ then the limiting equation

$$B(u) = 0 \tag{5}$$

may have no solution (e.g., $B(u) = e^u$).We prove that if (5) has a solution, then the DSM allows one to construct a solution to (5). We assume

$$\epsilon(t) = \frac{c_1}{(c_0 + t)^b}, \quad c_0 > 0,\ c_1 > 0,\ 0 < b < 1, \tag{6}$$

where c_0, c_1 and b are constants. Consider the problem:

$$\dot{u} = -B(u) - \epsilon(t)u,\ \ u(0) = u_0. \tag{7}$$

Our results are stated in theorems 2 and 3:

Theorem 2. *If assumption A) hold, equation (5) is solvable, and (6) holds, then problem (7) has a unique global solution $u(t)$, there exists strong limit $u(\infty) := \lim_{t\to\infty} u(t)$, $B(u(\infty)) = 0$, and $u(\infty) := y$ is the unique minimal-norm element in the set of all solutions to (5).*

Theorem 3. *If assumptions A) hold and $\epsilon = const > 0$, then problem (2) has a unique global solution $w(t)$, there exists strong limit $w(\infty)$, and $w(\infty)$ solves (1).*

So far we have assumed that the operators F or B are everywhere defined and bounded. In applications the nonlinear map F may be defined on a dense in H linear set. For example, one may have $F(u) := Au + g(u)$, where A is a linear, closed, densely defined operator, and g is a nonlinear operator. If one assumes that A^{-1} is a bounded operator defined on all of H, then the equation $F(u) = 0$ is equivalent to the equation $\phi(u) := u + A^{-1}g(u) = 0$. If one assumes that g is locally twice Fréchet differentiable, then the results of Theorem S can be applied to the equation $\phi(u) = 0$. In particular, one may have in mind an application of these results to semilinear elliptic boundary-value problems. A possible problem is the one where $A = -\Delta$ is the Dirichlet Laplacian in a bounded domain with a smooth boundary, and $g(u)$ is a twice differentiable function. As an example of an abstract result, obtained by the DSM, one gets the following:

Theorem 4. *Assume that $\phi(u) = u + A^{-1}g(u)$, that $A^{-1}g(u)$ satisfies assumption (F), and $\sup_{u\in B(u_0,R)} ||[\phi'(u)]^{-1}|| \le m(R)$. If*

$$||\phi(u_0)||m(R) \le R,$$

then the problem

$$\dot{u} = -[\phi'(u)]^{-1}\phi(u), \qquad u(0) = u_0$$

has a unique global solution $u(t)$, there exists $u(\infty)$, and $F(u(\infty)) = 0$. Moreover, the trajectory $u(t) \in B(u_0, R)$ and $||u(t) - u(\infty)|| = O(e^{-ct})$ with some constant $c > 0$, so the convergence is at an exponential rate.

In Section 2 proofs are given.

2 Proofs

1. *Proof of Theorem S.* Consider the problem:

$$\dot{u} = -[F'(u)]^{-1}[F(u) - f],\ u(0) = 0. \tag{$*$}$$

Our assumptions imply that in the right-hand side of the equation one has a locally Lipschitz operator, so there exists a local solution $u(t)$ to the above problem, and this solution is unique. We prove that this solution is global under the assumption of Theorem S, that $\lim_{t\to\infty}||\dot{u}|| = 0$, that $u(t) \in B(u_0, R)$ and $u(\infty) := \lim_{t\to\infty} u(t)$ exists provided that $g_0 m(R) \leq R$, where $g_0 := g(0)$ and $g(t) := ||F(u(t)) - f||$, and passing to the limit $t \to \infty$ in (*) one gets $F(u(\infty)) = f$, so F is surjective.

