

## Uniqueness Implies Existence for Four-Point Nonlocal Boundary Value Problems for Second Order Equations

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**Abstract** Shooting methods are employed to obtain solutions of the four-point nonlocal boundary value problem for the second order equation,  $y'' = f(x, y, y')$ ,  $y(x_1) = y_1$ ,  $y(x_4) - y(x_2) - y(x_3) = y_2$ , where  $f : (a, b) \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous,  $a < x_1 < x_2 < x_3 < x_4 < b$ , and  $y_1, y_2 \in \mathbb{R}$ , and it is assumed that solutions of such problems are unique, when they exist.

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## 1 Introduction

In this paper, we are concerned with the question of the uniqueness of solutions implying the existence of solutions for four-point nonlocal boundary value problems for the second order ordinary differential equation,

$$y'' = f(x, y, y'), \quad a < x < b. \quad (1)$$

In particular, given points  $a < x_1 < x_2 < x_3 < x_4 < b$ , and  $y_1, y_2 \in \mathbb{R}$ , we shall consider uniqueness implies existence results for solutions of (1) satisfying the boundary conditions,

$$y(x_1) = y_1, \quad y(x_4) - y(x_2) - y(x_3) = y_2. \quad (2)$$

We assume throughout the following:

- (A)  $f : (a, b) \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous.
- (B) Solutions of initial value problems for (1) are unique and exist on all of  $(a, b)$ .

This paper is motivated from a recent work by Henderson [10], in which it was proved that, under (A) and (B), if solutions of certain nonlocal three-point boundary value problems for (1) are unique, then solutions of those same three-point nonlocal boundary value problems also exist. This paper is devoted to showing that uniqueness of solutions of (1), (2) implies existence of solutions of (1), (2).

Interest in multipoint boundary value problems for second order differential equations has been ongoing for several years, with much of the attention given to positive solutions. For a small sample of such work, we refer the reader to works by Bai and Fang [1], Gupta and Trofimchuk [5], Ma [17, 18] and Yang [21].

This question of uniqueness implying existence for solutions of boundary value problems enjoys quite a history for ordinary differential equations as well as finite difference equations, and recently such questions have been addressed for dynamic equations on time scales. For ordinary differential equations, the papers by Hartman [6], Klaasen [15], Lasota and Luczynski [16], Henderson [7], Jackson and Schrader [14], Henderson, Karna and Tisdell [11], and Peterson [19] are significant contributions, and for finite difference equations, the papers by Davis and Henderson [4] and Henderson [8, 9] dealt with several analogues from the differential equations including both conjugate problems and right focal problems. For these questions in the context of dynamic equations on time scales, we mention the results by Bohner and Peterson [2], Chyan [3] and Henderson and Yin [12, 13].

Our uniqueness condition on solutions of the boundary value problem (1), (2) takes the following form:

- (C) Given points  $a < x_1 < x_2 < x_3 < x_4 < b$ , if  $y$  and  $z$  are solutions of (1) such that

$$\begin{aligned} y(x_1) &= z(x_1), \\ y(x_4) - y(x_2) - y(x_3) &= z(x_4) - z(x_2) - z(x_3), \end{aligned}$$

then  $y(x) = z(x)$ , for all  $a < x < b$ .

**Remark 1** We notice that, if (A) – (C) are satisfied, then for any points  $a < r_1 < r_2 < r_3 < b$ , the three-point nonlocal boundary value problem for (1) satisfying

$$y(r_1) = s_1, \quad y(r_3) - y(r_2) = s_2, \quad (3)$$

$s_1, s_2 \in \mathbb{R}$ , has at most one solution on  $(a, b)$ . To see this, suppose that for some points  $a < r_1 < r_2 < r_3 < b$ , there are distinct solutions  $y(x)$  and  $z(x)$  of (1) so that  $y(r_1) = z(r_1)$  and  $y(r_3) - y(r_2) = z(r_3) - z(r_2)$ . Set  $w = y - z$ . By (B), we may assume  $w'(r_1) > 0$ . Now, there exists  $\tau \in (r_1, r_3)$ , such that  $w(\tau)$  is a local positive maximum, and  $w'(x) > 0$  on  $(x_1, \tau)$ . Choose points  $r_1 < \tau_1 < \tau_2 < \tau$  such that  $w(\tau_1) < \frac{w(\tau)}{2}$  and  $w(\tau_2) < \frac{w(\tau)}{2}$ . Then there exists  $\tau_2 < \tau_3 < \tau$  so that  $w(\tau_3) - w(\tau_1) - w(\tau_2) = 0$ . This is a contradiction to (C).

