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A General Discrete Measure of Dependence

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Abstract Here the discrete Csiszar's f-Divergence or discrete Csiszar's Discrimination, the most general distance among discrete probability measures, is established as the most general measure of Dependence between two discrete random variables which is a very important matter of stochastics. Many discrete estimates of it are given, leading to optimal or nearly optimal discrete probabilistic inequalities. That is we are comparing and connecting this general discrete measure of dependence to known other specific discrete entities by involving basic parameters of our setting. This article is an expository and survey one where we apply the author's earlier continuous case results on Csiszar's distance, especially about dependence.

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1. Basics

The basic general background here has as follows.

Let f by a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$.

Let (Ω, A, λ) be a measure space, where λ is a finite or a σ-finite measure on (Ω, A). Let μ_1, μ_2 be two probability measures on (Ω, A) such that $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$.

Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon–Nikodym derivatives of μ_1, μ_2 with respect to λ. Typically we assume that

$$0 < a \leq \frac{p}{q} \leq b, \quad \text{a.e. on } \Omega \text{ and } a \leq 1 \leq b.$$

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_\Omega q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \tag{1}$$

was introduced by I. Csiszar in 1967, see [12], and is called *f-divergence* of the probability measures μ_1 and μ_2.

By Lemma 1.1 of [12], the integral (1) is well-defined and $\Gamma_f(\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. Furthermore by [12], [6] $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of λ. The concept of f-divergence was introduced first in [11] as a generalization of Kullback's "*information for discrimination*" or *I-divergence* (*generalized entropy*) [15], [16], and of Rényi's "*information gain*" (*I-divergence*) *of order* α [18]. In fact the *I-divergence of order* 1 equals

$$\Gamma_{u\log_2 u}(\mu_1,\mu_2).$$

The choice $f(u)=(u-1)^2$ produces again a known measure of difference of distributions that is called χ^2-*divergence*. Of course the *total variation* distance

$$|\mu_1-\mu_2|=\int_\Omega |p(x)-q(x)|\,d\lambda(x)$$

is equal to $\Gamma_{|u-1|}(\mu_1,\mu_2)$. Here by assuming $f(1)=0$ we can consider $\Gamma_f(\mu_1,\mu_2)$ the f-divergence as a measure of the difference between the probability measures μ_1, μ_2.

This is an expository, survey and applications article where we apply to the discrete case a large variety of interesting results of the above continuous case coming from the author's earlier but recent papers [5], [6], [7], [8], [9]. Namely here we present a complete study of discrete Csiszar's f-*divergence* as a *measure of dependence* of two discrete random variables.

Throughout this article we use the following.

Let again f be a convex function from $(0,+\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1)=0$.

Let the probability space (Ω,P) and X, Y be discrete random variables from Ω into $\mathbb{R}$ such that their ranges $\mathcal{R}_X=\{x_1,x_2,\ldots\}$, $\mathcal{R}_Y=\{y_1,y_2,\ldots\}$, respectively, are finite or countable subsets of $\mathbb{R}$.

Let the probability functions of X and Y be

$$\begin{aligned} p_i &= p(x_i)=P(X=x_i), \quad i=1,2,\ldots,\\ q_i &= q(y_j)=P(Y=y_j), \quad j=1,2,\ldots, \end{aligned} \tag{2}$$

respectively. Without loss of generality we assume that all $p_i,q_j>0$, $i,j=1,2,\ldots$.

Consider the probability distributions μ_{XY} and $\mu_X\times\mu_Y$ on $\mathcal{R}_X\times\mathcal{R}_Y$, where μ_{XY}, μ_X, μ_Y stand for the joint probability distribution of X and Y and their marginal distributions, respectively, while $\mu_X\times\mu_Y$ is the product probability distribution of X and Y. Here μ_{XY} is uniquely determined by the double sequence

$$t_{ij}=t(x_i,y_j)=P(X=x_i,Y=y_j)=\mu_{XY}(\{(x_i,y_j)\}), \quad i,j=1,2,\ldots, \tag{3}$$

which t is the probability function of (X,Y).

Again without loss of generality we assume that $t_{ij}>0$ for all $i,j=1,2,\ldots$. Clearly also here μ_X, μ_Y are uniquely determined by $p_i=\mu_X(\{x_i\})$, $q_j=\mu_Y(\{y_j\})$, respectively, $i,j=1,2,\ldots$. It is obvious that the product probability distribution $\mu_X\times\mu_Y$ has associated probability function pq.

In this article we assume that

$$0<a\le\frac{t_{ij}}{p_iq_j}\le b, \quad \text{all } i,j=1,2,\ldots \quad \text{and } a\le 1\le b, \tag{4}$$

where a,b are fixed.

Let $\mathcal{P}$ be the set of all subsets of $\mathcal{R}_X\times\mathcal{R}_Y$. We define λ to be a finite or σ-finite measure on $(\mathcal{R}_X\times\mathcal{R}_Y,\mathcal{P})$ such that $\lambda(\{(x_i,y_i)\})=1$, for all $i,j=1,2,\ldots$. We notice that μ_{XY} and $\mu_X\times\mu_Y$ are discrete probability measures on $(\mathcal{R}_X\times\mathcal{R}_Y,\mathcal{P})$ and are absolutely continuous ($\ll$) with respect to λ.

By applying (1) we obtain the special case

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)=\sum_{i=1}^\infty\sum_{j=1}^\infty p_iq_jf\left(\frac{t_{ij}}{p_iq_j}\right). \tag{5}$$

This is the *discrete Csiszar's distance* or *f-divergence between μ_{XY} and $\mu_X \times \mu_Y$*. The random variables X, Y are the less dependent the closer the distributions μ_{XY} and $\mu_X \times \mu_Y$ are, thus $\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y)$ can be considered as a *measure of dependence* of X and Y, see also [17], [12], [9].

