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On solving a rolling frictional contact problem using BEM and Mathematical Programming

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Abstract This work deals with the rolling frictional contact problem between an elastic cylinder and a flat rigid body. A simple approach to solve the quasistatic case through the results of the static one is presented. The discretization is based on the boundary element method. The unilateral frictional contact problem (nonsmooth but monotone) is formulated in a compact form as a nonsymmetric linear complementarity problem which is solved using the Lemke's algorithm.

Keywords: Inequality problems; Frictional contact problems; Rolling contact; Boundary Element Method; Complementarity Problems; Mathematical programming.

1 Introduction

There is a considerable interest in contact problems with friction both from the mathematical point of view and from applied practice. Indeed, the noise characteristics of engineering systems are increasing factors in product marketability. For instance, in the automotive industry, the custumer's perception of the car noise is very important in the purchasing decision. A considerable part of this noise results from frictional contact effects arising between moving parts.

In this work, we are interested in the noise generated by the contact between a rolling elastic cylinder and a flat rigid body. The investigation is actually focused on the vibration of wheels which causes a change of the surrounding air pressure. Nevertheless, the noise generated by the frictional contact itself could be significant. This part of the noise is related to the relative slip velocity in the contact area. The idea is to compute first this velocity in the quasistatic case. Abascal et al.

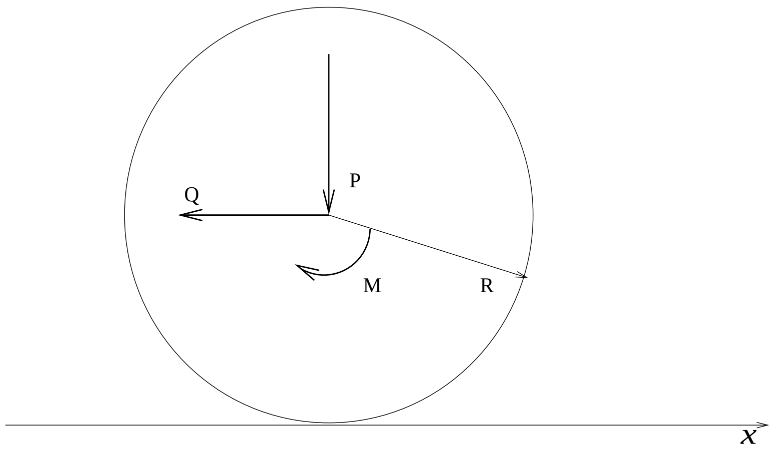

Figure 1: Rolling cylinder in contact with a rigid body

[1, 2, 3] used two approaches to solve the quasistatic problem considering directly the velocity as unknown variable.

Here, we propose another approach that allows to use results concerning the existence and the uniqueness of the solution. Indeed, we solve the static frictional contact problem and, afterwards, we compute the velocity using the appropriate relations. This approach is also motivated by the fact that it can also help modelling the noise caused by the vibrations of the wheel. Indeed, these vibrations are due to instabilities and impacts observed in the slip area. A possible measure for their minimization would be the reduction of area of these sources, namely, of the slip zone.

Here, the boundary element method (BEM) is used to solve the rolling problem. This method is not often applied to solve those problems with contact: one can cite Abascal et al.[1, 2, 3], Wang et al. [4], and Kong et al. [5]. In this paper, the elastostatic, frictional contact, rolling problem is solved for the first time using BEM and the compact LCP formulation using Lemke's algorithm.

2 Rolling problem

Let us consider an elastic rolling cylinder in unilateral contact with a flat rigid body. The Coulomb's law is taken to describe the friction phenomenon.

Since the displacements in the direction of cylindrical axis and the spin rotation about the axes normal to that of the cylinder are excluded, the problem is reduced to a plane strain state (see, e.g., [1, 2, 3]).

Friction is assumed to follow the dry Coulomb's law where normal and tangential tractions on the boundaries of the contact zone are related via a simple coefficient of friction. Under this assumption, two points belonging to the cylinder A and the rigid body B can be in three different states relative to each other:

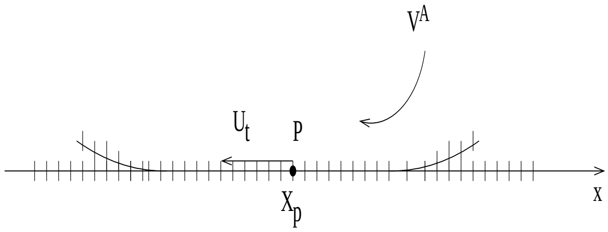

Figure 2: Contact zone

1. Stick state

$$\begin{array}{llr} r_N^{A,B} \leq 0, & r_N^A = r_N^B, & \delta_n = 0, \\ r_T^A = r_T^B, & & s_t = 0, \end{array} \tag{1}$$

2. Slip state

$$\begin{array}{lll} r_N^{A,B} \leq 0, & r_N^A = r_N^B, & \delta_n = 0, \\ |r_T^{A,B}| = \mu |r_N^{A,B}|, & & sgn(s_t) = -sgn(r_T^A), \end{array} \tag{2}$$

3. Separation state

$$r_N^{A,B} = 0, \quad r_T^{A,B} = 0, \quad \delta_n \geq 0. \tag{3}$$

Here, μ is the constant coefficient of friction, $r_N^{A,B}$ and $r_T^{A,B}$ are the normal and tangential tractions of cylinder A and rigid body B, respectively, at contacting points, u_N and u_T are the cylinder surface displacements, s_t is the slip velocity and δ_n is the normal separation given by:

$$\delta_n = \delta_{n0} - u_N \tag{4}$$

The initial separation, δ_{n0}, for a cylinder and a flat body in contact can be approximated by

$$\delta_{n0} = \frac{x^2}{2R} \tag{5}$$

where R is the cylinder radius and x is the Eulerian coordinate along the contact zone (see, Fig. 2) used to position each pair of points at each time relative to a rigid body position of the cylinder.

When applying an Eulerian description of particles moving through the contact area, the relative tangential slip velocity of each cylinder surface point is defined as

$$s_t = \dot{\delta}_t = \frac{d\delta_t(x,\tau)}{d\tau} \tag{6}$$

τ is the time coordinate, $x = x(x,\tau)$ is the Cartesian coordinate of each point relative to fixed axes and varying time τ, and δ_t is the tangential separation given by

$$\delta_t = (x^A - x^B) + u_T^A + u_T^B \tag{7}$$

where A and B indicate the cylinder and the flat body, respectively.
Substituting (7) into (6) leads to

$$s_t = V^A - V^B + V^A u^A_{T,x} + V^B u^B_{T,x} + u^A_{T,\tau} + u^B_{T,\tau} \tag{8}$$

where V^α is the velocity of a point of the rigid body α in the x-direction.

Under steady-state rolling conditions, the variation with time vanishes, so

$$s_t = V^A - V^B + V^A u^A_{T,x} + V^B u^B_{T,x} \tag{9}$$

which is usually approximated [6] by

$$s_t = |V| \left(\xi + sgn(V)(u^A_{T,x} + u^B_{T,x})\right) \tag{10}$$

where V is given by $(V^A + V^B)/2$, and ξ is the normalized relative rigid slip velocity (creepage), defined as $\xi = \dfrac{V^A - V^B}{|V|}$. In fact, we compute only the dimensionless part s^*_t of s_t [3]:

$$s^*_t = \xi + sgn(V)(u^A_{T,x} + u^B_{T,x}) \tag{11}$$

When the displacement derivatives are approximated using a finite difference scheme, the tangential slip velocity for a point located at the coordinate x_i is, under steady-state rolling conditions, expressed as

$$s^*_t(x_i) = \xi + sgn(V) \left(\frac{u_T(x_{i+1}) - u_T(x_i)}{h_i} \right) \tag{12}$$

where h_i denotes the distance between two adjacent boundary points x_{i+1} and x_i.

To analyse coupled 2D rolling contact between two cylinders, Abascal et al. have formulated the problem considering the slip velocity as unknown variable. Indeed, in [3], they used the NORM-TANG iteration to solve two complementarity problems. In [1], the problem is analysed by minimizing a function representing the equilibrium equation and the contact restrictions.

In our case, the static contact-friction problem between an elastic cylinder and a flat rigid body is first solved and, subsequently, the slip velocity is computed explicitly using the above formula.

3 Discretization and Condensation

The model is formulated by means of the boundary element method. This formulation is more suitable for this class of problem where the nonlinearity is confined to the boundary of the body. The application of mathematical programming techniques for unilateral contact problems together with a boundary element discretization has been considered, among others, in [7], [8], [9], [10]. The here adopted formulation follows the lines of [11], [12], [13], [14], [15], [16] and [17].