It follows in turn from [10] that, for any  $a < r_1 < r_2 < r_3 < b$  and any  $s_1, s_2 \in \mathbb{R}$ , there exists a unique solution of the three-point nonlocal boundary value problem (1), (3).

**Remark 2** As was pointed out in [10], it follows from Remark 1 that, under (C), for any  $a < t_1 < t_2 < b$ , the two-point conjugate boundary value problem for (1) satisfying

$$y(t_1) = v_1, \quad y(t_2) = v_2, \quad (4)$$

$v_1, v_2 \in \mathbb{R}$ , also has a unique solution on  $(a, b)$ .

In establishing our uniqueness implies existence result for solutions of (1), (2), we will make use of a precompactness condition on sequences of bounded solutions of (1). We now state the condition as a theorem, the proof of which is straight-forward.

**Theorem 1** *Assume that with respect to (1), condition (A) holds. If  $\{y_k(x)\}$  is a sequence of solutions of (1) for which there exists an interval  $[c, d] \subset (a, b)$  and there exists an  $M > 0$  such that  $|y_k(x)| \leq M$ , for all  $x \in [c, d]$  and for all  $k \in \mathbb{N}$ , then there exists a subsequence  $\{y_{k_j}(x)\}$  such that, for  $i = 0, 1$ ,  $\{y_{k_j}^{(i)}(x)\}$  converges uniformly on each compact subinterval of  $(a, b)$ .*

In Section 2, we prove that solutions of (1), (2) depend continuously on boundary conditions. In Section 3, we make use of Theorem 1 and the continuous dependence result to yield existence of solutions of (1), (2).

## 2 Continuous dependence

In this section, we will show that solutions of (1), (2) depend continuously on boundary conditions.

**Theorem 2** *Assume that conditions (A), (B) and (C) hold. Given a solution  $y(x)$  of (1) on  $(a, b)$ , an interval  $[c, d] \subset (a, b)$ , points  $c < x_1 < x_2 < x_3 < x_4 < d$ , and an  $\epsilon > 0$ , there exists a  $\delta(\epsilon, [c, d]) > 0$  such that, if  $|x_i - t_i| < \delta$ ,  $i = 1, 2, 3, 4$ , and  $c < t_1 < t_2 < t_3 < t_4 < d$ , and if  $|y(x_1) - z_1| < \delta$  and  $|y(x_4) - y(x_2) - y(x_3) - z_2| < \delta$ , then there exists a solution  $z(x)$  of (1) satisfying  $z(t_1) = z_1$ ,  $z(t_4) - z(t_2) - z(t_3) = z_2$ , and  $|y^{(i)}(x) - z^{(i)}(x)| < \epsilon$  on  $[c, d]$ ,  $i = 0, 1$ .*

**Proof:** Fix a point  $p_0 \in (a, b)$ . Next define the set

$$G = \{(x_1, x_2, x_3, x_4, C_1, C_2) \mid a < x_1 < x_2 < x_3 < x_4 < b, \text{ and } C_1, C_2 \in \mathbb{R}\}.$$

$G$  is an open subset of  $\mathbb{R}^6$ . Next, define a mapping  $\phi : G \rightarrow \mathbb{R}^6$  by

$$\phi(x_1, x_2, x_3, x_4, C_1, C_2) = (x_1, x_2, x_3, x_4, u(x_1), u(x_4) - u(x_2) - u(x_3)),$$

where  $u(x)$  is the solution of (1) satisfying the initial conditions  $u(p_0) = C_1$ ,  $u'(p_0) = C_2$ . Condition (B) implies the continuity of solutions of initial value problems for (1) with respect to initial conditions, from whence we may derive the continuity of  $\phi$ . Moreover, condition (C) implies that  $\phi$  is one-one. It follows from the Brouwer theorem on invariance of domain [20, page 199] that  $\phi(G)$  is an open subset of  $\mathbb{R}^6$ , and that  $\phi$  is a homeomorphism from  $G$  to  $\phi(G)$ . The statement of the theorem is a direct result of the continuity of  $\phi^{-1}$  and the fact that  $\phi(G)$  is open. The proof is complete.

