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On the Dynamics of $x_{n+1} = \dfrac{\alpha + x_{n-2}}{x_{n-1}}$

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Abstract We show that the third order rational difference equation

$$x_{n+1} = \frac{\alpha + x_{n-2}}{x_{n-1}} \quad , \quad n = 0, 1, \ldots$$

where the parameter α and the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers possesses unbounded solutions. In addition we show that every bounded solution converges to a finite limit.

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1 Introduction

We show that the third order rational difference equation

$$x_{n+1} = \frac{\alpha + x_{n-2}}{x_{n-1}}, \; n = 0, 1, \ldots . \tag{1}$$

where the parameter α and the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers possesses unbounded solutions. In addition we show that every bounded solution converges to a finite limit.

After a thorough search in the literature, to the best of our knowledge, the only rational difference equations of higher order that have been investigated and it has been shown that possess unbounded solutions are the following:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n}, \quad n = 0, 1, \ldots \quad . \tag{2}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots \tag{3}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + Dx_{n-2}}, \quad n = 0, 1, \dots \quad . \tag{4}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \dots \ . \tag{5}$$

$$x_{n+1} = \frac{\alpha + x_{n-1} + x_{n-2}}{x_n}, \quad n = 0, 1, \dots \tag{6}$$

$$x_{n+1} = \frac{\alpha + x_{n-2}}{x_n}, \quad n = 0, 1, \dots \ . \tag{7}$$

$$x_{n+1} = \frac{\gamma x_{n-1} + x_{n-2}}{x_n + x_{n-2}}, \quad n = 0, 1, \dots \ . \tag{8}$$

Theorem A (See [12],[13], and [15]). *Assume that $B > 0$. Then the following period-two trichotomy result holds for Eq.(2):*

(a) Every solution of Eq.(2) has a finite limit if and only if

$$\gamma < \beta + A.$$

(b) Every solution of Eq.(2) converges to a (not necessarily prime) period-two solution of Eq.(2) if and only if

$$\gamma = \beta + A.$$

(c) Eq.(2) has unbounded solutions if and only if

$$\gamma > \beta + A.$$

Theorem B (See [1],[9], and [14]). *Assume that $\gamma + \delta + A > 0$. Then the following period-two trichotomy result holds for Eq.(3):*

(a) Every solution of Eq.(3) has a finite limit if and only if

$$\gamma < \delta + A.$$

(b) Every solution of Eq.(3) converges to a (not necessarily prime) period-two solution of Eq.(3) if and only if

$$\gamma = \delta + A.$$

(c) Eq.(3) has unbounded solutions if and only if

$$\gamma > \delta + A.$$

Theorem C (See [10])**.** *Assume that* $\gamma + A + B > 0$. *Then the following period-two trichotomy result holds for Eq.(4):*

(a) Every solution of Eq.(4) has a finite limit if and only if

$$\gamma < A.$$

(b) Every solution of Eq.(4) converges to a (not necessarily prime) period-two solution of Eq.(4) if and only if

$$\gamma = A.$$

(c) Eq.(4) has unbounded solutions if and only if

$$\gamma > A.$$

Theorem D (See [8])**.** *Assume that* $\gamma > \beta + \delta + A$. *Then Eq.(5) has unbounded solutions. More precisely let* k *be a number such that*

$$0 < k < \gamma - \beta - \delta - A$$

and let $\{x_n\}_{n=-2}^{\infty}$ *be any solution of Eq.(5) with*

$$x_{-2}, x_0 \in (0, \gamma - A) \text{ and } x_{-1} > \frac{\alpha + \gamma(\gamma - A)}{k}\ .$$

Then

$$\lim_{n\to\infty} x_{2n+1} = \infty \text{ and } \lim_{n\to\infty} x_{2n} = \frac{\beta\gamma + \delta A}{\gamma - \delta}\ .$$

Theorem E (See [8])**.** *Assume that* $\gamma = \beta + \delta + A$ *and* $\beta + A > 0$. *Then every solution of Eq.(5) converges to a (not necessarily prime) period-two solution of Eq.(5).*