One starts from the matrix formulation of displacement boundary element in elastostatics [18]:

$$Hu = Gt \tag{13}$$

Here, u is the vector of the nodal boundary displacements, t is the vector of element boundary tractions, and H, G are the appropriate influence matrices.

The number of equations in (13) depends on the number of nodes on the discretized boundary. Let us note that the number of boundary element tractions (i.e., the dimension of t) depends on the nature of the boundary elements used.

The classical approach for the solution of the bilaterally constrained structures goes through the specification of appropriate, known boundary displacements or tractions, the rearrangement of the system (13), and finally, the formulation of a nonsymmetric system of equations, after reordering:

$$Ay = b \tag{14}$$

The 2n-dimensional vector y contains all the unknown boundary displacements or tractions of the problem. In the contact area, both displacements and tractions are unknown. They must be kept in the formulation and connected with the inequality and complementarity relations of the unilateral contact mechanism. After solving the arising LCP, one knows which of these variables vanishes. Thus, we proceed with condensation and, then, formulate the linear complementarity problem.

Let us consider u_c and t_c as being the boundary nodal displacements and tractions, respectively, at the frictional unilateral contact boudary of the cylinder. After partitioning the boundary of the cylinder, the equation (14) gives:

$$\begin{bmatrix} H_{ff} & H_{fc} \\ H_{cf} & H_{cc} \end{bmatrix} \begin{bmatrix} x \\ u_c \end{bmatrix} = \begin{bmatrix} f_f \\ f_c \end{bmatrix} + \begin{bmatrix} G_{fc} \\ G_{cc} \end{bmatrix} \begin{bmatrix} t_c \end{bmatrix} \tag{15}$$

When n_c is the number of the nodes at the unilateral contact boundary, then u_c and t_c have $2n_c$ elements, H_{cc} and G_{cc} have $2n_c \times 2n_c$ elements, H_{ff} has $2(n - n_c) \times 2(n - n_c)$ elements, G_{fc}, H_{fc} and H_{cf} have $2(n - n_c) \times 2n_c$ elements.

The next step is to perform a local coordinate transformation so that normal and tangential to the unilateral boundary quantities appear in the formulation. Therefore, let us consider w and r as the natural local coordinates (normal and tangential coordinates) of the displacements and the tractions in the contact boundary, respectively:

$$u_c^i = \begin{bmatrix} u_{cx}^i \\ u_{cy}^i \end{bmatrix}, \quad t_c^i = \begin{bmatrix} t_{cx}^i \\ t_{cy}^i \end{bmatrix}, \quad w^i = \begin{bmatrix} u_N^i \\ u_T^i \end{bmatrix}, \quad r^i = \begin{bmatrix} r_N^i \\ r_T^i \end{bmatrix} \tag{16}$$

The transformation for a single unilateral boundary node i reads:

$$C_i u_c^i = w^i, \quad -C_i t_c^i = r^i \tag{17}$$

with

$$C_i = \begin{bmatrix} cos\phi_i & sin\phi_i \\ -sin\phi_i & cos\phi_i \end{bmatrix} \tag{18}$$

Since $C_i^{-1} = C_i^T$, relations (17) can be inverted:

$$u_c^i = C_i^T w, \quad t_c^i = -C_i^T r \tag{19}$$

Taking into acount these transformations, one has

$$\begin{bmatrix} H_{ff} & H_{fc}C^T \\ H_{cf} & H_{cc}C^T \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} f_f \\ f_c \end{bmatrix} - \begin{bmatrix} G_{fc}C^T \\ G_{cc}C^T \end{bmatrix} \begin{bmatrix} t_c \end{bmatrix} \tag{20}$$

Then, the problem will be condensed, i.e., in order to get the flexibility matrix, the unknown variable x will be reduced by considering the relation (20):

$$x = H_{ff}^{-1} f_f - H_{ff}^{-1} G_{fc} C^T r - H_{ff}^{-1} H_{fc} C^T w \tag{21}$$

Hence, w becomes:

$$\begin{aligned}\left[H_{cc}C^T - H_{cf}H_{ff}^{-1}H_{fc}C^T\right] w = \; & \left[f_c - H_{cf}H_{ff}^{-1} f_f\right] \\ & - \left[G_{cc}C^T - H_{cf}H_{ff}^{-1}G_{fc}C^T\right] r\end{aligned} \tag{22}$$

Thus, w can be written as:

$$w = w_0 + Fr \tag{23}$$

or:

$$\begin{bmatrix} w_n \\ w_t \end{bmatrix} = \begin{bmatrix} w_{n0} \\ w_{t0} \end{bmatrix} + \begin{bmatrix} F_{nn} & F_{nt} \\ F_{tn} & F_{tt} \end{bmatrix} \begin{bmatrix} r_N \\ r_T \end{bmatrix} \tag{24}$$

4 Problem formulation

ic version of the Coulomb's frictional contact law is adopted here. Moreover, following [19], a direct nonsymmetric LCP formulation of the frictional unilateral contact problem is used.

We follow the formulation of [14] and [19] for the case of three-dimensional problems, where the friction cone is linearized by means of a convex polyhedron. Let the normal forces and the friction forces be assembled in vectors $\mathbf{r}_N = \{r_{N1}, \ldots, r_{Nn}\}^T$ and $\mathbf{r}_T = \{r_{T1}, , r_{T2} \ldots, r_{Tn}\}^T$, respectively, where, e.g., r_{Ti} is the frictional force of the i-th contact node.

Coulomb's law of dry friction connects the tangential (frictional) forces with the normal (contact) forces by the relation

$$\gamma_i = \mu |r_{Ni}| - |r_{Ti}|, \; i = 1, \ldots, n, \; \gamma_i \geq 0 \tag{25}$$

Here $| * |$ denotes the absolute value and μ is the friction coefficient (anisotropic friction may also be considered). The friction mechanism is considered to work in the following way: If $|r_{Ti}| < \mu |r_{Ni}|$ (i.e., $\gamma_i > 0$), the slipping value γ_{iT} must be equal

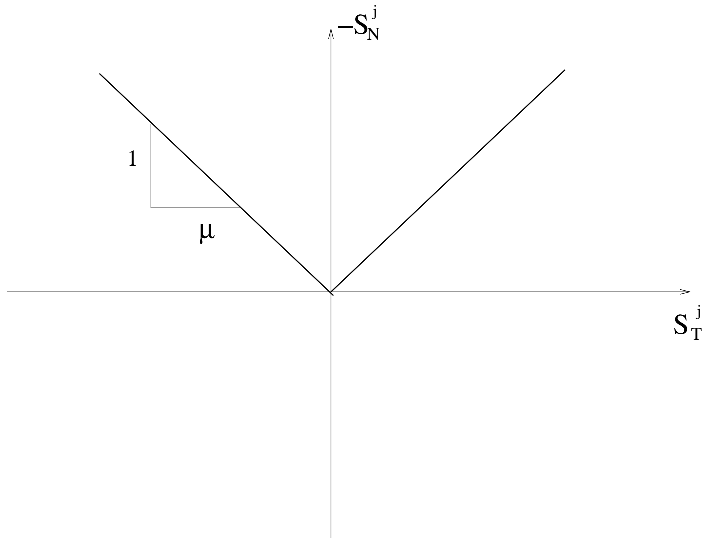

Figure 3: Coulomb's friction law in 2-D

to zero, and if $|r_{Ti}| = \mu |r_{Ni}|$ (i.e., $\gamma_i = 0$), then we have slipping in the opposite direction of r_{Ti}:

$$\begin{array}{lll} \text{if} \quad \gamma_i > 0, & \text{then} & y_{Ti} = 0 \\ \text{if} \quad \gamma_i = 0, & \text{then} & \text{there exists } \sigma > 0 \text{ such that } y_{Ti} = -\sigma r_{Ti} \end{array} \tag{26}$$

By assembling the contributions of all (n) unilateral nodes, relation (26) reads:

$$\gamma = \mathbf{T}_N^T \mathbf{r}_N + \mathbf{T}_T^T \mathbf{r}_T \tag{27}$$

with the matrices $\mathbf{T}_T$ and $\mathbf{T}_N$

$$\mathbf{T}_T \ = \ \text{diag} \ \left[\mathbf{T}_T^1, \mathbf{T}_T^2, \ldots, \mathbf{T}_T^n\right], \ \mathbf{T}_N \ = \ \text{diag} \ \left[\mathbf{T}_N^1, \mathbf{T}_N^2, \ldots, \mathbf{T}_N^n\right]. \tag{28}$$

These matrices are obtained from the linearized friction law considered in 2D and have the following form (Fig.3) [11, 19].

$$\mathbf{T}_T^j \ = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{T}_N^j \ = \ [\mu, \mu]. \tag{29}$$

Finally, the slip value in (25), (26) is written as

$$\mathbf{y}_T = \mathbf{T}_T \lambda, \ \lambda \geq 0 \tag{30}$$

where λ is a vector of nonnegative slipping parameters. Then, γ and λ fulfil the following complementarity condition:

$$\gamma^T \lambda = 0 \tag{31}$$

To formulate the linear complementarity problem, two approaches are used: rigid body displacements and rigid body loading.