Let us give details. With $g(t)$ defined above, using (*) one gets $g\dot{g} = -g^2$, so $g(t) = g_0 e^{-t}$. This and (*) imply $||\dot{u}|| \leq m(R) g_0 e^{-t}$. Therefore, if $m(R)g_0 \leq R$, then $||u(t)-u_0|| \leq \int_0^t m(R) g_0 e^{-t} dt \leq m(R) g_0 \leq R$, so $u(t) \in B(u_0, R)$ for all $t > 0$. Thus, $u(t)$ is a global solution, $u(\infty$ exists, and $||u(t) - u(\infty)|| \leq m(R) g_0 e^{-t}$. Therefore $\lim_{t\to\infty} ||\dot{u}|| = 0$ and passing to the limit in (*), one gets $F(u(\infty)) = f$. Theorem S is proved. □

Note that the inequality $m(R)g_0 \leq R$ is satisfied for some $R > 0$ whatever the value g_0 is, i.e., whatever the initial approximation u_0 is, provided that the basic assumption $\sup_{R>0} \frac{R}{m(R)} = \infty$ holds.

Proof of Proposition. One argues as above, and gets $||\dot{u}|| \leq (a||u|| + b) g_0 e^{-t}$. Let $h := ||u(t)||$. Then $\dot{h} \leq (h + p) a g_0 e^{-t}$, where $p := \frac{b}{a}$, and we assume $a > 0$. since if $a = 0$ the argument is simpler. Integrating one gets $\sup_{t>0} h(t) \leq c_1 := (h(0) + p)e^{ag_0} - p$, and $||\dot{u}|| \leq c_2 e^{-t}$, where $c_2 := (ac_1 + b) g_0$. Thus $u(t) \in B(u_0, c_2)$ for all $t > 0$. From Theorem S it follows that F is surjective, and estimate $||[F'(u(t))]^{-1}|| \leq c$ for all $u(t) \in B(u_0, c_2)$ and all $t > 0$ implies that F is a local homeomorphism. If one proves that $F(\eta) = F(y)$ implies $\eta = y$, then F is a global homeomorphism and the Proposition is proved. Let $u(\infty) := u(\infty, u_0) := \eta$. Consider the elements $w(s) := (1-s)u_0 + sy$. Then $w(0) = u_0$, $w(1) = y$. Note that $u(t, y) = y$ for all $t > 0$. One checks that small changes of s lead to small changes of $u(\infty, w(s))$. Since F is a local homeomorphism, and $F(u(\infty, u_0) = f = F(u(\infty, w(s))$, one concludes that $u(\infty, u_0) = u(\infty, w(s)$ for s sufficiently close to 0, and using this argument finitely many times, one shows that $u(\infty, w(s)) = u(\infty, w(s + \sigma)$ for any $s \in [0, 1 - \sigma]$ and $\sigma > 0$ independent of s. This implies that $\eta = u(\infty, u_0) = u(\infty, w(1)) = y$. To complete the proof it is sufficient to check that for an arbitrary small $\delta > 0$ one has $\sup_{t>0} ||u(t, w(s)) - u(t, w(s + \sigma)|| \leq c\delta$, if $||u(0, w(s)) - u(0, w(s + \sigma)|| = \delta$ and σ is sufficiently small and independent of s.

Let $x(t) := u(t, w(s + \sigma) - u(t, w(s)) := p - \mu$ and $||x(t)|| := q(t)$. Then

$$q\dot{q} = -([F'(p)]^{-1}[F(p) - f] - [F'(\mu)]^{-1}[F(\mu) - f], x(t)) =$$

$$-(([F'(p)]^{-1} - [F'(\mu)]^{-1})[F(p) - f], x(t)) - ([F'(\mu)]^{-1}(F(p) - F(\mu)), x(t)) \leq$$

$$\leq ce^{-t} q^2(t) - q^2(t) + cq^3(t),$$

where $c = const > 0$ and we have used the formulas:

$$F(p) - F(\mu) = F'(\mu) q(t) + K, \qquad ||K|| \leq \frac{1}{2} M_2 q^2(t).$$

Since $q \geq 0$, one gets the inequality:

$$\dot{q} \leq -q + ce^{-t} q + cq^2, \qquad q(0) = \delta.$$

Setting $h := qe^t$, one obtains the following inequality for h:

$$\dot{h} \leq ce^{-t}(h + h^2), \quad h(0) = \delta.$$

After an integration, one gets $h(t) \leq c\delta$, $c = const > 0$, so

$$q(t) \leq ce^{-t}\delta,$$

where $c > 0$ stands for various constants independent of t and δ. Proposition is proved. □.