### 3 Existence of solutions

In this section, we employ the method of shooting to obtain solutions of (1), (2) under our uniqueness assumptions.

**Theorem 3** *Assume that with respect to (1), conditions (A), (B), and (C) are satisfied. Given points  $a < x_1 < x_2 < x_3 < x_4 < b$ , and  $y_1, y_2 \in \mathbb{R}$ , there exists a unique solution of (1), (2) on  $(a, b)$ .*

**Proof:** Let  $a < x_1 < x_2 < x_3 < x_4 < b$  and  $y_1, y_2 \in \mathbb{R}$  be selected. Then, employing Remark 1 from the Introduction, let  $z(x)$  denote the solution of the three-point nonlocal boundary value problem for (1) satisfying the boundary conditions at  $x_2, x_3$  and  $x_4$ ,

$$z(x_2) = 0, \quad z(x_4) - z(x_3) = y_2.$$

Next, define the set

$$S = \{y(x_1) \mid y(x) \text{ is a solution (1) and } y(x_4) - y(x_2) - y(x_3) = z(x_4) - z(x_2) - z(x_3)\}.$$

We observe first that  $S$  is nonempty, since  $z(x_1) \in S$ . In addition, by invoking Theorem 2, we conclude that  $S$  is also an open subset of  $\mathbb{R}$ .

The remainder of the argument is devoted to showing that  $S$  is also a closed subset of  $\mathbb{R}$ . To that end, we assume for the purpose of contradiction that  $S$  is not closed. Then there exists an  $r_0 \in \overline{S} \setminus S$  and a strictly monotone sequence  $\{r_k\} \subset S$  such that  $\lim_{k \rightarrow \infty} r_k = r_0$ .

We may assume, without loss of generality, that  $r_k \uparrow r_0$ . By the definition of  $S$ , we denote, for each  $k \in \mathbb{N}$ , by  $u_k(x)$  the solution of (1) satisfying

$$u_k(x_1) = r_k, \text{ and } u_k(x_4) - u_k(x_2) - u_k(x_3) = z(x_4) - z(x_2) - z(x_3).$$

By (B) and since  $r_{k+1} > r_k$ , we have

$$u_k(x) < u_{k+1}(x) \text{ on } (a, x_2).$$

Consequently, from Theorem 1 and the fact that  $r_0 \notin S$ , we may conclude that  $\{u_k(x)\}$  is not uniformly bounded above on each compact subinterval of each of  $(a, x_1)$  and  $(x_1, x_2)$ .

Now, let  $w(x)$  be the solution of the initial value problem for (1) satisfying the initial conditions at  $x_1$ ,

$$w(x_1) = r_0, \quad w'(x_1) = 0.$$

It follows that, for some  $K$  large, there exist points,

$$a < \tau_1 < x_1 < \tau_2 < x_2,$$

such that

$$y_K(\tau_1) = w(\tau_1), \quad y_K(\tau_2) = w(\tau_2),$$

which is a contradiction to the uniqueness of solutions of (1), (4) as recorded in the Introduction's Remark 2. Thus,  $S$  is also a closed subset of  $\mathbb{R}$ .

In summary,  $S$  is a nonempty subset of  $\mathbb{R}$  that is both open and closed. We have  $S \equiv \mathbb{R}$ .

By choosing  $r = y_1 \in S$ , there is a corresponding solution  $y(x)$  of (1) such that

$$\begin{aligned}y(x_1) &= y_1, \\y(x_4) - y(x_2) - y(x_3) &= y_2.\end{aligned}$$

This completes the proof.

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