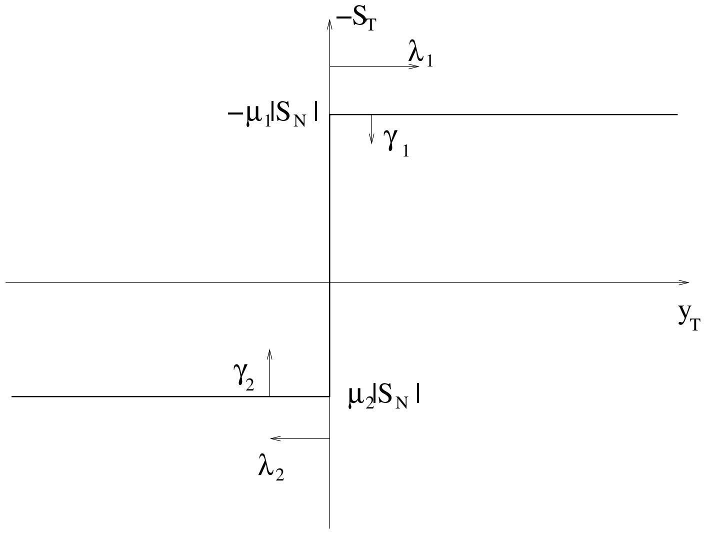

Figure 4: Decomposition of a frictional law

4.1 Rigid body displacement approach

In the rigid body displacement approach, the slipping value λ and the tangential displacements $\mathbf{u}_T$ are related by the compatibility relation:

$$\mathbf{T}_T\lambda - \mathbf{u}_T = \mathbf{d}_T \tag{32}$$

where $\mathbf{d}_T$ denotes the initial tangential distance.

The general decomposition scheme with slack variables can be found in more details for two-dimensional friction problems in [14] and for three-dimensional applications in [11] and [19]. In fact, for three-dimensional problems, the submatrices of (28) are constructed from a linearization of the friction cone. Note also that, in general, anisotropic friction effects may be considered as well. For the two-dimensional case, which is considered in this paper, the previously used decomposition scheme for one frictional joint is explained in (Fig.3 and Fig.4)

Now, a linear elastic behaviour of the structure is assumed which, on the assumption that everything outside of the frictional contact interfaces has been condensed out (elimination of d.o.f's), reads:

$$\tilde{\mathbf{u}} = \tilde{\mathbf{F}}\widetilde{\mathbf{r}} \tag{33}$$

where

$$\tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_N \\ \mathbf{u}_T \end{bmatrix}, \; \tilde{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_{NN} & \mathbf{F}_{NT} \\ \mathbf{F}_{TN} & \mathbf{F}_{TT} \end{bmatrix}, \; \widetilde{r} = \begin{bmatrix} \mathbf{r}_N \\ \mathbf{r}_T \end{bmatrix}. \tag{34}$$

Here, $\tilde{\mathbf{F}}$ is the symmetric flexibility matrix where $\mathbf{F}_{NN}$ is an $n \times n$ nonsingular matrix with the mechanical meaning of being the normal flexibility matrix, $\mathbf{F}_{TT}$ is a $2n \times 2n$ nonsingular matrix (the tangential flexibility) and $\mathbf{F}_{NT}$, $\mathbf{F}_{TN}$ are the corresponding couple flexibility matrices.

By using the previous relations, the unilateral kinematic conditions normal and tangential to the interface, take the form:

$$\begin{aligned} \mathbf{y}_N - \mathbf{F}_{NN}\mathbf{r}_N - \mathbf{F}_{NT}\mathbf{r}_T &= \mathbf{d}_N, \\ \mathbf{T}_T\lambda - \mathbf{F}_{TN}\mathbf{r}_N - \mathbf{F}_{TT}\mathbf{r}_T &= \mathbf{d}_T. \end{aligned} \tag{35}$$

A standard LCP formulation is derived by means of the following change of variables. First, from the second relation in (35), $\mathbf{r}_T$ is expressed as follows:

$$\mathbf{r}_T = -\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN}\mathbf{r}_N + \mathbf{F}_{TT}^{-1}\mathbf{T}_T\lambda - \mathbf{F}_{TT}^{-1}\mathbf{d}_T \tag{36}$$

Then, by eliminating $\mathbf{r}_T$ from equations (35), we obtain

$$\begin{aligned} \mathbf{y}_N - \left(\mathbf{F}_{NN} - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN}\right)\mathbf{r}_N - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{T}_T\lambda &= \mathbf{d}_N - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{d}_T \\ \gamma + \left(\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN} - \mathbf{T}_N^T\right)\mathbf{r}_N - \mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{T}_T\lambda &= -\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{d}_T \end{aligned} \tag{37}$$

Finally, a standard LCP is obtained from equations (37):

$$\begin{aligned} \mathbf{w} - \mathbf{M}\mathbf{z} &= \mathbf{b} \\ \mathbf{w} &\geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0},\ \mathbf{w}^T\mathbf{z} = 0 \end{aligned} \tag{38}$$

with

$$\begin{aligned} \mathbf{w} &= \begin{bmatrix} \mathbf{y}_N \\ \gamma \end{bmatrix},\ \mathbf{z} = \begin{bmatrix} \mathbf{r}_N \\ \lambda \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} \mathbf{d}_N - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{d}_T \\ -\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{d}_T \end{bmatrix}, \\ \mathbf{M} &= \begin{bmatrix} (\mathbf{F}_{NN} - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN}) & \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{T}_T \\ -(\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN} - \mathbf{T}_N^T) & \mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{T}_T \end{bmatrix} \end{aligned}$$

4.2 Rigid body loading approach

In this approch, we follow the ideas of [11] but a two-dimensional model is adopted. First, the global equilibrium equations are written as

$$\mathbf{G}_N\mathbf{r}_N + \mathbf{G}_T\mathbf{r}_T = \mathbf{L} \tag{39}$$

where $\mathbf{G}_N$ and $\mathbf{G}_T$ are the the equilibrium matrices corresponding to the normal and the tangential contact reactions, respectively. $L = \{P, Q, M\}^T$ is the vector of the external loadings P and Q, and the moment M (Fig.2).

The slipping value λ and the tangential displacements $\mathbf{u}_T$ are related by the compatibility relation:

$$\mathbf{T}_T\lambda - \mathbf{u}_T + \mathring{\mathbf{u}}_\mathbf{T} = \mathbf{d}_T \tag{40}$$

$\mathring{\mathbf{u}}_\mathbf{T}$ is the vector of the tangential displacements of the adjacent points in the contact area due to the rigid body motion. The linear elastic behaviour (13) is here available and the flexibility matrices are the same.

By means of the principle of virtual work one can find that :

$$\mathring{\mathbf{u}}_\mathbf{N} = -G_N^T\mathring{\mathbf{u}} \tag{41}$$

$$\overset{\circ}{\mathbf{u}}_{\mathbf{T}} = -G_T^T \overset{\circ}{\mathbf{u}} \tag{42}$$

where $\overset{\circ}{\mathbf{u}}$ is the vector of the rigid body displacements and rotations with respect to coordinate system. For two-dimensional problems, $\overset{\circ}{\mathbf{u}}$ reads as

$$\overset{\circ}{\mathbf{u}} = \left\{ \overset{\circ}{u}_x, \overset{\circ}{u}_y, \overset{\circ}{\varphi} \right\}^T \tag{43}$$

where $\overset{\circ}{u}_x$ and $\overset{\circ}{u}_y$ are the rigid body displacements along the directions OX and OY, respectively, and $\overset{\circ}{\varphi}$ denotes the rigid body rotation. By means of the relations (41), (42), (33), (24) and (40), we obtain

$$\begin{aligned} \mathbf{y}_N - \mathbf{F}_{NN}\mathbf{r}_N - \mathbf{F}_{NT}\mathbf{r}_T - G_N^T \overset{\circ}{\mathbf{u}} &= \mathbf{d}_N, \\ \mathbf{T}_T\lambda - \mathbf{F}_{TN}\mathbf{r}_N - \mathbf{F}_{TT}\mathbf{r}_T - G_T^T \overset{\circ}{\mathbf{u}} &= \mathbf{d}_T. \end{aligned} \tag{44}$$