Proof of Theorem 1. Consider the problem

$$\dot{u} = -A_\epsilon^{-1}[B(u) + \epsilon u], \quad u(0) = u_0, \quad A := B'(u),\ A_\epsilon := A + \epsilon I. \qquad (8a)$$

From spectral assumption it follows that the operator in the right-hand side in (8a) is locally Lipschitz, so there exists and is unique local solution $u(t)$ to (8a). We establish a uniform with respect to $t > 0$ bound $\sup_{t>0} ||u(t)|| \leq c$, and this bound implies global existence of the solution to (8a). Then we prove that $||\dot{u}|| \in L^1(0, \infty)$. This implies by the Cauchy criterion the existence of $u_\epsilon := u(\infty) := \lim_{t\to\infty} u(t)$. Passing to the limit $t \to \infty$ in (8a) shows that u_ϵ solves equation (1). The Cauchy criterion says that $u(\infty)$ exists if and only if for any $\eta > 0$, however small, there exists $t_\eta > 0$, such that for any $t \geq t_\eta > 0$ and for any $h > 0$ one has $||u(t+h) - u(t)|| \leq \eta$. If $||\dot{u}|| \in L^1(0, \infty)$, then $||u(t + h) - u(t)|| \leq \int_t^{t+h} ||\dot{u}(s)||ds \leq \eta$ if $t \geq t_\eta > 0$ and $h > 0$ is arbitrary. So the Cauchy criterion is satisfied.

Let us give the details. Define $g(t) := ||B(u(t)) + \epsilon u(t)||$. Then equation (8a) implies: $g\dot{g} = -g^2$, so $g(t) = g(0)e^{-t}$. Using equation (8a) again together with the spectral assumption SA), one gets: $||\dot{u}|| \leq cg(0)e^{-t}/\epsilon$. This implies $||\dot{u}|| \in L^1(0, \infty)$, and the existence of $u_\epsilon := u(\infty) := \lim_{t\to\infty} u(t)$ follows.

Moreover,

$$||u(t) - u(\infty)|| \leq cg(0)e^{-t}/\epsilon,$$

and

$$||u(t) - u(0)|| \leq cg(0)/\epsilon.$$

Passing to the limit $t \to \infty$ in (8a) yields $0 = B(u_\epsilon) + \epsilon u_\epsilon$, i.e., $u_\epsilon := u(\infty)$ solves equation (1).

Let us now consider the equation $B(u_\epsilon) + \epsilon(u_\epsilon - z) = 0$ which is solvable for any $z \in H$, and choose z such that $y - z = A\nu$, where

$$A := B'(y), \quad B(y) = 0, \quad ||\nu|| < \frac{1}{2cM}, \quad ||B''(u)|| \leq M_2 := M,$$

and c is the constant from the assumption SA). We assume that B is thrice Fréchet differentiable in a neighborhood of the solution y. Then $A_\epsilon\psi + K(\psi) + A\nu = 0$, where

$$\psi := u_\epsilon - y, \quad B(u_\epsilon) - B(y) = A\psi + K(\psi), \quad ||K|| \leq 0.5M||\psi||^2, \quad A_\epsilon := A + \epsilon I,$$

and one gets:

$$\psi = -A_\epsilon^{-1}K(\psi) - \epsilon A_\epsilon^{-1}A\nu := T(\psi).$$

Let us check that T is a contraction map on the ball $||\psi|| \leq R$ and T maps this ball B_R into itself. Here $R = O(\epsilon)$. Then there is a unique solution $\psi \in B_R$, and $||\psi|| = ||u_\epsilon - y|| = O(\epsilon)$.