Eq.(6) was investigated in [2] and [11] and it was shown that possesses unbounded solutions. Indeed, it follows from Eq.(6) that

$$x_{n+2} - x_n = \frac{1}{x_{n+1}}\left(x_n - x_{n-2}\right), \quad n \geq 0.$$

Therefore, in this case,

$$x_{2n} = x_0, \quad n \geq 0$$

and so from Eq.(6) we see that

$$x_{2n+1} = \frac{\alpha + x_0}{x_0} + \frac{1}{x_0}x_{2n-1} \to \infty \quad \text{as } n \to \infty.$$

Theorem G (See [3]). *Assume that $\alpha > 0$. Then the following period-two trichotomy result holds for Eq.(7):*

(a) Every solution of Eq.(7) has a finite limit if

$$\alpha \geq 2.$$

(b) Every solution of Eq.(7) converges to a (not necessarily prime) period-two solution of Eq.(7) if and only if

$$\alpha = 1.$$

(c) Eq.(7) has unbounded solutions if

$$\alpha < 1.$$

Theorem H (See [4]). *Assume that $\gamma > 1$. Then Eq.(8) has unbounded solutions.*

Theorem I (See [4]). *Assume that $\gamma = 1$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(8) such that*

$$x_{-2}, x_{-1}, x_0 \in (\frac{1}{2}, \infty) .$$

Then the solution $\{x_n\}_{n=-2}^{\infty}$ of Eq.(8) converges to a period-two solution of Eq.(8).

2 The Main Result

The main results are described in the following theorems:

Theorem 2.1. *Every solution of Eq.(1) is either unbounded or converges to the unique positive equilibrium $\bar{x}$ of Eq,(1).*

Theorem 2.2. *Let x_n, $n = 0, 1...$ be a nontrivial oscillatory solution of Eq.(1) such that*

$$\sup x_n \neq x_i, \ i = 0, 1, 2, 3.$$

Then the solution x_n is unbounded.

The proofs of Theorems 2.1 and 2.2 are long and tedious. A series of lemmas will be used for the proofs of the theorems.

Lemma 2.3. *Every non oscillatory solution $\{x_n\}$ of Eq.(1) converges to the unique positive equilibrium of Eq.(1).*

Proof. Assume that the solution $\{x_n\}$ is above the positive equilibrium $\bar{x}$ of Eq.(1). The case where the solution is below the positive equilibrium is similar and will be omitted.
We will show that the solution which stays above the equilibrium $\bar{x}$ is decreasing. For the sake of contradiction assume that there exists N such that $x_N < x_{N+1}$. Then

$$x_{N+3} = \frac{\alpha + x_N}{x_{N+1}} < \frac{\alpha + x_{N+1}}{x_{N+1}} \leq \bar{x}$$

which is a contradiction. The proof is complete. □

Lemma 2.4. *The positive and the negative semicycle of every oscillatory solution of Eq.(1) contains at least one and at most two terms.*

Proof. The proof will be given in the case of a positive semicycle. In the case of a negative semicycle the proof is similar and will be omitted.
Assume that for some $N > 0$, it holds that $x_{N-1} < \bar{x}$ and $x_N \geq \bar{x}$. In addition assume that $x_{N+1} \geq \bar{x}$. Then

$$x_{N+2} = \frac{\alpha + x_{N-1}}{x_N} < \frac{\alpha + \bar{x}}{\bar{x}} = \bar{x} \ .$$

The proof is complete. □

In view of Lemma 2.4 and without loss of generality, when we refer to an oscillatory solution of Eq.(1), we will assume that the first semicycle of that solution, positive or negative, will contain at least one and at most two terms.

The following two lemmas which we state without proof will also be useful in the sequel.