From the above derives that $\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \geq 0$ with equality only when $\mu_{XY} = \mu_X \times \mu_Y$, i.e. when X, Y are independent discrete random variables. As related references see here [11], [12], [13], [14], [15], [16], [17], [18], [5], [6], [7], [8], [9].

For $f(u) = u \log_2 u$ we obtain the *mutual information* of X and Y,

$$I(X, Y) = I(\mu_{XY} \parallel \mu_X \times \mu_Y) = \Gamma_{u \log_2 u}(\mu_{XY}, \mu_X \times \mu_Y),$$

see [12]. For $f(u) = (u-1)^2$ we get the *mean square contingency*:

$$\varphi^2(X, Y) = \Gamma_{(u-1)^2}(\mu_{XY}, \mu_X \times \mu_Y),$$

see [12].

In Information Theory and Statistics many various divergences are used which are special cases of the above Γ_f divergence. Γ_f has many applications also in Anthropology, Genetics, Finance, Economics, Political Science, Biology, Approximation of Probability Distributions, Signal Processing and Pattern Recognition.

In the next the proofs of all formulated and presented results are based a lot on this Section 1 especially on the transfer from the continuous to the discrete case.

2. Results

Part I. Here we apply results from [5], [9]. We present the following

Theorem 1. *Let $f \in C^1([a, b])$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \|f'\|_{\infty,[a,b]} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |t_{ij} - p_i q_j| \right). \tag{6}$$

Inequality (6) is sharp. The optimal function is $f^*(y) = |y-1|^\alpha$, $\alpha > 1$ under the condition, either

i)

$$\max(b-1, 1-a) < 1 \quad \textit{and} \quad C := \sup_{i,j \in \mathbb{N}} (p_i q_j) < +\infty, \tag{7}$$

or

ii)

$$|t_{ij} - p_i q_j| \leq p_i q_j \leq 1, \quad \textit{all } i, j = 1, 2, \ldots. \tag{8}$$

Proof. Based on Theorem 1 of [5] and Theorem 1 of [9]. □

Next we give the more general

Theorem 2. *Let $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$, such that $f^{(k)}(1) = 0$, $k = 0, 2, 3, \ldots, n$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\|f^{(n+1)}\|_{\infty,[a,b]}}{(n+1)!} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i^{-n} q_j^{-n} |t_{ij} - p_i q_j|^{n+1} \right). \tag{9}$$

Inequality (9) is sharp. The optimal function is $\tilde{f}(y) := |y-1|^{n+\alpha}$, $\alpha > 1$ under the condition (i) or (ii) of Theorem 1.

Proof. As in Theorems 2 of [5], [9], respectively. ☐

As a special case we have

Corollary 1 (to Theorem 2). *Let* $f \in C^2([a,b])$. *Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \frac{\|f^{(2)}\|_{\infty,[a,b]}}{2} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i^{-1} q_j^{-1} (t_{ij} - p_i q_j)^2 \right). \tag{10}$$

Inequality (10) is sharp. The optimal function is $\tilde{f}(y) := |y-1|^{1+\alpha}$, $\alpha > 1$ *under the condition (i) or (ii) of Theorem 1.*

Proof. By Corollaries 1 of [5] and [9]. ☐

Next we connect Csiszar's f-divergence to the usual first modulus of continuity ω_1.

Theorem 3. *Suppose that*

$$0 < h := \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |t_{ij} - p_i q_j| \right) \le \min(1-a, b-1). \tag{11}$$

Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \omega_1(f, h). \tag{12}$$

Inequality (12) is sharp, i.e. attained by

$$f^*(x) = |x-1|.$$

Proof. By Theorems 3 of [5] and [9]. ☐

Part II. Here we apply results from [6] and [9]. But first we need to give some basic background from [10], [2]. Let $\nu > 0$, $n := [\nu]$ and $\alpha = \nu - n$ $(0 < \alpha < 1)$. Let $x, x_0 \in [a,b] \subseteq \mathbb{R}$ such that $x \ge x_0$, where x_0 is fixed and $f \in C([a,b])$ and define

$$(J_\nu^{x_0} f)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-s)^{\nu-1} f(s) ds, \quad x_0 \le x \le b, \tag{13}$$

the *generalized Riemann–Liouville integral*, where Γ stands for the gamma function.

We define the subspace

$$C_{x_0}^{\nu}([a,b]) := \left\{ f \in C^n([a,b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0,b]) \right\}. \tag{14}$$

For $f \in C_{x_0}^{\nu}([a,b])$ we define the *generalized ν-fractional derivative* of f over $[x_0,b]$ as

$$D_{x_0}^{\nu} f = D J_{1-\alpha}^{x_0} f^{(n)} \quad \left(D := \frac{d}{dx} \right). \tag{15}$$

We present the following

Theorem 4. *Let* $a < b$, $1 \le \nu < 2$, $f \in C_a^{\nu}([a,b])$. *Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{1-\nu} (t_{ij} - a p_i q_j)^{\nu} \right). \tag{16}$$

Proof. It follows from Theorem 2 of [6] and Theorem 4 of [9]. ☐

The counterpart of the previous result comes next.

Theorem 5. *Let $a < b$, $\nu \geq 2$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\|D_a^\nu f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{1-\nu} (t_{ij} - a p_i q_j)^\nu \right). \tag{17}$$

Proof. By Theorem 3 of [6] and Theorem 5 of [9]. □

Next we given an $L_{\tilde{\alpha}}$ estimate.