Let us now express the rigid body displacements as a difference of two nonnegative values [20],

$$\overset{\circ}{\mathbf{u}} = \overset{\circ}{\mathbf{u}}^{+} - \overset{\circ}{\mathbf{u}}^{-} \tag{45}$$

where

$$\overset{\circ}{\mathbf{u}}^{+} = \frac{\overset{\circ}{\mathbf{u}} + |\overset{\circ}{\mathbf{u}}|}{2} \geq 0 \quad \text{and} \quad \overset{\circ}{\mathbf{u}}^{-} = \frac{-\overset{\circ}{\mathbf{u}} + |\overset{\circ}{\mathbf{u}}|}{2} \geq 0$$

Substituting (45) in (44), we have

$$\begin{aligned} \mathbf{y}_N - \mathbf{F}_{NN}\mathbf{r}_N - \mathbf{F}_{NT}\mathbf{r}_T - G_N^T \overset{\circ}{\mathbf{u}}^{+} + G_N^T \overset{\circ}{\mathbf{u}}^{-} &= \mathbf{d}_N, \\ \mathbf{T}_T\lambda - \mathbf{F}_{TN}\mathbf{r}_N - \mathbf{F}_{TT}\mathbf{r}_T - G_T^T \overset{\circ}{\mathbf{u}} + G_T^T \overset{\circ}{\mathbf{u}}^{-} &= \mathbf{d}_T. \end{aligned} \tag{46}$$

Let us now introduce the nonnegative slack variables vectors $\mathbf{v}^+ \geq 0$ and $\mathbf{v}^- \geq 0$ in the equation (39), which can be put in the form

$$\mathbf{v}^+ + \mathbf{G}_N\mathbf{r}_N + \mathbf{G}_T\mathbf{r}_T = \mathbf{L}, \mathbf{v}^- - \mathbf{G}_N\mathbf{r}_N - \mathbf{G}_T\mathbf{r}_T = -\mathbf{L} \tag{47}$$

From equations (39) and (47), we note that at the equilibrium state the nonnegative slack variables vectors $\mathbf{v}^+ \geq 0$ and $\mathbf{v}^- \geq 0$ must be equal to zero and thus the following orthogonalities hold:

$$\mathbf{v}^+ \overset{\circ}{\mathbf{u}}^{+} = 0, \quad \mathbf{v}^- \overset{\circ}{\mathbf{u}}^{-} = 0 \tag{48}$$

Following the same way as in the previous section, a Linear Complementarity Problem (LCP) is obtained:

$$\begin{aligned} \mathbf{w} - \mathbf{Mz} &= \mathbf{b} \\ \mathbf{w} &\geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0},\ \mathbf{w}^T\mathbf{z} = 0 \end{aligned} \tag{49}$$

with

$$\mathbf{w} = \begin{bmatrix} \mathbf{y}_N \\ \mathbf{v}^+ \\ \mathbf{v}^- \\ \gamma \end{bmatrix}, \ \mathbf{z} = \begin{bmatrix} \mathbf{r}_N \\ \overset{\circ}{\mathbf{u}}{}^+ \\ \overset{\circ}{\mathbf{u}}{}^- \\ \lambda \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} \mathbf{d}_N - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{d}_T \\ L + \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{d}_T \\ -L - \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{d}_T \\ -\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{d}_T \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} (\mathbf{F}_{NN} - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN}) & (\mathbf{G}_N - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T) & -(\mathbf{G}_N - \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{G}_T) & \mathbf{F}_{NT}\mathbf{F}_{TT}^{-1}\mathbf{T}_T \\ -(\mathbf{G}_N - \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN}) & \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T & -\mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T & -\mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{T}_T \\ (\mathbf{G}_N - \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN}) & -\mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T & \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T & \mathbf{G}_T\mathbf{F}_{TT}^{-1}\mathbf{T}_T \\ -(\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{F}_{TN} - \mathbf{T}_N^T) & -\mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T & \mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{G}_T^T & \mathbf{T}_T^T\mathbf{F}_{TT}^{-1}\mathbf{T}_T \end{bmatrix}$$

The LCP (38) is defined on $\mathbb{R}^{n+m}$ where n is the number of discrete unilateral joints, $m = n \times l$ and l is the number of faces in the polyhedral approximation of the friction cone. Note that, here, we get a nonsymmetric LCP due to the nonsymmetric (but positive semidefinite) matrix $\mathbf{M}$ in (38) (a well-known fact from analogous cases of nonassociated laws in friction, plasticity etc, see, e.g., [21]).

5 Lemke's Algorithm

There are many results on the existence and the uniqueness of the solution. We make especially reference to Al-Fahed et al. [11] where the conditions guaranteeing the solvability of the LCP are based on the theory of Fichera [22] for nonsymmetric variational inequalities.

Lemke's algorithm [23] is used here to solve the LCP (38) and (49) which are in the general following form:

$$\begin{aligned} \mathbf{w} - \mathbf{Mz} &= \mathbf{b} \\ \mathbf{w} &\geq \mathbf{0}, \ \mathbf{z} \geq \mathbf{0}, \ \mathbf{w}^T\mathbf{z} = \mathbf{0} \end{aligned} \tag{50}$$

We start from a feasible point which is not easy to obtain. In order to overcome this difficulty, the algorithm which we use introduces a 'measure of feasability' expressed by an additional nonnegative $\mathbf{z}_0$. Problem (50) is then written in the following form

$$\begin{aligned} \mathbf{w} - \mathbf{Mz} - \mathbf{e}_0\mathbf{z}_0 &= \mathbf{b} \\ \mathbf{w} &\geq \mathbf{0}, \ \mathbf{z} \geq \mathbf{0}, \ \mathbf{z}_0 \geq \mathbf{0}, \ \mathbf{w}^T\mathbf{z} = \mathbf{0} \end{aligned} \tag{51}$$

where

$$\mathbf{e}_0 = [1, 1, ..., 1]^T$$

Let us describe the complementarity pivot algorithm [24] where we consider m to be the number of variables contained in each vector $\mathbf{w}$ and $\mathbf{z}$ in (51). Recall that an m-vector composed of certain nonzero elements of the vectors $\mathbf{w}$ and $\mathbf{z}$ is called a basic vector. Let $\mathbf{w}$ be the initial base vector of problem (51), while $\mathbf{A}$ means the $m \times (2m+1)$ matrix and $\mathbf{h}$ the m-vector for writing the first equation in (51) in tableau form. Initially, $\mathbf{A}$ and $\mathbf{h}$ take the forms :

$$\mathbf{A} = [\mathbf{I} \quad -\mathbf{M} \quad -\mathbf{e}_0], \quad \mathbf{h} = \mathbf{b} \tag{52}$$

The algorithm has the following steps.

Step 1. Initialization: Set $k = 1$, let $\mathbf{w}$ be the initial basic vector, and find t^k such that

$$b_{t^k} = \min_j b_j : j = 1, ..., m$$

If $b_{t^k} \leq 0$, then continue with Step 2, otherwise the algorithm stops with $\mathbf{z} = 0$ (unique if $b_{t^k} \geq 0$)

Step 2. Initial infeasability $z_0^{(1)}$: Drop from the basic vector the basic variable which corresponds to index t^k and insert z_0 in its position. This is performed by going directly to Step 4.

Step 3. Complementarity change of basic variables: Set $k = k + 1$, let s be the column of the nonbasic variables complementary to the dropped variable which has to enter the basis according to the complementary pivoting rule, and find the basic variable, to be dropped, by means of the minimum ratio law, i.e., find t^k such that

$$R_{t^k} = \left\{ \min_j \frac{h_j}{a_{js}} : a_{js} \geq 0, \ j = 1, ..., m \right\},$$

where a_{js}, h_j, $i = 1, ..., 2m+1$ and $j = 1, ..., m$ are the components of the current tableau $\mathbf{A}$ and $\mathbf{h}$. If $\{a_{js}\} \leq 0$ for all j, i.e., the algorithm is unable to introduce the variable s into the base without violating the nonnegativity restriction, then go to Step 6, otherwise continue with Step 4. This is done in order to drop from the basic vector the variable which corresponds to index t^k, and to insert in its position the nonbasic variable which correponds to column s.

Step 4. Apply a Gauss-Jordan pivot elimination step (first phase of the Simplex method of linear programming) with pivot row the $t^k th$ row and pivot column the sth column (respectively, the $(2m+1)th$ column) if $k > 1$ (respectively if $k = 1$).