Lemma 2.5. *Let $\{x_n\}$ be a solution of Eq.(1). Then the following hold:*

$$x_{n+1}x_{n-1} = \alpha + x_{n-2}, n = 0, 1, ... \tag{9}$$

and

$$x_{n+3} - x_{n-1} = \frac{x_n - x_{n-2}}{\alpha + x_{n-2}} x_{n-1} \tag{10}$$

Lemma 2.6. *Let $\{x_n\}$ be a solution of Eq.(1) which is bounded from above. Then the solution is also bounded from below.*

Lemma 2.7. *Let $\{x_n\}$ be a positive oscillatory solution of Eq.(1) which is bounded from above and below. Let l_4, l_5, l_6 be the limits of three consecutive subsequences x_{n_i+4}, x_{n_i+5} and x_{n_i+6} of the solution $\{x_n\}$. These limits cannot all be less than $\bar{x}$. In addition they cannot all be greater or equal to $\bar{x}$, unless they are all equal to $\bar{x}$.*

Proof. **Case 1** $l_4, l_5, l_6 \neq \bar{x}$. Let $0 < \epsilon < \min(|l_4 - \bar{x}|, |l_5 - \bar{x}|, |l_6 - \bar{x}|)$. Then there exists N such that

$$l_4 - \epsilon < x_{n_N+4} < l_4 + \epsilon, \ l_5 - \epsilon < x_{n_N+5} < l_5 + \epsilon, \ l_6 - \epsilon < x_{n_N+6} < l_6 + \epsilon \ .$$

Suppose for the sake of contradiction that either $\max(l_4, l_5, l_6) < \bar{x}$, or $\min(l_4, l_5, l_6) \geq \bar{x}$. It follows that either

$$x_{n_N+4} < \bar{x}, \ x_{n_N+5} < \bar{x}, \ x_{n_N+6} < \bar{x}$$

or

$$x_{n_N+4} > \bar{x}, \ x_{n_N+5} > \bar{x}, \ x_{n_N+6} > \bar{x}$$

which in view of Lemma 2.4 yields a contradiction.
Case 2 $l_4 = \bar{x}$. We will show that $l_5 < \bar{x}$ or $l_6 < \bar{x}$ or $l_5 = l_6 = \bar{x}$. There exist subsequences $\{x_{n_i+k}\}$, $k = 0, 1, 2, 3, 4, 5, 6$ such that

$$\lim_{i\to\infty} x_{n_i+k} = l_k \ .$$

Suppose for the sake of contradiction that $l_5 \geq \bar{x}$ and $l_6 \geq \bar{x}$ and $l_5 \neq \bar{x}$ or $l_6 \neq \bar{x}$. Then it follows that either

$$l_5 = l_6 = \bar{x}$$

or

$$l_i > \bar{x}$$

or

$$l_{i+1} > \bar{x}$$

for $i = 0, 1, 2$, which yields a contradiction.
The cases $l_5 = \bar{x}$, $l_6 = \bar{x}$ are similar. The proof is complete. □

Note that the results of the previous Lemma apply for any three consecutive terms of any positive oscillatory solution of Eq.(1).

Lemma 2.8. *Let $\{x_n\}$ be a solution of Eq.(1) which is bounded from above and below. Let*

$$s = \limsup_{n\to\infty} x_n, \text{ and } i = \liminf_{n\to\infty} x_n.$$

Then it holds that $s = \bar{x}$ if and only if $i = \bar{x}$.

Proof. We will give the proof in the case where $s = \bar{x}$. the case where $i = \bar{x}$ is similar and will be omitted.
There exist subsequences $\{x_{n_i+k}\}$, $k = -3, -2, ...$ of the solution $\{x_n\}$ such that

$$\lim_{i\to\infty} x_{n_i} = i_0 = i \leq i_k = \lim_{i\to\infty} x_{n_i+k} \leq \bar{x} .$$

In addition $\{i_k\}$ is a solution of Eq.(1) and so

$$i_0 = \frac{\alpha + i_{-3}}{i_{-2}} \geq \frac{\alpha + i_0}{\bar{x}}$$

which implies that $i = i_0 = \bar{x}$. The proof is complete. □

Lemma 2.9. *Let $\{x_n\}_{n=0}^{\infty}$ be an oscillatory solution of Eq.(1) for which*