First, let us check that $TB_R \subset B_R$. One has $||T\psi|| \leq \frac{c}{\epsilon}\frac{1}{2}MR^2 + \epsilon||\nu|| \leq R$ provided that $R = R(\epsilon) := \frac{\epsilon}{cM}(1-\rho)$, where $\rho := (1-2cM||\nu||)^{1/2}$, $0 < \rho < 1$. Note that $R = O(\epsilon)$.

Secondly, let us check that T is a contraction map on B_R. In this argument we use the boundedness of the third derivative of B, $\sup_{||u-y||\leq R(\epsilon)} ||B^{(3)}(u)|| \leq M_3$. One has

$$||T(\psi)-T(\phi)|| \leq \frac{c}{\epsilon}||K(\psi)-K(\phi)|| \leq \int_0^1 ds(1-s)||B''(y+s\psi)\psi\psi - B''(y+s\phi)\phi\phi|| \leq$$

$$\frac{c}{\epsilon}(\frac{1}{6}M_3R^2 + MR)||\psi - \phi||,$$

where $\psi, \phi \in B_R$ are arbitrary. Moreover, for $R = R(\epsilon) = \frac{\epsilon}{cM}(1-\rho)$ one gets $\frac{c}{\epsilon}(\frac{1}{6}M_3R^2 + MR) = 1 - \rho + O(\epsilon) < 1$ for sufficiently small $\epsilon > 0$. Thus T is a contraction on B_R.

Theorem 1 is proved. □

Remark: If B is monotone and twice Fréchet differentiable, then spectral assumption SA) is satisfies with $c = 1$ and $\epsilon_0 = \infty$. Thus, Theorem 1 implies existence and uniqueness of the solution to (1) for monotone operators which are twice Fréchet differentiable. A stronger result is known: the Fréchet differentiability assumption can be dropped, and the conclusion holds under the assumption A). This result is proved below by the DSM method.

2. In Lemma 1 the existence of the unique global solutions to problems (2) and (7) is claimed. The result of Lemma 1 is known, but we prove this Lemma in Section 2.5 to make the presentation self-contained for convenience of the reader. Our proof is based on the Peano approximation and follows the proof in [1].

Lemma 1. *If $\epsilon = const > 0$ and assumptions A) hold, then problem (2) has a unique global solution. If assumptions A) and (6) hold, then (7) has a unique global solution.*

Proof of Theorem 3. The proof consists of the following steps:

a) we prove:

$$\sup_{t\geq 0} ||w(t)|| < c < \infty; \quad g(t) \leq g(0)e^{-\epsilon t}, \quad g(t) := ||w(t+h) - w(t)||, \tag{8}$$

where $c > 0$ stands for various estimation constants, and $h > 0$ is an arbitrary number.

This and the Cauchy test imply the existence of $V_\epsilon := w(\infty)$.

b) we prove that

$$||\dot{w}(t)|| \leq ||\dot{w}(0)||e^{-\epsilon t}. \tag{9}$$

Thus, $\lim_{t\to\infty} ||\dot{w}(t)|| = 0$. Estimate (9) implies $\int_0^\infty ||\dot{w}(t)||dt < \infty$. This again, independently, implies the existence of $V_\epsilon := w(\infty)$.

From a), b), and from (2), one concludes that V_ϵ solves equation (1). To pass to the limit in (2) one uses demicontinuity of hemicontinuous, monotone, defined on all of H, operators, i.e., the property which says that $w \to V$ implies $B(w) \rightharpoonup B(V)$, (cf, e.g., [1], p.98). Here and below $\rightharpoonup$ denotes weak convergence in H.

The proof of (4) is given in subsection 2.6.

To complete the proof of Theorem 3, let us prove (8) and (9).

Let $z := w(t+h)-w(t)$ and $g := ||z||$. From equation (2) and from the monotonicity of B one gets:

$$g\dot{g} = -(B(w(t+h)) - B(w(t)) + \epsilon z, z) \le -\epsilon g^2.$$

Since $g \ge 0$, this implies the second half of (8). Its first half, namely the estimate $\sup_{t\ge0} ||w(t)|| < c < \infty$, is proved below formula (16) for the more general case when ϵ depends on t.