Theorem 6. *Let $a < b$, $\nu \geq 1$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$ and let $\tilde{\alpha}, \beta > 1$: $\frac{1}{\tilde{\alpha}} + \frac{1}{\beta} = 1$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\|D_a^\nu f\|_{\tilde{\alpha},[a,b]}}{\Gamma(\nu)(\beta(\nu-1)+1)^{1/\beta}} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{2-\nu-\frac{1}{\beta}} (t_{ij} - a p_i q_j)^{\nu-1+\frac{1}{\beta}} \right). \tag{18}$$

Proof. Based on Theorem 4 of [6] and Theorem 6 of [9]. □

It follows an L_∞ estimate.

Theorem 7. *Let $a < b$, $\nu \geq 1$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{2-\nu} (t_{ij} - a p_i q_j)^{\nu-1} \right). \tag{19}$$

Proof. Based on Theorem 5 of [6] and Theorem 7 of [9]. □

We continue with

Theorem 8. *Let $f \in C^1([a,b])$, $a < b$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{1}{b-a} \int_a^b f(x)dx - \left(\sum_{i,j=1}^{\infty} f'\left(\frac{t_{ij}}{p_i q_j}\right) \left(\frac{p_i q_j (a+b)}{2} - t_{ij} \right) \right). \tag{20}$$

Proof. Based on Theorem 6 of [6] and Theorem 8 of [9]. □

Furthermore we get

Theorem 9. *Let n odd and $f \in C^{n+1}([a,b])$, $a < b$, such that $f^{(n+1)} \geq 0$ (≤ 0). Then*

$$\begin{aligned} \Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq (\geq) & \frac{1}{b-a} \int_a^b f(x)dx \\ & - \sum_{i=1}^{n} \frac{1}{(i+1)!} \left[\sum_{k=0}^{i} \left(\sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} f^{(i)}\left(\frac{t_{\ell j}}{p_\ell q_j}\right) \right.\right. \\ & \left.\left. \cdot (p_\ell q_j)^{1-i} (b p_\ell q_j - t_{\ell j})^{(i-k)} (a p_\ell q_j - t_{\ell j})^k \right) \right]. \end{aligned} \tag{21}$$

Proof. Based on Theorem 7 of [6] and Theorem 9 of [9]. □

Part III. Here we apply results from [7] and [9].

We start with

Theorem 10. *Let $f, g \in C^1([a,b])$ where f as in this article, $g' \neq 0$ over $[a,b]$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \left\| \frac{f'}{g'} \right\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left| g\left(\frac{t_{ij}}{p_i q_j} \right) - g(1) \right| \right). \tag{22}$$

Proof. Based on Theorem 1 of [7] and Theorem 10 of [9]. □

We give

Examples 1 (to Theorem 10).

1) Let $g(x) = \frac{1}{x}$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \|x^2 f'\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} \frac{p_i q_j}{t_{ij}} |t_{ij} - p_i q_j| \right). \tag{23}$$

2) Let $g(x) = e^x$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \|e^{-x} f'\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left| e^{t_{ij}/p_i q_j} - e \right| \right). \tag{24}$$

3) Let $g(x) = e^{-x}$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \|e^x f'\|_{\infty[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left| e^{-t_{ij}/p_i q_j} - e^{-1} \right| \right). \tag{25}$$

4) Let $g(x) = \ln x$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \|x f'\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left| \ln\left(\frac{t_{ij}}{p_i q_j} \right) \right| \right). \tag{26}$$

5) Let $g(x) = x \ln x$, with $a > e^{-1}$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \left\| \frac{f'}{1 + \ln x} \right\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} t_{ij} \left| \ln\left(\frac{t_{ij}}{p_i q_j} \right) \right| \right). \tag{27}$$

6) Let $g(x) = \sqrt{x}$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq 2\|\sqrt{x} f'\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} \sqrt{p_i q_j} \left| \sqrt{t_{ij}} - \sqrt{p_i q_j} \right| \right). \tag{28}$$

7) Let $g(x) = x^\alpha$, $\alpha > 1$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{1}{\alpha} \left\| \frac{f'}{x^{\alpha-1}} \right\|_{\infty,[a,b]} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{1-\alpha} |t_{ij}^\alpha - (p_i q_j)^\alpha| \right). \tag{29}$$

Next we give

Theorem 11. *Let $f \in C^1([a,b])$, $a<b$. Then*

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq \frac{\|f'\|_{\infty,[a,b]}}{(b-a)}\left[\left(\frac{a^2+b^2}{2}\right)-(a+b)+\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}t_{ij}^2\right)\right]. \tag{30}$$

Proof. By Theorem 2 of [7] and Theorem 11 of [9]. □

It follows

Theorem 12. *Let $f \in C^{(2)}([a,b])$, $a<b$. Then*

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{1}{b-a}\int_a^b f(s)ds-\left(\frac{f(b)-f(a)}{b-a}\right)\left(1-\frac{a+b}{2}\right)\right| \\ \leq \frac{(b-a)^2}{2}\|f''\|_{\infty,[a,b]}\left\{\left\{\sum_{i,j=1}^{\infty}p_iq_j\left[\frac{\left(\frac{t_{ij}}{p_iq_j}-\left(\frac{a+b}{2}\right)\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2\right\}+\frac{1}{12}\right\}. \tag{31}$$

Proof. Based on (33) of [7] and Theorem 12 of [9]. □

Working more generally we have

Theorem 13. *Let $f \in C^1([a,b])$ and $g \in C([a,b])$ of bounded variation, $g(a) \neq g(b)$. Then*