Step 5. Stopping criterion: If z_0 is the droped basic variable (i.e., $z_0 = 0$), then go to Step 7, otherwise continue with Step 3.

Step 6. Ray termination: The algorithm is terminated with ray termination ($z_0 > 0$) and the LCP has no solution.

Step 7. Normal exit: The algorithm is terminated with $z_0 = 0$ and the current basic vector contains the solution of the LCP.

From all the above steps we conclude that if ray termination occurs, the LCP has no solution. In this case, the data of the problem are incompatible, which means that the structure cannot be supported from the existing unilateral joints and for the applied loading and the geometry of the assumed problem. We could summurize that the Lemke's algorithm reduces the initial residual $|\mathbf{w} - \mathbf{M}\mathbf{z} - \mathbf{b}|$ for $\mathbf{w} \geq \mathbf{0}, \ \mathbf{z} \geq \mathbf{0}, \ \mathbf{w}^T\mathbf{z} = \mathbf{0}$ to zero.

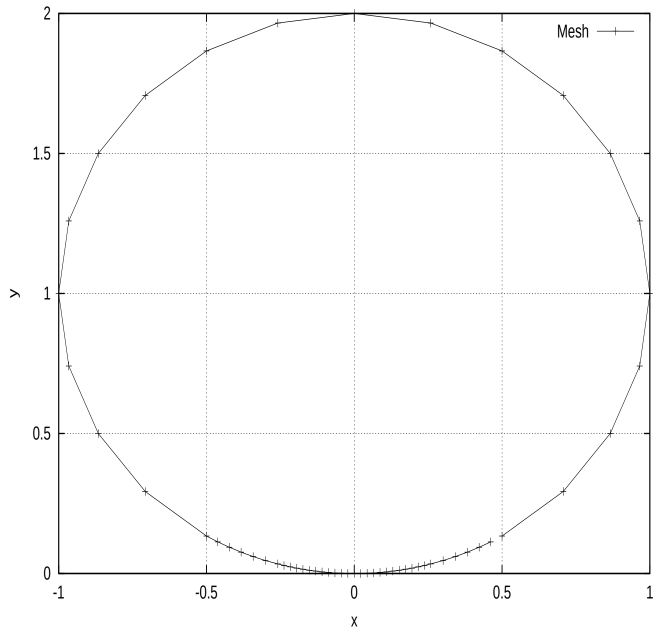

Figure 5: The mesh around the cylinder contour.

6 Numerical results

To discretize the rolling problem with BEM, constant as well as discontinuous quadratic elements are used. The computations with the both kinds of elements are presented only in the frictionless case. In the other cases, we use constant elements which give better results. In fact, seeing some small difference in traction results according to the used element, that point will be investigated more deeply. Indeed, in the literature, there is almost no reference to this statement; we can only cite Karami [25] and Stavroulakis et al. [16] who remarked the existence of oscillations in the traction behaviour with quadratic elements.

For all the computations, a unit circle is divided into 56 elements as shown in Figure 5. Young modulus and Poisson coefficient are taken to be 94500 and 0.1, respectively. The rigid body normal and tangential displacement used were $d_N = R/2$ and $d_T = 0.1 d_N$ where R is the radius of the cylinder. The numerical simulations are made with $|Q/P| = 0.1$ where Q and P are the resultants of tangential and normal forces respectively. The creepage used[1] is $\xi = 1/500$ as in [3].

Firstly, the results concerning the frictionless unilateral contact are presented in order to check our solution by comparing it to the analytical one due to the Hertz theory [25, 26].

Around the contact zone, we approximate the 1/12 of the cylinder contour with 24 elements.

In Figure 6, the agreement between the Hertz theory solution and the results of

[1] personal communication

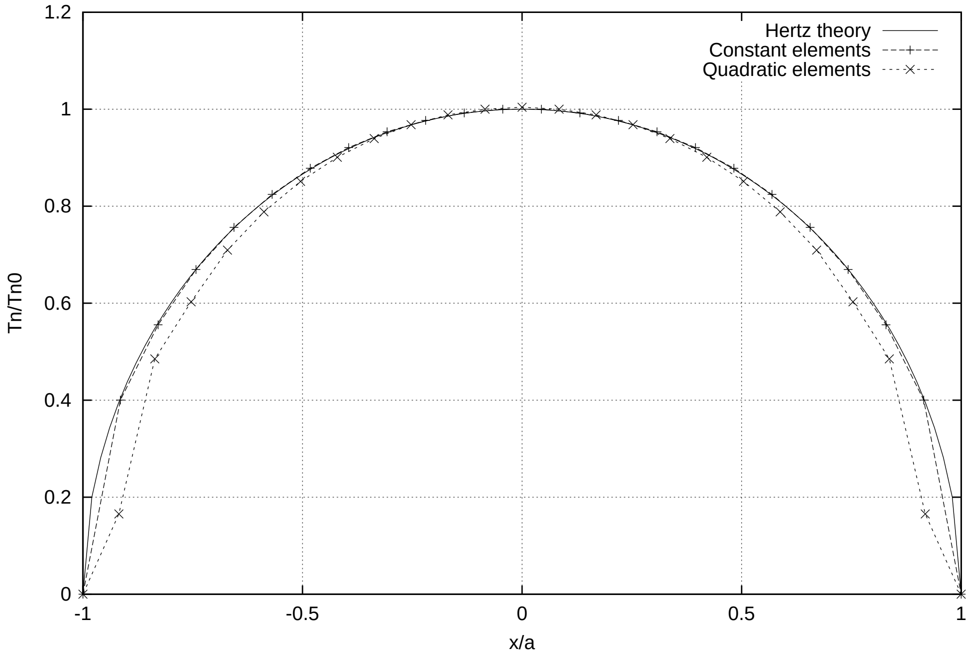

Figure 6: Normal tractions in frictionless unilateral contact: Comparison of different approximations with the analytic solution (Hertz theory)

our computations using constant and quadratic elements is presented. The residual of the Lemke algorithm is about 10^{-18}.

Figures 7, 8, and 9 show the behaviour of the normal and the tangential tractions for different values of the friction coefficient, while Figure 10 gives the corresponding tangential velocity. These results have been obtained using only a normal rigid body displacement approach.

For the following analysis, both the rigid body displacement and rigid body loading approaches have been implemented. In fact, here, the tangential effects become active such that the potential contact area is bigger than in the previous analysis. Hence, $1/6th$ of the cylinder contour is approximated in order to be able to see the three parts of the contact area: separation, slip and stick.

The results shown in the Figures 11, 12, 13, and 14 are obtained by applying a normal and a tangential rigid body displacements approach. Figure 11 shows the existence of a separation area in the potential contact zone. The stick/slip area is not centered because of the tangential effect. In the Figures 12 and 13, the normal and the tangential tractions in the frictional contact case are plotted. Figure 14 shows the velocity behaviour only in the effective contact zone.

The residual of the computations in the frictional contact case considering the rigid body displacement method are about 10^{-11}.

The last four figures show the results of the rigid body loading approach. The computed residual, related to this approach, with different values of the friction coefficient is about 10^{-7}.

All results of the frictional contact problem case satisfy the equations (2) and

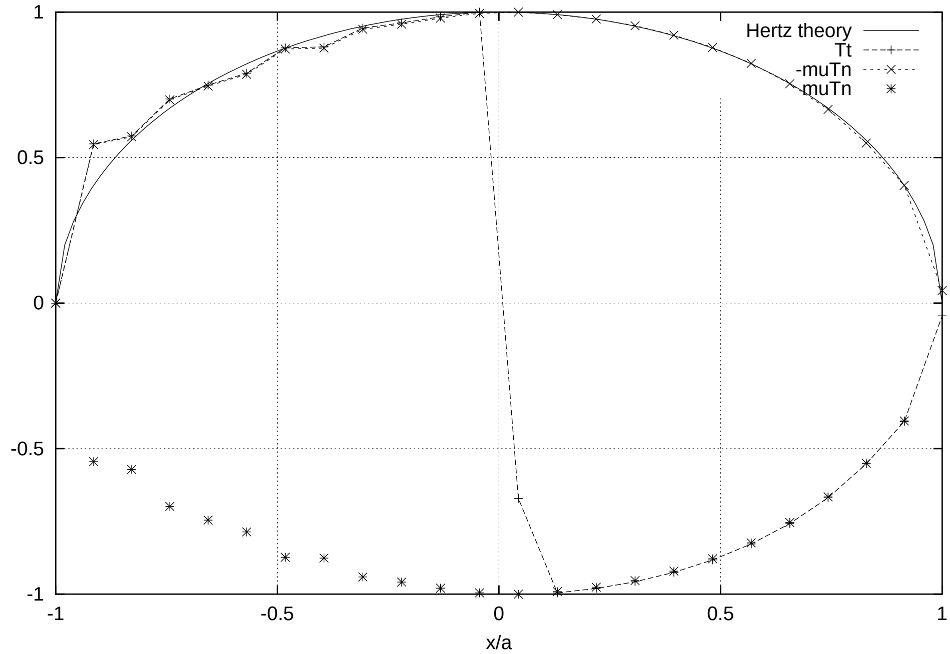

Figure 7: Normal and tangential tractions in the frictional contact case, $\mu = 0.02$: Comparison with the analytic solution