Let $\psi := \frac{||z||}{h}$. Then, as above, one gets $\psi\dot{\psi} \le -\epsilon\psi^2$. Thus, $\psi(t) \le \psi(0)e^{-\epsilon t}$. Let $h \to 0$ and get (9). Theorem 3 is proved. □

3. **Proof of Theorem 2.** The scheme of the proof is similar to the one used above, but there are new points due to the dependence of $\epsilon(t)$ on t now. Denote $g(t) := ||u(t+h) - u(t)||$ and $z := u(t+h) - u(t)$. From (7) one gets

$$g\dot{g} = -(B(u(t+h)) - B(u(t)) + \epsilon(t)z, z) - (\epsilon(t+h) - \epsilon(t))(u(t+h), u(t)) \le$$

$$-\epsilon(t)g^2 + |\epsilon(t+h) - \epsilon(t)|||u(t+h)||g. \quad (10)$$

We prove below that

$$\sup_{t>0} ||u(t)|| \le c < \infty, \quad c = const > 0. \quad (11)$$

From (10), (11) and (6) one gets the following differential inequality:

$$\dot{g} \le -\epsilon(t)g + hc|\dot{\epsilon}(t)|, \quad (12)$$

where c is defined in (11).

From (12) one gets:

$$g(t) \le e^{-\int_0^t \epsilon(s)ds}[g(0) + hc\int_0^t e^{\int_0^s \epsilon(x)dx}|\dot{\epsilon}(s)|ds]. \quad (13)$$

From (6) and (13) one gets

$$\lim_{t\to\infty} g(t) = 0, \quad \forall h > 0. \quad (14)$$

Indeed, if $a(t) := e^{\int_0^t \epsilon(s)ds}$, then $a^{-1}(t)\int_0^t a(s)|\dot{\epsilon}(s)|ds = O(\frac{1}{t})$ as $t \to \infty$, as one derives from assumption (6). In fact, the rate $O(\frac{1}{t})$ holds under weaker assumption $\frac{|\dot{\epsilon}(t)|}{\epsilon(t)} = O(\frac{1}{t})$ as $t \to \infty$, as one can see easily.

From (11) it follows that there exists a sequence $t_n \to \infty$, such that $u(t_n) \rightharpoonup v$, where $\rightharpoonup$ stands for the weak convergence, and $v \in H$ is some element. We prove below that $B(v) = 0$ by passing to the limit $t_n \to \infty$ in equation (7), using assumption (6), inequality (11), and relation (15), which one obtains dividing (14) by h and letting $h \to 0$:

$$\lim_{t\to\infty} ||\dot{u}(t)|| = 0. \tag{15}$$

Passing to the limit $t_n \to \infty$ in (7), proves that $u(\infty) := v$ solves (5).

Let us prove (11). In this proof we use the assumption that (5) has a solution y.

Denote $u(t) - y := p(t)$ and $||p|| := q$. Then

$$\dot{p} = -(B(u) - B(y)) - \epsilon(t)p - \epsilon(t)y. \tag{16}$$

Multiplying this by p and using the monotonicity of B, one gets:

$$\dot{q} \leq -\epsilon(t)q + \epsilon(t)||y||. \tag{17}$$

This implies $q(t) \leq a^{-1}(t)[q(0) + ||y|| \int_0^t a(s)\epsilon(s)ds]$. Thus,

$$||u(t) - y|| := q(t) \leq c, \tag{18}$$

so (11) follows (with a different c).

Let us now prove the existence of the strong limit $u(\infty)$ and the relation $u(\infty) = y$, where y is the unique minimal-norm solution to (5).

From (11) it follows that there is a sequence $t_n \to \infty$ such that $u(t_n) \rightharpoonup v$. From (6), (11), (15), and (7) one gets $\lim_{n\to\infty} B(u(t_n)) = 0$. This and assumption A) imply $B(v) = 0$ (cf Lemma 3 below).