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{1}{g(b)-g(a)}\int_a^b f(s)dg(s)\right| \\ \leq \|f''\|_{\infty,[a,b]}\left\{\sum_{i,j=1}^{\infty}p_iq_j\left(\int_a^b\left|P\left(g\left(\frac{t_{ij}}{p_iq_j}\right),g(s)\right)\right|ds\right)\right\}, \tag{32}$$

where

$$P(g(z),g(s)) := \begin{cases} \dfrac{g(s)-g(a)}{g(b)-g(a)}, & a\leq s\leq z, \\ \dfrac{g(s)-g(b)}{g(b)-g(a)}, & z<s\leq b. \end{cases} \tag{33}$$

Proof. By Theorem 5 of [7] and Theorem 13 of [9]. □

We give

Examples 2 (to Theorem 13). Let $f \in C^1([a,b])$ and $g(x)=e^x, \ln x, \sqrt{x}, x^\alpha$, $\alpha>1$; $x>0$. Then by (71)–(74) of [7] and (30)–(33) of [9] we obtain

1)

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{1}{e^b-e^a}\int_a^b f(s)e^s ds\right| \\ \leq \frac{\|f'\|_{\infty,[a,b]}}{(e^b-e^a)}\left\{2\sum_{i,j=1}^{\infty}p_iq_je^{t_{ij}/p_iq_j}-e^a(2-a)+e^b(b-2)\right\}, \tag{34}$$

2)

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\left(\ln\left(\frac{b}{a}\right)\right)^{-1}\int_a^b\frac{f(s)}{s}ds\right|$$
$$\leq\|f'\|_{\infty,[a,b]}\left(\ln\left(\frac{b}{a}\right)\right)^{-1}\left\{2\left(\sum_{i,j=1}^{\infty}t_{ij}\ln\left(\frac{t_{ij}}{p_iq_j}\right)\right)-\ln(ab)+(a+b-2)\right\},\quad(35)$$

3)

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{1}{2(\sqrt{b}-\sqrt{a})}\int_a^b\frac{f(s)}{\sqrt{s}}ds\right|$$
$$\leq\frac{\|f'\|_{\infty[a,b]}}{(\sqrt{b}-\sqrt{a})}\left\{\frac{4}{3}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1/2}t_{ij}^{3/2}\right)+\frac{(a^{3/2}+b^{3/2})}{3}-(\sqrt{a}+\sqrt{b})\right\},\quad(36)$$

4) ($\alpha>1$)

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{\alpha}{(b^\alpha-a^\alpha)}\int_a^b f(s)s^{\alpha-1}ds\right|$$
$$\leq\frac{\|f'\|_{\infty,[a,b]}}{(b^\alpha-a^\alpha)}\left\{\frac{2}{(\alpha+1)}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-\alpha}t_{ij}^{\alpha+1}\right)\right.$$
$$\left.+\left(\frac{\alpha}{\alpha+1}\right)(a^{\alpha+1}+b^{\alpha+1})-(a^\alpha+b^\alpha)\right\}\quad(37)$$

We continue with

Theorem 14. *Let $f\in C^{(2)}([a,b])$, $g\in C([a,b])$ and of bounded variation, $g(a)\neq g(b)$. Then*

$$\left|\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)-\frac{1}{(g(b)-g(a))}\int_a^b f(s)dg(s)-\frac{1}{(g(b)-g(a))}\left(\int_a^b f'(s_1)dg(s_1)\right)\right.$$
$$\left.\cdot\left\{\sum_{i,j=1}^{\infty}\left\{p_iq_j\left(\int_a^b P\left(g\left(\frac{t_{ij}}{p_iq_j}\right),g(s)\right)ds\right)\right)\right\}\right|$$
$$\leq\|f''\|_{\infty,[a,b]}\left\{\sum_{i,j=1}^{\infty}\left(p_iq_j\left(\int_a^b\int_a^b\left|P\left(g\left(\frac{t_{ij}}{p_iq_j}\right),g(s)\right)\right||P(g(s),g(s_1))|ds_1ds\right)\right)\right\}(38)$$

Proof. Apply (79) of [7] and Theorem 14 of [9]. □

Remark 1. Next we define

$$P^*(z,s):=\begin{cases}s-a, & s\in[a,z],\\ s-b, & s\in(z,b].\end{cases}\quad(39)$$

Let $f\in C^1([a,b])$. Then by (25) of [7] we get the representation

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)=\frac{1}{b-a}\left(\int_a^b f(s)ds+\mathcal{R}_1\right),$$

where according to (26) of [7] and (36) of [9] we have

$$\mathcal{R}_1 = \sum_{i,j=1}^{\infty} p_i q_j \left(\int_a^b P^* \left(\frac{t_{ij}}{p_i q_j}, s \right) f'(s) ds \right). \tag{40}$$

Let again $f \in C^1([a,b])$, $g \in C([a,b])$ and of bounded variation, $g(a) \neq g(b)$. Then by (47) of [7] we get the representation

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \frac{1}{(g(b)-g(a))} \int_a^b f(s) dg(s) + \mathcal{R}_4, \tag{41}$$

where by (48) of [7] and (38) of [9] we obtain

$$\mathcal{R}_4 = \sum_{i,j=1}^{\infty} p_i q_j \left(\int_a^b P \left(g\left(\frac{t_{ij}}{p_i q_j} \right), g(s) \right) f'(s) ds \right). \tag{42}$$