$$x_4 = \max x_i, \; i = 0, 1, ... \tag{11}$$

Then for $j = 1, 2...$ it holds that

$$x_{4j+4k} \geq \bar{x} \geq x_{4j-4k+5}, \; k = 1, 2, ...j \tag{12}$$

$$x_{4j+4} \geq x_{4j+7} \geq x_{4j+3} \geq \bar{x} \tag{13}$$

$$x_{4j+4k+4} \geq x_{4j-4k+7}, \; x_{4j-4k+9} \geq x_{4j+4k+2}, \; x_{4j+4k+1} \geq x_{4j-4k+6}, \; k = 1, 2, ...j \tag{14}$$

$$x_{8j-4k+8} \geq \bar{x} \geq x_{4k+1}, \; k = 1, 2, ...j \tag{15}$$

$$x_{8j+5} \geq x_2, \; x_5 \geq x_{8j+6}, \; x_{8j+8} \geq x_3, \; x_{8j+9} \geq x_6, \; x_8 \geq x_{8j+7} \tag{16}$$

$$x_{4k+4} \geq x_{8j+11-4k}, k = 2, 3, ...2j + 1 \tag{17}$$

$$x_{8j-4k+9} \geq x_{4k+6}, \; k = 1, 2, ...2j \tag{18}$$

$$x_{8j-4k+9} \geq x_{4k+2}, \; k = 1, 2, ...j \tag{19}$$

$$x_{8j+9} \geq x_2, \; x_5 \geq x_{8j+10}, \; x_{8j+13} \geq x_6, \; x_{8j+12} \geq x_3, \; x_8 \geq x_{8j+11} \tag{20}$$

$$x_{4k+8} \geq x_{8j-4k+11}, \; k = 1, 2, ...j \tag{21}$$

$$x_{8j-4k+13} \geq x_{4k+6}, \; k = 1, 2, ...j . \tag{22}$$

Proof. The proof will be by induction. Assume that $j = 1$. In view of (11) we have

$$x_4 \geq x_8$$

and in view of (10) we have $x_3 \geq x_5$. In addition

$$x_7 = \frac{\alpha + x_4}{x_5} \geq \frac{\alpha + x_0}{x_3} = x_1 .$$

Also

$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_7 = \frac{\alpha + x_4}{x_5} \geq \frac{\alpha + x_3}{x_5}$$

and so

$$x_5 \geq x_6.$$

In addition

$$\frac{\alpha + x_1}{x_2} = x_4 \geq x_7 = \frac{\alpha + x_4}{x_5} \geq \frac{\alpha + x_3}{x_5}$$

which implies that

$$x_5 \geq x_2 \ .$$

Hence

$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_2}{x_5} = x_3 \ .$$

Furthermore

$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_{11} = \frac{\alpha + x_8}{x_9} \geq \frac{\alpha + x_3}{x_9}$$

which implies that

$$x_9 \geq x_6$$

and so

$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_6}{x_9} = x_7$$

Furthermore

$$\frac{\alpha + x_1}{x_2} = x_4 \geq x_{11} = \frac{\alpha + x_8}{x_9} \geq \frac{\alpha + x_7}{x_9} \geq \frac{\alpha + x_1}{x_9}$$

which implies that

$$x_9 \geq x_2 \ .$$

Also note that since $x_4 \geq x_3 \geq x_5$ and the solution is oscillatory we have that $x_5 \leq \bar{x}$. In addition since $x_5 \geq x_6$ we get that $x_6 \leq \bar{x}$. Therefore

$$x_7 = \frac{\alpha + x_4}{x_6} \geq \frac{\alpha + \bar{x}}{\bar{x}} = \bar{x}$$

and since $x_8 \geq x_7$ we get that $x_8 \geq \bar{x}$. Therefore (12) holds for $j = 1$. Also

$$x_5 = \frac{\alpha + x_4}{x_7} \geq \frac{\alpha + x_7}{x_8} = x_{10}$$

and

$$x_{12} = \frac{\alpha + x_9}{x_{10}} \geq \frac{\alpha + x_2}{x_5} = x_3 \ .$$

It also holds that

$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_{15} = \frac{\alpha + x_{12}}{x_{13}} \geq \frac{\alpha + x_3}{x_{13}}$$

which implies that $x_{13} \geq x_6$. In addition

$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_{10}}{x_{13}} = x_{11} \ .$$