Let us prove that $u(t_n) \to v$. Since $u(t_n) \rightharpoonup v$, one gets

$$||v|| \leq \liminf_{n\to\infty} ||u(t_n)||.$$

If $\limsup_{n\to\infty} ||u(t_n)|| \leq ||v||$, then $\lim_{n\to\infty} ||u(t_n)|| = ||v||$, and together with the weak convergence $u(t_n) \rightharpoonup v$ this implies strong convergence $u(t_n) \to v$ (cf Lemma 4 below).

To prove that $\limsup_{n\to\infty} ||u(t_n)|| \leq ||v||$, we need some preparations. First, (6) implies that $\int_0^t \epsilon(s)ds \sim \frac{t^a}{a}$ as $t \to \infty$, where $a := 1 - b$, $0 < a < 1$. Second, (13) implies

$$||\dot{u}(t)|| \leq \frac{c}{t} \quad as \quad t \to \infty,$$

where $c > 0$ is a constant. Indeed, if (6) holds, then

$$a^{-1}(t) \int_0^t a(s)|\dot{\epsilon}(s)|ds = O(\frac{1}{t}), \quad t \to \infty.$$

Equations $B(v) = 0$ and (7) imply

$$(B(u(t_n)) - B(v), u(t_n) - v) + \epsilon(t_n)(u(t_n), u(t_n) - v) = -(\dot{u}(t_n), u(t_n) - v).$$

Since B is monotone, it follows that

$$(u(t_n), u(t_n) - v) \leq \frac{c}{t_n\epsilon(t_n)}.$$

Thus, $\limsup_{n\to\infty} ||u(t_n)|| \le ||v||$, because $\lim_{n\to\infty} t_n\epsilon(t_n) = \infty$, as follows from (6). Therefore

$$||v|| \le \lim \inf_{n\to\infty} ||u(t_n)|| \le \lim \sup_{n\to\infty} ||u(t_n)|| \le ||v||.$$

This implies

$$\lim_{n\to\infty} ||u(t_n)|| = ||v||.$$

Let us prove that $v = y$, where y is the unique minimal-norm solution to (5). Replacing v by y in the above argument yields

$$(u(t_n), u(t_n) - y) \le \frac{c}{t_n\epsilon(t_n)},$$

so

$$||v|| = \lim_{n\to\infty} ||u(t_n)|| \le ||y||.$$

Since y is the unique minimal-norm solution to (5), and v solves (5), it follows that $v = y$.

Since the limit $\lim_{n\to\infty} u(t_n) = v = y$ is the same for every subsequence $t_n \to \infty$, for which the weak limit of $u(t_n)$ exists, one concludes that the strong limit $\lim_{t\to\infty} u(t) = y$. Indeed, assuming that for some sequence $t_n \to \infty$ the limit of $u(t_n)$ does not exist, one selects a subsequence, denoted again t_n, for which the weak limit of $u(t_n)$ does exist, and proves as before that this limit is y, thus getting a contradiction. Theorem 1 is proved. □

4. **Proof of Lemma 1.**

Let F be a nonlinear map satisfying assumptions A). In Lemma 1 this map is $F(u) := B(u) + \epsilon u$ for equation (2) and $F(u) := B(u) + \epsilon(t)u$ for equation (7). Our argument holds for any F satisfying assumptions A). We want to prove that the problem

$$\dot{w} = -F(w), \quad w(0) = w_0, \tag{19}$$

has a unique global solution.

Uniqueness of the solution is immediate: if w and v are solutions to (19), and $z := w - v$, then $\dot{z} = -[F(w) - F(v)]$, $z(0) = 0$. Multiplying by z and using the monotonicity of F, one gets $(\dot{z}, z) \le 0$, so $||z(t)|| \le 0$, and the uniqueness follows.

Let us now prove the global existence of the solution to (19).