Let $\tilde{\alpha}, \beta > 1$: $\frac{1}{\tilde{\alpha}} + \frac{1}{\beta} = 1$, then by (89) of [7] and (39) of [9] we have

$$|\mathcal{R}_1| \leq \|f'\|_{\tilde{\alpha},[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left\| P^* \left(\frac{t_{ij}}{p_i q_j}, \cdot \right) \right\|_{\beta,[a,b]} \right), \tag{43}$$

and by (90) of [7] and (40) of [9] we get

$$|\mathcal{R}_1| \leq \|f'\|_{1,[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left\| P^* \left(\frac{t_{ij}}{p_i q_j}, \cdot \right) \right\|_{\infty,[a,b]} \right). \tag{44}$$

Furthermore by (95), (96) of [7] and (41), (42) of [9] we find

$$|\mathcal{R}_4| \leq \|f'\|_{\tilde{\alpha},[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left\| P \left(g\left(\frac{t_{ij}}{p_i q_j} \right), g(\cdot) \right) \right\|_{\beta,[a,b]} \right), \tag{45}$$

and

$$|\mathcal{R}_4| \leq \|f'\|_{1,[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \left\| P \left(g\left(\frac{t_{ij}}{p_i q_j} \right), g(\cdot) \right) \right\|_{\infty,[a,b]} \right), \tag{46}$$

Remark 2. Let $a < b$.

(i) Here by (105) of [7] and (43) of [9] we obtain

$$|\mathcal{R}_1| \leq \|f'\|_{1,[a,b]} \left(\sum_{i,j=1}^{\infty} p_i q_j \max \left(\frac{t_{ij}}{p_i q_j} - a, b - \frac{t_{ij}}{p_i q_j} \right) \right), \quad f \in C^1([a,b]). \tag{47}$$

(ii) Let g be strictly increasing and continuous over $[a,b]$, e.g., $g(x) = e^x$, $\ln x$, $\sqrt{x}$, x^α with $\alpha > 1$, and $x > 0$ whenever is needed. Also $f \in C^1([a,b])$. Then by (106) of [7] and (44) of [9] we find

$$|\mathcal{R}_4| \leq \frac{\|f'\|_{1,[a,b]}}{(g(b)-g(a))} \left(\sum_{i,j=1}^{\infty} p_i q_j \max \left(g\left(\frac{t_{ij}}{p_i q_j} \right) - g(a), g(b) - g\left(\frac{t_{ij}}{p_i q_j} \right) \right) \right). \tag{48}$$

In particular via (107)–(110) of [7] and (45)–(48) of [9] we obtain

1)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{(e^b - e^a)} \left(\sum_{i,j=1}^{\infty} p_i q_j \max\left(e^{t_{ij}/p_i q_j} - e^a, e^b - e^{t_{ij}/p_i q_j}\right) \right), \tag{49}$$

2)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{(\ln b - \ln a)} \left(\sum_{i,j=1}^{\infty} p_i q_j \max\left(\ln\left(\frac{t_{ij}}{p_i q_j}\right) - \ln a, \ln b - \ln\left(\frac{t_{ij}}{p_i q_j}\right)\right)\right), \tag{50}$$

3)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{\sqrt{b} - \sqrt{a}} \left(\sum_{i,j=1}^{\infty} p_i q_j \max\left(\sqrt{\frac{t_{ij}}{p_i q_j}} - \sqrt{a}, \sqrt{b} - \sqrt{\frac{t_{ij}}{p_i q_j}}\right)\right), \tag{51}$$

and finally for $\alpha > 1$ we obtain

4)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{b^\alpha - a^\alpha} \left(\sum_{i,j=1}^{\infty} p_i q_j \max\left(\frac{t_{ij}^\alpha}{(p_i q_j)^\alpha} - a^\alpha, b^\alpha - \frac{t_{ij}^\alpha}{(p_i q_j)^\alpha}\right)\right). \tag{52}$$

At last

iii) Let $\tilde{\alpha}, \beta > 1$ such that $\frac{1}{\tilde{\alpha}} + \frac{1}{\beta} = 1$, then by (115) of [7] and (49) of [9] we have

$$|\mathcal{R}_1| \le \|f'\|_{\tilde{\alpha},[a,b]} \left(\sum_{i,j=1}^{\infty} \left(\sqrt[\beta]{\frac{(t_{ij} - a p_i q_j)^{\beta+1} + (b p_i q_j - t_{ij})^{\beta+1}}{(\beta+1) p_i q_j}} \right)\right). \tag{53}$$

Part IV. Here we apply results from [8] and [9].

We start with

Theorem 15. *Let $a < b$, $f \in C^n([a,b])$, $n \ge 1$ with $|f^{(n)}(s) - f^{(n)}(1)|$ be a convex function in s. Let $0 < h < \min(1-a, b-1)$ be fixed. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i,j=1}^{\infty} (p_i q_j)^{1-k} (t_{ij} - p_i q_j)^k \right| + \frac{\omega_1(f^{(n)}, h)}{(n+1)!h} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n} |t_{ij} - p_i q_j|^{n+1} \right). \tag{54}$$

Here ω_1 is the usual first modulus of continuity. If $f^{(k)}(1) = 0$, $k = 2, \dots, n$, then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \frac{\omega_1(f^{(n)}, h)}{(n+1)!h} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n} |t_{ij} - p_i q_j|^{n+1} \right). \tag{55}$$

Inequalities (54) and (55) when n is even are attained by

$$\tilde{f}(s) = \frac{|s-1|^{n+1}}{(n+1)!}, \quad a \le s \le b. \tag{56}$$

Proof. Apply Theorem 1 of [8] and Theorem 15 of [9]. □

We continue with the general

Theorem 16. *Let $f \in C^n([a,b])$, $n \geq 1$. Assume that $\omega_1(f^{(n)}, \delta) \leq w$, where $0 < \delta \leq b-a$, $w > 0$. Let $x \in \mathbb{R}$ and denote by*

$$\phi_n(x) := \int_0^{|x|} \left\lceil \frac{s}{\delta} \right\rceil \frac{(|x|-s)^{n-1}}{(n-1)!} ds, \tag{57}$$

where $\lceil \cdot \rceil$ is the ceiling of the number. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i,j=1}^{\infty} (p_i q_j)^{1-k} (t_{ij} - p_i q_j)^k \right| + w \left(\sum_{i,j=1}^{\infty} p_i q_j \phi_n \left(\frac{t_{ij} - p_i q_j}{p_i q_j} \right) \right). \tag{58}$$

Inequality (58) is sharp, namely it is attained by the function

$$\tilde{f}_n(s) := w \phi_n(s-1), \quad a \leq s \leq b, \tag{59}$$

when n is even.