Furthermore since $x_8 \geq \bar{x} \geq x_6$ and in view of (10) we have $x_{11} \geq x_7$ and so (13) holds for $j = 1$. Since $x_8 \geq x_{11}$, we have

$$\frac{\alpha + x_7}{x_{10}} = x_8 \geq x_{11} = \frac{\alpha + x_8}{x_9} \geq \frac{\alpha + x_7}{x_9}$$

and

$$\frac{\alpha + x_5}{x_6} = x_8 \geq x_{11} = \frac{\alpha + x_8}{x_9} \geq \frac{\alpha + \bar{x}}{x_9} \geq \frac{\alpha + x_5}{x_9}$$

which implies that
$$x_9 \geq x_{10},\ \ x_9 \geq x_6\ .$$
In addition
$$x_{12} = \frac{\alpha + x_9}{x_{10}} \geq \frac{\alpha + x_6}{x_9} = x_7$$
and so (14) and (15)hold for $j = 1$. Also since $x_4 \geq x_{15}$, we have
$$\frac{\alpha + x_1}{x_3} = x_4 \geq x_{15} = \frac{\alpha + x_{12}}{x_{13}} \geq \frac{\alpha + x_7}{x_{13}} \geq \frac{\alpha + x_1}{x_{13}}$$
which implies $x_{13} \geq x_2$. Also
$$x_5 = \frac{\alpha + x_4}{x_7} \geq \frac{\alpha + x_{11}}{x_{12}} = x_{14}$$
and
$$x_{16} = \frac{\alpha + x_{13}}{x_{14}} \geq \frac{\alpha + x_2}{x_5} = x_3\ .$$
Since $x_4 \geq x_{19}$, we have
$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_{19} = \frac{\alpha + x_{16}}{x_{17}} \geq \frac{\alpha + x_3}{x_{17}}$$
which implies $x_{17} \geq x_6$ and
$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_{14}}{x_{17}} = x_{15}$$
and so (16) holds for $j = 1$. Since $x_8 \geq x_{15}$
$$\frac{\alpha + x_7}{x_{10}} = x_8 \geq x_{15} = \frac{\alpha + x_{12}}{x_{13}} \geq \frac{\alpha + x_7}{x_{13}}$$
and
$$\frac{\alpha + x_5}{x_6} = x_8 \geq x_{15} = \frac{\alpha + x_{12}}{x_{13}} \geq \frac{\alpha + \bar{x}}{x_{13}} \geq \frac{\alpha + x_5}{x_{13}}$$
and so
$$x_{13} \geq x_{10},\ \ x_{13} \geq x_6\ .$$
Also
$$x_{12} = \frac{\alpha + x_9}{x_{10}} \geq \frac{\alpha + x_{10}}{x_{13}} = x_{11}$$
and
$$x_9 = \frac{\alpha + x_8}{x_{11}} \geq \frac{\alpha + x_{11}}{x_{12}} = x_{14}$$
$$x_{16} = \frac{\alpha + x_{13}}{x_{14}} \geq \frac{\alpha + x_6}{x_9} = x_7$$
which proves (17), (18)and (19) for $j = 1$. Furthermore since $x_4 \geq x_{19}$ we have
$$\frac{\alpha + x_1}{x_2} = x_4 \geq x_{19} = \frac{\alpha + x_{16}}{x_{17}} \geq \frac{\alpha + x_7}{x_{17}} \geq \frac{\alpha + x_1}{x_{17}}$$
which implies that $x_{17} \geq x_2$. Also
$$x_5 = \frac{\alpha + x_4}{x_7} \geq \frac{\alpha + x_{15}}{x_{16}} = x_{18}$$

and

$$x_{20} = \frac{\alpha + x_{17}}{x_{18}} \geq \frac{\alpha + x_2}{x_5} = x_3 \ .$$

Since $x_4 \geq x_{23}$, we get

$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_{23} = \frac{\alpha + x_{20}}{x_{21}} \geq \frac{\alpha + x_3}{x_{21}}$$

and so $x_{21} \geq x_6$ and

$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_{18}}{x_{21}} = x_{19}$$

which proves (20) for $j = 1$. The fact that $x_8 \geq x_{19}$ implies

$$\frac{\alpha + x_7}{x_{10}} = x_8 \geq x_{19} = \frac{\alpha + x_{16}}{x_{17}} \geq \frac{\alpha + x_7}{x_{17}}$$

and so $x_{17} \geq x_{10}$. Finally

$$x_{12} = \frac{\alpha + x_9}{x_{10}} \geq \frac{\alpha + x_{14}}{x_{17}} = x_{15}$$

which proves (21) and (22) for $j = 1$. The proof of the induction is complete in the case where $j = 1$.