Consider the equation:

$$w_n(t) = w_0 - \int_0^t F(w_n(s - \frac{1}{n}))ds, \;\; t > 0; \quad w_n(t) = w_0, \;\; t \le 0. \tag{20}$$

We wish to prove that

$$\lim_{n\to\infty} w_n(t) = w(t), \quad \forall t > 0, \tag{21}$$

where w solves (19). Recall that assumptions A) imply demicontinuity of F.

Fix an arbitrary $T > 0$, and let $B(w_0, r)$ be the ball centered at w_0 with radius $r > 0$. We may assume that $\sup_{u\in B(w_0,r)} ||F(u)|| := c$ because of the local boundedness of a monotone, hemicontinuous operator F defined on all of H (Kato's theorem, see [1], p.97). Then (20) implies $||w_n(t) - w_0|| \leq ct$. If $t \leq r/c$, then $w_n(t) \in B(w_0, r)$, and $||\dot{w}_n(t)|| \leq c$. Define

$$z_{nm}(t) := w_n(t) - w_m(t), \quad ||z_{nm}(t)|| := g_{nm}(t).$$

From (20) one gets:

$$g_{nm}\dot{g}_{nm} = -(F(w_n(t - \frac{1}{n})) - F(w_m(t - \frac{1}{m})), w_n(t) - w_m(t)) := I.$$

One has:

$$I = -(F(w_n(t - \frac{1}{n})) - F(w_m(t - \frac{1}{m})), w_n(t - \frac{1}{n}) - w_m(t - \frac{1}{m}))$$

$$-(F(w_n(t - \frac{1}{n})) - F(w_m(t - \frac{1}{m})), w_n(t) - w_n(t - \frac{1}{n}) - (w_m(t) - w_m(t - \frac{1}{m}))).$$

Using the monotonicity of F, the estimate $\sup_{w \in B(w_0,r)} ||F(w)|| \leq c$, and the estimate $||\dot{w}_n(t)|| \leq c$, one gets:

$$I \leq 4c^2(\frac{1}{n} + \frac{1}{m}).$$

Therefore

$$g_{nm}\dot{g}_{nm} \leq 4c^2(\frac{1}{n} + \frac{1}{m}) \to 0 \quad as \quad n, m \to \infty. \tag{22}$$

This implies

$$\lim_{n,m\to\infty} g_{nm}(t) = 0, \quad 0 \leq t \leq \frac{r}{c}. \tag{23}$$

Therefore, there exists the strong limit $w(t)$:

$$\lim_{n\to\infty} w_n(t) = w(t), \quad 0 \leq t \leq \frac{r}{c}. \tag{24}$$

A monotone, hemicontinuous, defined on all of H operator F is demicontinuous on H, that is, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$ (see e.g. [1], p.98). This allows one to pass to the limit in (20) and conclude that the function w, defined in (24), satisfies the integral equation:

$$w(t) = w_0 - \int_0^t F(w(s))ds. \tag{25}$$

The function w is weakly differentiable because of the weak convergence $F(w_n) \rightharpoonup F(w)$. Therefore this function solves problem (19), where the derivative of w is unerstood in the weak sense. If F is continuous, then problem (19) and equation (25) are equivalent and the derivative in (19) can be understood as the strong derivative. If F is demicontinuous, then they are also equivalent, but the derivative in (19) should be understood in the weak sense. We have proved the existence of the unique local solution to (19).

To prove that the solution to (19) exists for any $t \in [0, \infty)$, let us assume that the solution exists on $[0, T)$, but not on a larger interval $[0, T + d)$, $d > 0$, and show that this leads to a contradiction. It is sufficient to prove that the finite limit:

$$\lim_{t\to T} w(t) \tag{26}$$

does exist, because then one can solve locally, on the interval $[T, T+d)$, equation (19) with the initial data $w(T) = \lim_{t\to T} w(t)$, and construct the solution to (19) on the interval $[0, T+d)$, thus getting a contradiction.