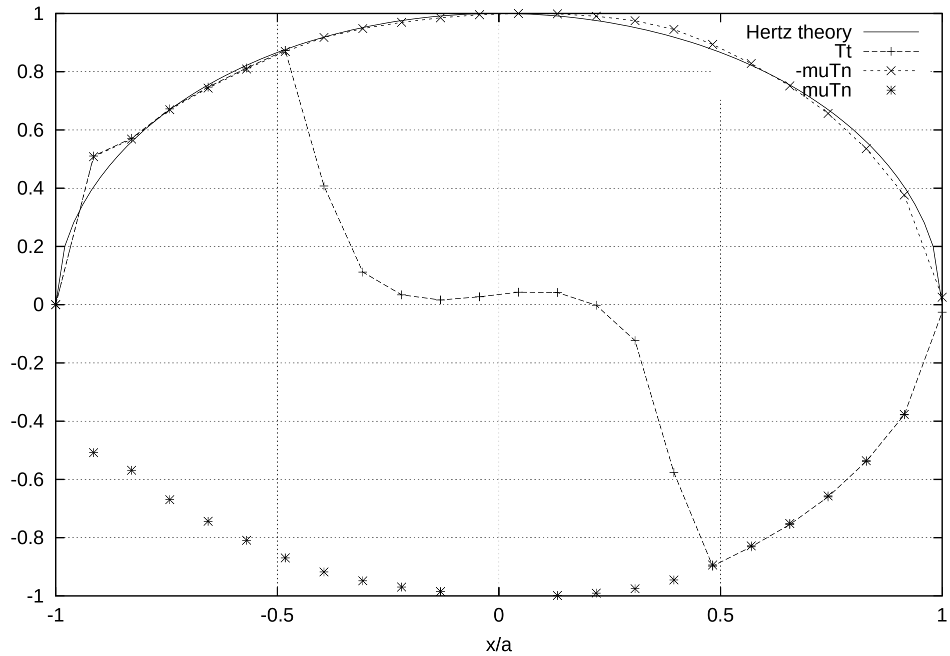

Figure 8: Normal and tangential tractions in the frictional contact case, $\mu = 0.1$: Comparison with the analytic solution

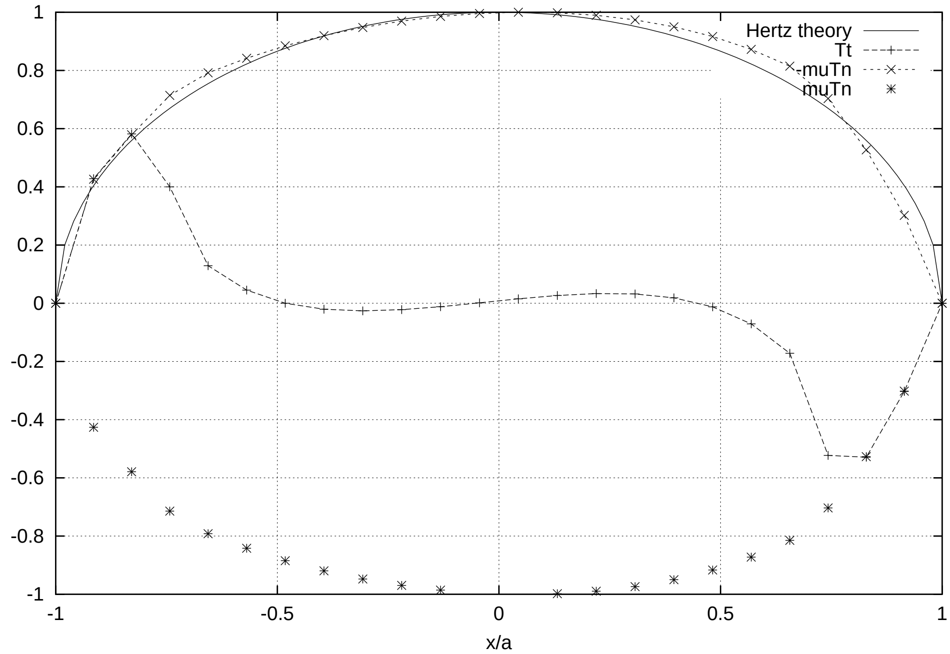

Figure 9: Normal and tangential tractions in the frictional contact case, $\mu = 0.5$: Comparison with the analytic solution

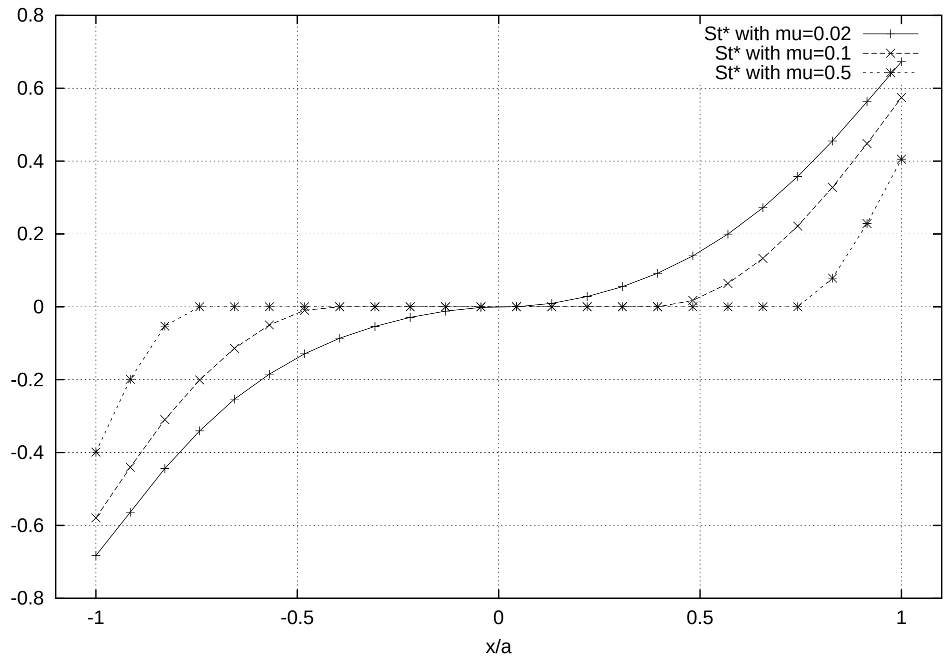

Figure 10: Tangential velocity: Dependence on the friction coefficient

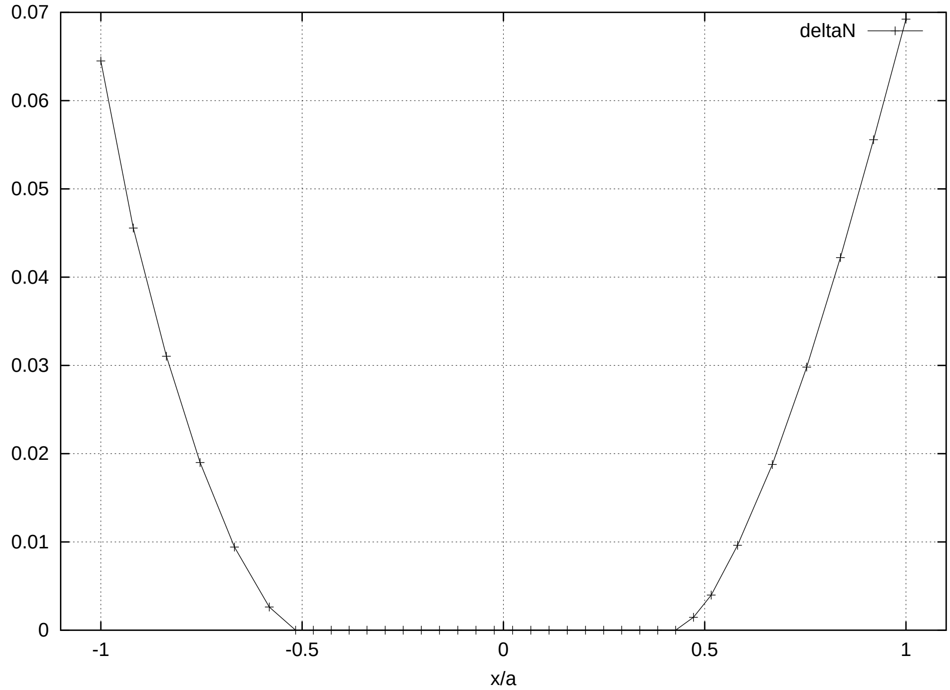

Figure 11: Distribution of stick/slip and normal separation in the potential contact zone applying the rigid body displacement approach