Proof. Based on Theorem 2 of [8] and Theorem 16 of [9]. □

It follows

Corollary 2 (to Theorem 16). *It holds* $(0 < \delta \leq b-a)$

$$\begin{aligned} \Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq& \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i,j=1}^{\infty} (p_i q_j)^{1-k} (t_{ij} - p_i q_j)^k \right| \\ &+ \omega_1(f^{(n)}, \delta) \Bigg\{ \frac{1}{(n+1)!\delta} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n} |t_{ij} - p_i q_j|^{n+1} \right) \\ &+ \frac{1}{2n!} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{1-n} |t_{ij} - p_i q_j|^{n} \right) \\ &+ \frac{\delta}{8(n-1)!} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{2-n} |t_{ij} - p_i q_j|^{n-1} \right) \Bigg\}, \end{aligned} \tag{60}$$

and

$$\begin{aligned} \Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq& \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i,j=1}^{\infty} (p_i q_j)^{1-k} (t_{ij} - p_i q_j)^k \right| \\ &+ \omega_1(f^{(n)}, \delta) \Bigg\{ \frac{1}{n!} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{1-n} |t_{ij} - p_i q_j|^{n} \right) \\ &+ \frac{1}{n!(n+1)\delta} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n} |t_{ij} - p_i q_j|^{n+1} \right) \Bigg\}. \end{aligned} \tag{61}$$

In particular we have

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)\le\omega_1(f',\delta)\Bigg\{\frac{1}{2\delta}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}(t_{ij}-p_iq_j)^2\right)+\frac{1}{2}\left(\sum_{i,j=1}^{\infty}|t_{ij}-p_iq_j|\right)+\frac{\delta}{8}\Bigg\}, \tag{62}$$

and

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)\le\omega_1(f',\delta)\Bigg\{\left(\sum_{i,j=1}^{\infty}|t_{ij}-p_iq_j|\right)+\frac{1}{2\delta}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}(t_{ij}-p_iq_j)^2\right)\Bigg\}. \tag{63}$$

Proof. By Corollary 1 of [8] and Corollary 2 of [9]. □

Also we have

Corollary 3 (to Theorem 16). *Assume here that $f^{(n)}$ is of Lipschitz type of order α, $0<\alpha\le 1$, i.e.*

$$\omega_1(f^{(n)},\delta)\le K\delta^{\alpha},\quad K>0, \tag{64}$$

for any $0<\delta\le b-a$. Then

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)\le\sum_{k=2}^{n}\frac{|f^{(k)}(1)|}{k!}\left|\sum_{i,j=1}^{\infty}(p_iq_j)^{1-k}(t_{ij}-p_iq_j)^k\right|+\frac{K}{\prod_{i=1}^{n}(\alpha+i)}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{1-n-\alpha}|t_{ij}-p_iq_j|^{n+\alpha}\right). \tag{65}$$

When n is even (65) is attained by $f^*(x)=c|x-1|^{n+\alpha}$, where $c:=K/\big(\prod_{i=1}^{n}(\alpha+i)\big)>0$.

Proof. Application of Corollary 2 of [8] and Corollary 3 of [9]. □

Next comes

Corollary 4 (to Theorem 16). *Assume that*

$$b-a\ge\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}t_{ij}^2\right)-1>0. \tag{66}$$

Then

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y)\le\omega_1\left(f',\Big(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}t_{ij}^2-1\Big)\right)\left\{\Big(\sum_{i,j=1}^{\infty}|t_{ij}-p_iq_j|\Big)+\frac{1}{2}\right\}. \tag{67}$$

Proof. Application of Corollary 3 of [8] and Corollary 4 of [9]. □

Corollary 5 (to Theorem 16). *Assume that*

$$\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| > 0. \tag{68}$$

Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \omega_1 \left(f', \left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) \right) \left\{ \left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) + \frac{b-a}{2} \right\}$$

$$\le \frac{3}{2}(b-a)\omega_1 \left(f', \left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) \right). \tag{69}$$

Proof. Application of Corollary 5 of [8] and Corollary 5 of [9]. □

We present

Proposition 1. *Let* $f \in C([a,b])$.

i) *Assume that*

$$\left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) > 0. \tag{70}$$

Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le 2\omega_1 \left(f, \left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) \right). \tag{71}$$

ii) *Let* $r > 0$ *and*

$$b - a \ge r \left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) > 0. \tag{72}$$

Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le \left(1 + \frac{1}{r} \right) \omega_1 \left(f, r \left(\sum_{i,j=1}^{\infty} |t_{ij} - p_i q_j| \right) \right). \tag{73}$$

Proof. By Propositions 1 of [8] and [9]. □

Also we give

Proposition 2. *Let f be a Lipschitz function of order α, $0 < \alpha \le 1$, i.e. there exists $K > 0$ such that*

$$|f(x) - f(y)| \le K|x-y|^{\alpha}, \quad \textit{all } x, y \in [a,b]. \tag{74}$$

Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \le K \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{1-\alpha} |t_{ij} - p_i q_j|^{\alpha} \right). \tag{75}$$

Proof. By Propositions 2 of [8] and [9]. □

Next we present some alternative type of results. We start with

Theorem 17. *Let* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, *then we get the representation*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \left(\sum_{i,j=1}^{\infty} f^{(k+1)} \left(\frac{t_{ij}}{p_i q_j} \right) (p_i q_j)^{-k} (t_{ij} - p_i q_j)^{k+1} \right) + \psi_2, \tag{76}$$

where

$$\psi_2 = \frac{(-1)^n}{n!}\left(\sum_{i,j=1}^{\infty}(p_iq_j)\left(\int_1^{\left(\frac{t_{ij}}{p_iq_j}\right)}(s-1)^n f^{(n+1)}(s)ds\right)\right). \tag{77}$$

Proof. By Theorem 12 of [8] and Theorem 17 of [9]. □

Next we estimate ψ_2. We give

Theorem 18. *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$. Then*

$$|\psi_2| \le \min\ of \begin{cases} B_1 := \dfrac{\|f^{(n+1)}\|_{\infty,[a,b]}}{(n+1)!}\left(\displaystyle\sum_{i,j=1}^{\infty}(p_iq_j)^{-n}|t_{ij}-p_iq_j|^{n+1}\right), \\ B_2 := \dfrac{\|f^{(n+1)}\|_{1,[a,b]}}{n!}\left(\displaystyle\sum_{i,j=1}^{\infty}(p_iq_j)^{1-n}|t_{ij}-p_iq_j|^{n}\right), \\ \text{and for } \tilde{\alpha},\beta>1\text{: } \dfrac{1}{\tilde{\alpha}}+\dfrac{1}{\beta}=1, \\ B_3 := \dfrac{\|f^{(n+1)}\|_{\tilde{\alpha},[a,b]}}{n!(n\beta+1)^{1/\beta}}\left(\displaystyle\sum_{i,j=1}^{\infty}(p_iq_j)^{1-n-1/\beta}|t_{ij}-p_iq_j|^{n+1/\beta}\right). \end{cases} \tag{78}$$

Also it holds

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y) \le \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left|\sum_{i,j=1}^{\infty} f^{(k+1)}\left(\frac{t_{ij}}{p_iq_j}\right)(p_iq_j)^{-k}(t_{ij}-p_iq_j)^{k+1}\right| + \min(B_1,B_2,B_3). \tag{79}$$

Proof. See Theorem 13 of [8] and Theorem 18 of [9]. □

We have

Corollary 6 (to Theorem 18). *Case of $n=1$. It holds*

$$|\psi_2| \le \min\ of \begin{cases} B_{1,1} := \dfrac{\|f''\|_{\infty,[a,b]}}{2}\left(\displaystyle\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}|t_{ij}-p_iq_j|^{2}\right), \\ B_{2,1} := \|f''\|_{1,[a,b]}\left(\displaystyle\sum_{i,j=1}^{\infty}|t_{ij}-p_iq_j|\right), \\ \text{and for } \tilde{\alpha},\beta>1,\ \dfrac{1}{\tilde{\alpha}}+\dfrac{1}{\beta}=1, \\ B_{3,1} := \dfrac{\|f''\|_{\tilde{\alpha},[a,b]}}{(\beta+1)^{1/\beta}}\left(\displaystyle\sum_{i,j=1}^{\infty}(p_iq_j)^{-1/\beta}|t_{ij}-p_iq_j|^{1+1/\beta}\right). \end{cases} \tag{80}$$

Also it holds

$$\Gamma_f(\mu_{XY},\mu_X\times\mu_Y) \le \left|\sum_{i,j=1}^{\infty} f'\left(\frac{t_{ij}}{p_iq_j}\right)(t_{ij}-p_iq_j)\right| + \min(B_{1,1},B_{2,1},B_{3,1}). \tag{81}$$

Proof. See Corollary 8 of [8] and Corollary 6 of [9]. □

We further present

Theorem 19. *Let* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, *then we get the representation*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \left(\sum_{i,j=1}^{\infty} f^{(k+1)}\left(\frac{t_{ij}}{p_i q_j}\right)(p_i q_j)^{-k}(t_{ij} - p_i q_j)^{k+1} \right) + \frac{(-1)^n}{(n+1)!} f^{(n+1)}(1) \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n}(t_{ij} - p_i q_j)^{n+1} \right) + \psi_4, \quad (82)$$

where

$$\psi_4 = \sum_{i,j=1}^{\infty} p_i q_j \psi_3\left(\frac{t_{ij}}{p_i q_j}\right), \quad (83)$$

with

$$\psi_3(x) := \frac{(-1)^n}{n!} \int_1^x (s-1)^n \left(f^{(n+1)}(s) - f^{(n+1)}(1)\right) ds. \quad (84)$$

Proof. Based on Theorem 14 of [8] and Theorem 19 of [9]. □

Next we present estimations of ψ_4.