Assume that (12), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22) hold for arbitrary $j > 1$. We will show that all the above hold for $j = j + 1$.

In view of (21) for $k = j$, $x_{4j+8} \geq x_{4j+11}$ and since in view of (17) for $k = j+1$, $x_{4j+8} \geq x_{4j+7}$ in view of (12) and Lemma (2.5) we conclude that $x_{4j+6} \leq \bar{x} \leq x_{4j+8}$. In addition and in view of (10) we get $x_{4j+8} \geq x_{4j+11} \geq x_{4j+7} \geq \bar{x}$ which proves (13) for $j = j + 1$.

In view of (13), $x_{4j+4} \geq x_{4j+3} \geq \bar{x}$ and so in view of Lemma (2.5), $x_{4j+5} \leq \bar{x}$. Also in view of (15), we have $x_{4j+8}, x_{4j+12}, ..., x_{8j+4} \geq \bar{x}$ and $x_9, ..., x_{4j+1} \leq \bar{x}$ which implies that (12) hold for $j = j + 1$ as long as we can show that $x_{8j+8} \geq \bar{x}$.

In view of (21) with $k = j - m$, $m = 0, 1, ...j - 1$ and (17) with $k = j - m + 1$, $m = 0, 1, ...j - 1$ we have

$$x_{4j-4m+8} \geq x_{4j+4m+11}, \ m = 0, 1, ...j - 1 \tag{23}$$

and

$$x_{4j-4m+8} \geq x_{4j+4m+7}, \ m = 0, 1, ...j - 1 \tag{24}$$

respectively. Furthermore it holds that

$$x_{4j+4m+8} \geq \bar{x} \geq x_{4j-4m+5}, \ m = 0, 1, ...j - 1 \ .$$

Hence

$$\frac{\alpha + x_{4j-4m+5}}{x_{4j-4m+6}} = x_{4j-4m+8} \geq x_{4j+4m+11} = \frac{\alpha + x_{4j+4m+8}}{x_{4j+4m+9}} \geq \frac{\alpha + \bar{x}}{x_{4j+4m+9}}$$

$$\geq \frac{\alpha + x_{4j-4m+5}}{x_{4j+4m+9}}$$

which implies that

$$x_{4j+4m+9} \geq x_{4j-4m+6}, \ m = 0, 1, ...j - 1 \ . \tag{25}$$

At this point we claim that

$$x_{4j+4t+8} \geq x_{4j-4t+11} \ t = 0, 1, ...j - 1 \tag{26}$$

and

$$x_{4j-4t+9} \geq x_{4j+4t+10}, \ t = 0, 1, ...j - 1 \ . \tag{27}$$

For $t=0$ the above relations become, $x_{4j+8} \geq x_{4j+11}$ which is true from (21) for $k=j$, and $x_{4j+9} \geq x_{4j+10}$. In view of (17) for $k=j+1$, we have $x_{4j+8} \geq x_{4j+7}$. Therefore

$$x_{4j+9} = \frac{\alpha + x_{4j+8}}{x_{4j+11}} \geq \frac{\alpha + x_{4j+7}}{x_{4j+8}} = x_{4j+10}$$

and so the claim is true for $t=0$. Assume that (26) and (27) hold for arbitrary $0 \leq t \leq j-2$. In view of of (25) and (27) we have

$$x_{4j+4t+12} = x_{4j+4(t+1)+8} = \frac{\alpha + x_{4j+4t+9}}{x_{4j+4t+10}}$$

$$\geq \frac{\alpha + x_{4j-4t+6}}{x_{4j-4t+9}} = x_{4j-4(t+1)+11} = x_{4j-4t+7} \ .$$