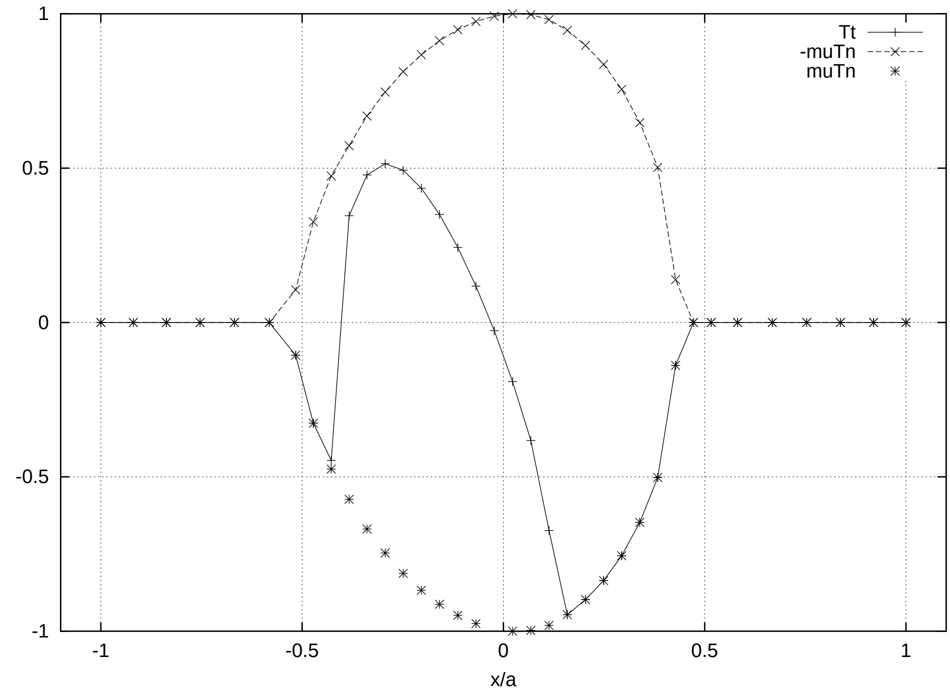

Figure 12: Normal and tangential tractions in the frictional contact case ($\mu = 0.1$) applying the rigid body displacement approach

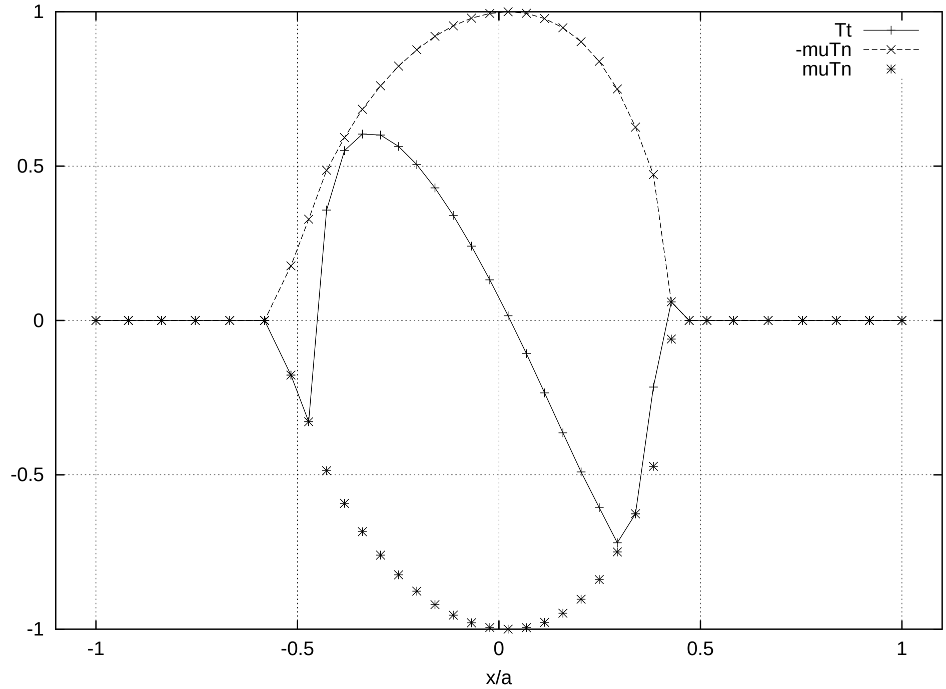

Figure 13: Normal and tangential tractions in the frictional contact case ($\mu = 0.3$) applying the rigid body displacement approach

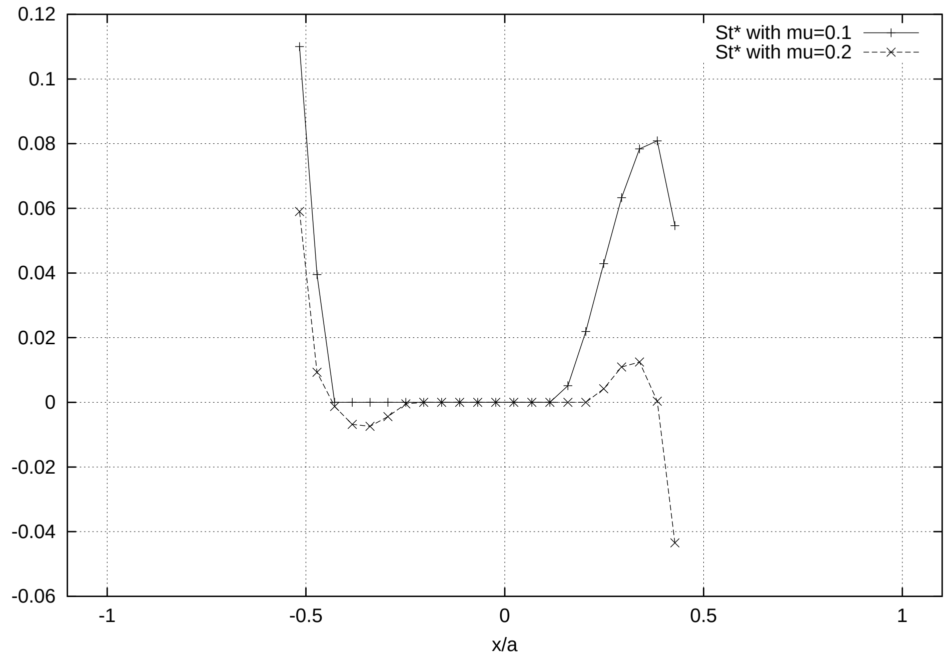

Figure 14: Tangential velocity in the effective contact zone applying the rigid body displacement approach: Dependence on the friction coefficient

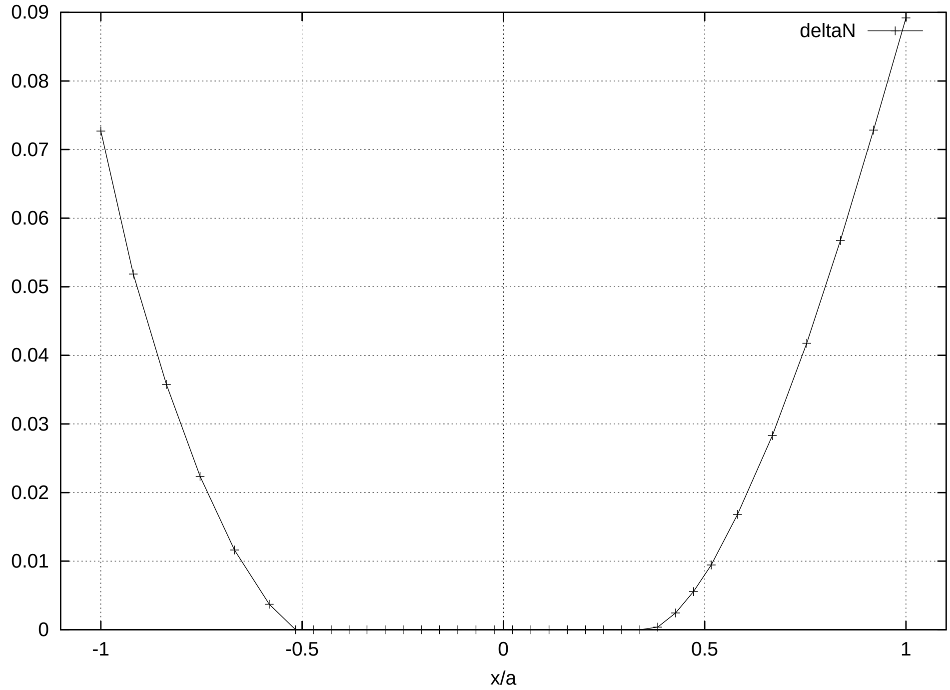

Figure 15: Distribution of stick/slip and normal separation in the potential contact zone applying the rigid body loading approach