To prove that the finite limit (26) exists, consider

$$w(t+h) - w(t) := z(t), \quad ||z|| := g.$$

One has $\dot{z} = -[F(w(t+h)) - F(w(t))]$. Using the monotonicity of F, one gets $(z, \dot{z}) \le 0$. Thus,

$$||w(t+h) - w(t)|| \le ||w(h) - w(0)||. \tag{27}$$

The right-hand side in (27) tends to zero as $h \to 0$. This, and the Cauchy test imply the existence of the finite limit (26).

Lemma 1 is proved. □

5. **Proof of (4).**

This proof requires the following lemmas in which assumptions A) hold and are not repeated:

Lemma 2. *If y solves (5) and V_ϵ solves (3), then:*

$$||V_\epsilon|| \le ||y||. \tag{28}$$

Lemma 3. *If $v_n \rightharpoonup v$ and $B(v_n) \to f$, then $B(v) = f$.*

Lemma 4. *If $v_n \rightharpoonup v$ and $||v_n|| \le ||v||$, then $v_n \to v$.*

Assuming these lemmas, let us prove (4). From (28) one gets $V_\epsilon \rightharpoonup v$ (by a subsequence denoted again V_ϵ). Equation (3) implies $B(V_\epsilon) \to 0$. Thus, Lemma 3 yields $B(v) = 0$. From $V_\epsilon \rightharpoonup v$ and from (26) one gets

$$||v|| \le \liminf_{\epsilon\to 0} ||V_\epsilon|| \le \limsup_{\epsilon\to 0} ||V_\epsilon|| \le ||y||.$$

Therefore $v = y$, since the solution to the equation (5), which has minimal norm, is unique, if A) holds. The weak convergence $V_\epsilon \rightharpoonup y$, inequality (28), and Lemma 4 imply (4).

Let us prove Lemmas 2-4.

Proof of Lemma 2. One has $B(V_\epsilon) + \epsilon V_\epsilon - B(y) = 0$. Multiply this by $V_\epsilon - y$ and use the monotonicity of B to get $\epsilon(V_\epsilon, V_\epsilon - y) \le 0$. Since $\epsilon > 0$, inequality (28) follows. □

Proof of Lemma 3. The monotonicity of B implies

$$(B(v_n) - B(v - tz), v_n - v + tz) \ge 0 \quad \forall z \in H \ \forall t > 0.$$

Letting $n \to \infty$ one gets $(f - B(v - tz), tz) \ge 0$, so

$$(f - B(v - tz), z) \ge 0.$$

Letting $t \to 0$, one gets $(f - B(v), z) \ge 0 \ \forall z$. Thus, $B(v) = f$. Lemma 3 is proved. □

Proof of Lemma 4. One has $||v|| \le \liminf_{n\to\infty} ||v_n|| \le \limsup_{n\to\infty} ||v_n|| \le ||v||$. Thus, $\lim_{n\to\infty} ||v_n|| = ||v||$. This and the weak convergence $v_n \rightharpoonup v$, imply: $||v - v_n||^2 = ||v_n||^2 + ||v||^2 - 2\Re(v_n, v) \to 0$. Lemma 4 is proved. □.

Proof of (4) is completed. □

We leave a proof of Theorem 4 to the reader, since this proof is similar to the proofs from [2].

Remark: One may aslo assume that the operator A in the representation $F(u) = Au + g(u)$ is not boundedly invertible, but, satifies Assumption SA), but we do not go into detail.

References

[1] Deimling, K., Nonlinear functional analysis, Springer, Berlin, 1985

[2] Ramm, A.G., Dynamical systems method for solving operator equations, Comm. Nonlinear Sci. and Numer. Simul., 9, N2, (2004), 383-402.

[3] Ortega, J., Rheinboldt, W., Iterative solution of non;inear equations in several variables, SIAM, Philadelphia, 2000.

[4] Ramm, A.G., Dynamical systems method and surjectivity of nonlinear maps, Comm. Nonlinear Sci. and Numer. Simul., 10, (2005), (to appear)

[5] Ramm, A.G., **Inverse Problems**, Springer, New York, 2005.