Since $1 \leq t+1 \leq j-1$, we replace in (24) $m = t+1$, to get $x_{4j-4t+4} \geq x_{4j+4t+11}$. Furthermore

$$x_{4j-4(t+1)+9} = \frac{\alpha + x_{4j-4t+4}}{x_{4j-4t+7}} \geq \frac{\alpha + x_{4j+4t+11}}{x_{4j+4t+12}} = x_{4j+4(t+1)+10}$$

and so the claim is true.
At this point it holds that for $t = 0, 1, ...j-1$

$$x_{4j+4t+8} \geq x_{4j-4t+11}, \ x_{4j-4t+9} \geq x_{4j+4t+10}, \ x_{4j+4t+9} \geq x_{4j-4t+6} \ .$$

It is also true that for $t = 0, 1, ...j-1$

$$x_{4j+4t+12} = \frac{\alpha + x_{4j+4t+9}}{x_{4j+4t+10}} \geq \frac{\alpha + x_{4j-4t+6}}{x_{4j-4t+9}} = x_{4j-4t+7} \ .$$

Hence

$$x_{4j+4(t+1)+8} \geq x_{4j-4(t+1)+11}, \ x_{4j-4(t+1)+13} \geq x_{4j+4(t+1)+6}, \ x_{4j+4(t+1)+5} \geq x_{4j-4(t+1)+10} \ .$$

Letting $t+1 = k$ in the inequalities above we have

$$x_{4(j+1)+4k+4} \geq x_{4(j+1)-4k+7}, \ x_{4(j+1)-4k+9} \geq x_{4(j+1)+4k+2}, \ x_{4(j+1)+4k+1} \geq x_{4(j+1)-4k+6}$$

which implies that (14) holds for $j = j+1$ and $k = 1, 2, ...j$. To prove that (14) holds for $j = j+1$ we need to show that the latter inequalities hold for $k = j+1$.
From (20), $x_8 \geq x_{8j+11}$ and also holds from the previous inequalities for $k=j$ that $x_{8j+8} \geq x_{11} \geq x_7 \geq \bar{x} \geq x_5$. First note that since $x_{8j+8} \geq \bar{x}$, (12) holds for $j = j+1$. On the other hand

$$\frac{\alpha + x_5}{x_6} = x_8 \geq x_{8j+11} = \frac{\alpha + x_{8j+8}}{x_{8j+9}} \geq \frac{\alpha + \bar{x}}{x_{8j+9}} \geq \frac{\alpha + x_5}{x_{8j+9}}$$

which implies

$$x_{8j+9} \geq x_6 \ .$$

Also

$$x_9 = \frac{\alpha + x_8}{x_{11}} \geq \frac{\alpha + x_{8j+7}}{x_{8j+8}} = x_{8j+10}$$

and

$$x_{8j+12} = \frac{\alpha + x_{8j+9}}{x_{8j+10}} \geq \frac{\alpha + x_6}{x_9} = x_7 \ .$$

and so (14) holds for $k = j + 1$. Also since $x_{8j+12} \geq \bar{x}$ we have that (15) also holds for $j = j + 1$.
Furthermore it holds that $x_4 \geq x_{8j+15}$ and so

$$\frac{\alpha + x_1}{x_2} = x_4 \geq x_{8j+15} = \frac{\alpha + x_{8j+12}}{x_{8j+13}} \geq \frac{\alpha + x_7}{x_{8j+13}} \geq \frac{\alpha + x_1}{x_{8j+13}}$$

and so

$$x_{8j+13} \geq x_2 \ .$$

Also

$$x_5 = \frac{\alpha + x_4}{x_7} \geq \frac{\alpha + x_{8j+11}}{x_{8j+12}} = x_{8j+14}$$

$$x_{8j+16} = \frac{\alpha + x_{8j+13}}{x_{8j+14}} \geq \frac{\alpha + x_2}{x_5} = x_3 \ .$$

In addition it holds that $x_4 \geq x_{8j+19}$ and so

$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_{8j+19} = \frac{\alpha + x_{8j+16}}{x_{8j+17}} \geq \frac{\alpha + x_3}{x_{8j+17}}$$

and so

$$x_{8j+17} \geq x_6$$

and

$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_{8j+14}}{x_{8j+17}} = x_{8j+15}$$

which proves (16) for $j = j + 1$.