Theorem 20. *Let* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, *and*

$$0 < \frac{1}{(n+2)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n-1} |t_{ij} - p_i q_j|^{n+2} \right) \le b - a. \quad (85)$$

Then (82) is again valid and

$$|\psi_4| \le \frac{1}{n!} \omega_1 \left(f^{(n+1)}, \frac{1}{(n+2)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n-1} |t_{ij} - p_i q_j|^{n+2} \right) \right) \cdot \left(1 + \frac{1}{(n+1)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n} |t_{ij} - p_i q_j|^{n+1} \right) \right). \quad (86)$$

Proof. By Theorem 15 of [8] and Theorem 20 of [9]. □

Also we have

Theorem 21. *Let* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, *and* $|f^{(n+1)}(s) - f^{(n+1}(1)|$ *is convex in* s *and*

$$0 < \frac{1}{n!(n+2)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n-1} |t_{ij} - p_i q_j|^{n+2} \right) \le \min(1-a, b-1). \quad (87)$$

Then (82) is again valid and

$$|\psi_4| \le \omega_1 \left(f^{(n+1)}, \frac{1}{n!(n+2)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n-1} |t_{ij} - p_i q_j|^{n+2} \right) \right). \quad (88)$$

The last (88) is attained by

$$\hat{f}(s) := \frac{(s-1)^{n+2}}{(n+2)!}, \quad a \le s \le b \quad (89)$$

when n is even.

Proof. By Theorem 16 of [8] and Theorem 21 of [9]. □

When $n = 1$ we obtain

Corollary 7 (to Theorem 20). *Let $f \in C^2([a,b])$ and*

$$0 < \frac{1}{3}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-2}|t_{ij}-p_iq_j|^3\right) \leq b-a. \tag{90}$$

Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \sum_{i,j=1}^{\infty} f'\left(\frac{t_{ij}}{p_iq_j}\right)(t_{ij}-p_iq_j) - \frac{f''(1)}{2}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}(t_{ij}-p_iq_j)^2\right) + \psi_{4,1}, \tag{91}$$

where

$$\psi_{4,1} := \sum_{i,j=1}^{\infty} p_iq_j\psi_{3,1}\left(\frac{t_{ij}}{p_iq_j}\right), \tag{92}$$

with

$$\psi_{3,1}(x) := -\int_1^x (s-1)(f''(s)-f''(1))ds, \quad x \in [a,b]. \tag{93}$$

Furthermore it holds

$$|\psi_{4,1}| \leq \omega_1\left(f'', \frac{1}{3}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-2}|t_{ij}-p_iq_j|^3\right)\right)\frac{1}{2}\left(\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-1}t_{ij}^2\right)+1\right). \tag{94}$$

Proof. See Corollary 9 of [8] and Corollary 7 of [9]. □

Also we have

Corollary 8 (to Theorem 21). *Let $f \in C^2([a,b])$, and $|f^{(2)}(s) - f^{(2)}(1)|$ is convex in s and*

$$0 < \frac{1}{3}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-2}|t_{ij}-p_iq_j|^3\right) \leq \min(1-a, b-1). \tag{95}$$

Then again (91) is true and

$$|\psi_{4,1}| \leq \omega_1\left(f'', \frac{1}{3}\left(\sum_{i,j=1}^{\infty}(p_iq_j)^{-2}|t_{ij}-p_iq_j|^3\right)\right). \tag{96}$$

Proof. See Corollary 10 of [8] and Corollary 8 of [9]. □

The last main result follows.

Theorem 22. *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$. Additionally assume that*

$$|f^{(n+1)}(x) - f^{(n+1)}(y)| \leq K|x-y|^{\alpha}, \tag{97}$$

$K > 0$, $0 < \alpha \leq 1$, all $x, y \in [a, b]$ with $(x-1)(y-1) \geq 0$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \left(\sum_{i,j=1}^{\infty} f^{(k+1)}\left(\frac{t_{ij}}{p_i q_j}\right)(p_i q_j)^{-k}(t_{ij} - p_i q_j)^{k+1} \right) + \frac{(-1)^n}{(n+1)!} f^{(n+1)}(1) \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n} (t_{ij} - p_i q_j)^{n+1} \right) + \psi_4. \quad (98)$$

Here we find that

$$|\psi_4| \leq \frac{K}{n!(n+\alpha+1)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-n-\alpha} |t_{ij} - p_i q_j|^{n+\alpha+1} \right). \quad (99)$$

Inequality (99) is attained when n is even by

$$f^*(x) = \tilde{c}|x-1|^{n+\alpha+1}, \quad x \in [a, b] \quad (100)$$

where

$$\tilde{c} := \frac{K}{\prod_{j=0}^{n}(n+\alpha+1-j)}. \quad (101)$$

Proof. By Theorem 17 of [8] and Theorem 22 of [9]. □

Finally we give

Corollary 9 (to Theorem 22). *Let $f \in C^2([a, b])$. Additionally assume that*

$$|f''(x) - f''(y)| \leq K|x-y|^{\alpha}, \quad K > 0, \ 0 < \alpha \leq 1, \quad (102)$$

all $x, y \in [a, b]$ with $(x-1)(y-1) \geq 0$. Then

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \sum_{i,j=1}^{\infty} f'\left(\frac{t_{ij}}{p_i q_j}\right)(t_{ij} - p_i q_j) - \frac{f''(1)}{2} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-1}(t_{ij} - p_i q_j)^2 \right) + \psi_{4,1}. \quad (103)$$

Here

$$\psi_{4,1} := \sum_{i,j=1}^{\infty} p_i q_j \psi_{3,1}\left(\frac{t_{ij}}{p_i q_j}\right) \quad (104)$$

with

$$\psi_{3,1}(x) := -\int_1^x (s-1)(f''(s) - f''(1))ds, \quad x \in [a, b]. \quad (105)$$

It holds that

$$|\psi_{4,1}| \leq \frac{K}{(\alpha+2)} \left(\sum_{i,j=1}^{\infty} (p_i q_j)^{-1-\alpha} |t_{ij} - p_i q_j|^{\alpha+2} \right). \quad (106)$$

Proof. See Corollary 11 of [8] and Corollary 9 of [9]. □

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