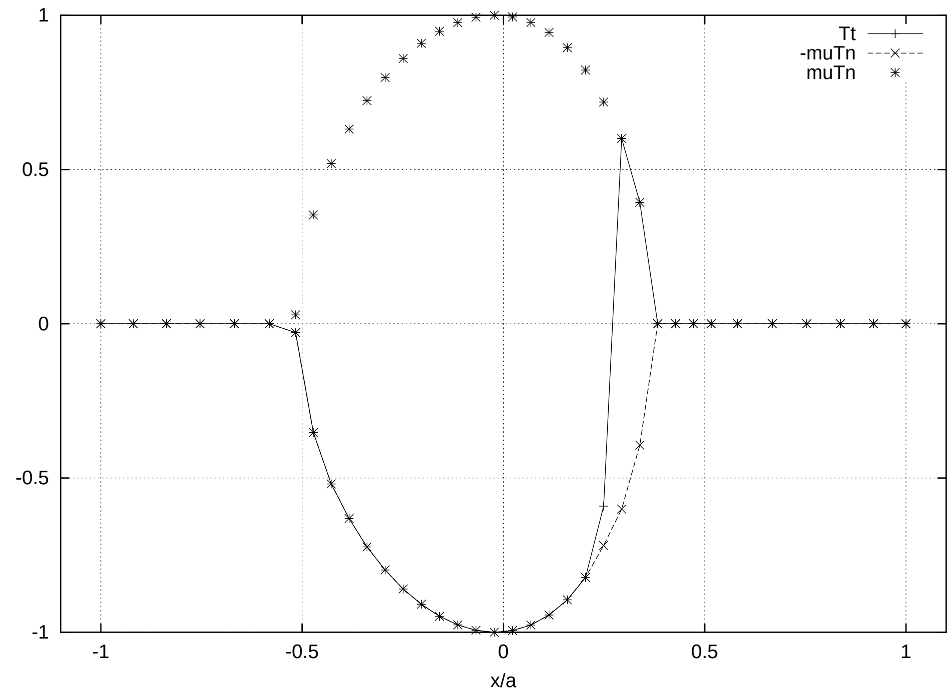

Figure 16: Normal and tangential tractions in the frictional contact case ($\mu = 0.1$) applying the rigid body loading approach

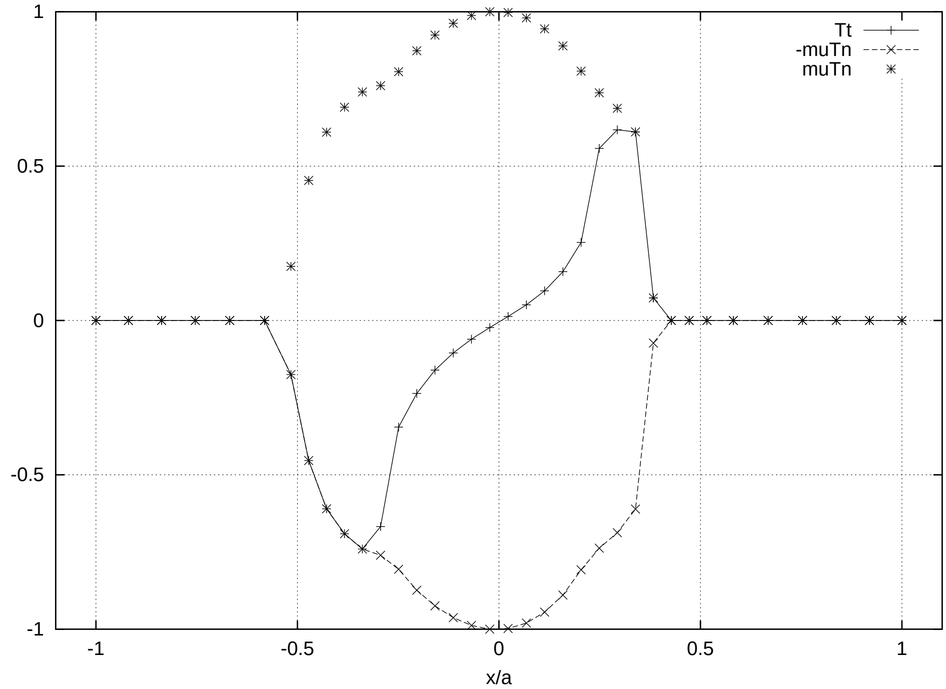

Figure 17: Normal and tangential tractions in the frictional contact case ($\mu = 0.7$) applying the rigid body loading approach

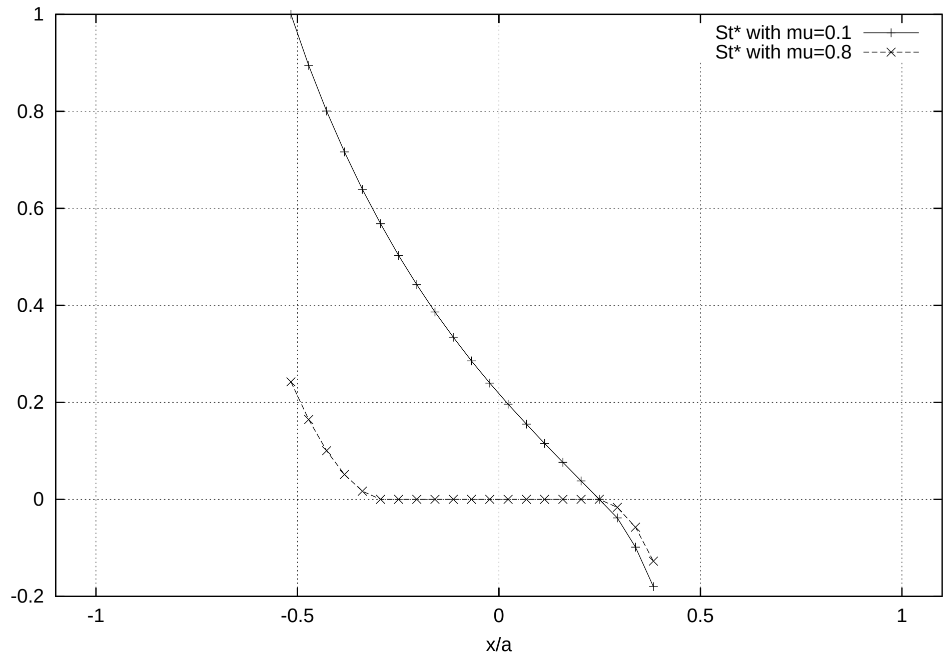

Figure 18: Tangential velocity in the effective contact zone applying the rigid body loading approach: Dependence on the friction coefficient

(26).

7 Conclusion

In the present work, the real rolling unilateral frictional contact problem has been considered, not its smoothed version, leading to a linear complementarity problem. Hence, this paper is somehow the extension of [3]: there, the assumption of a half-space has been used while, here, the real domain geometry has been discretized not the half-space. Thus, in the quasistatic frictional contact problem, an alternative to compute the velocity has been proposed. The influence matrices representing the elastostatic equations were determined using the Boundary Element Method which allows to make the CPU time much smaller than when the Finite Element Method would be applied. The nonsymmetric linear complementarity problem is solved with the Lemke's algorithm. The numerical results show a solution behaviour which is verified by the mathematical modelling and the mechanical meaning, and they have been compared with other results [1], [2] ,[3], [5], [7], [19], [25], [27] and [28]: the trend is the same.

Finally, one can state that, although the actually solved problem, namely the static problem, is well known theoretically, the presented method for solving the LCP is an effective alternative to the existing methods for solving rolling problems, namely the semi-analytical or trial and error methods, or solving the LCP by means of equivalent non-linear equations.

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