In view of (14) with $j = j + 1$, we have for $k = 1, 2, ...j + 1$

$$x_{4j+4k+8} \geq x_{4j-4k+11}, \ x_{4j-4k+13} \geq x_{4j+4k+6}, \ x_{4j+4k+5} \geq x_{4j-4k+10} \ .$$

We replace $k = j - m + 1$, $m = 1, 2, ...j + 1$ in the first inequality above (note that for $k = 0$, the first inequality becomes $x_{4j+8} \geq x_{4j+11}$ which is true) and also $k = j - m + 2$, $m = 1, 2, ...j + 1$ in the middle inequality above to get

$$x_{8j-4m+12} \geq x_{4m+7} \tag{28}$$

and

$$x_{4m+5} \geq x_{8j-4m+14} \ . \tag{29}$$

Furthermore it holds that for $k = 1, 2, ...j + 1$

$$x_{8j-4k+12} \geq \bar{x} \geq x_{4k+5} \ . \tag{30}$$

At this point we claim that for $t = 1, 2, ...j + 1$ we have

$$x_{4t+4} \geq x_{8j-4t+19}, \ x_{8j-4t+17} \geq x_{4t+6}, \ x_{8j-4t+17} \geq x_{4t+2} \ . \tag{31}$$

For $t = 1$ the above inequalities become

$$x_8 \geq x_{8j+15}, \ x_{8j+13} \geq x_{10}, \ x_{8j+13} \geq x_6$$

and in order to prove the claim for $t = 1$ it suffices to show that $x_{8j+13} \geq x_{10}$ and $x_{8j+13} \geq x_6$.

Since $x_8 \geq x_{8j+15}$, we have

$$\frac{\alpha + x_7}{x_{10}} = x_8 \geq x_{8j+15} = \frac{\alpha + x_{8j+12}}{x_{8j+13}} \geq \frac{\alpha + x_7}{x_{8j+13}}$$

$$\frac{\alpha + x_5}{x_6} = x_8 \geq x_{8j+15} = \frac{\alpha + x_{8j+12}}{x_{8j+13}} \geq \frac{\alpha + x_7}{x_{8j+13}} \geq \frac{\alpha + x_5}{x_{8j+13}}$$

and so

$$x_{8j+13} \geq x_{10}, \ x_{8j+13} \geq x_6 \ .$$

Assume that the claim is true for arbitrary $1 \leq t \leq j$. We will show that the claim is true for $t = t+1$. First note that in view of (29) and (31) we have

$$x_{4t+8} = x_{4(t+1)+4} = \frac{\alpha + x_{4t+5}}{x_{4t+6}} \geq \frac{\alpha + x_{8j-4t+14}}{x_{8j-4t+17}} = x_{8j-4(t+1)+19} = x_{8j-4+15}$$

which in view of (28) and (30) implies

$$\frac{\alpha + x_{4t+7}}{x_{4(t+1)+6}} = x_{4t+8} \geq x_{8j-4t+15} = \frac{\alpha + x_{8j-4t+12}}{x_{8j-4t+13}} \geq \frac{\alpha + x_{4t+7}}{x_{8j-4(t+1)+17}}$$

and

$$\frac{\alpha + x_{4t+5}}{x_{4(t+1)+2}} = x_{4t+8} \geq x_{8j-4t+15} = \frac{\alpha + x_{8j-4t+12}}{x_{8j-4t+13}} \geq \frac{\alpha + \bar{x}}{x_{8j-4(t+1)+17}} \geq \frac{\alpha + x_{4t+5}}{x_{8j-4(t+1)+17}}$$

and so $x_{8j-4(t+1)+17} \geq x_{4(t+1)+6}$ and $x_{8j-4(t+1)+17} \geq x_{4(t+1)+2}$. The claim is true.
in view of (31), (19) holds for $j = j+1$.
Also in view of (31) we have that (17) holds for $j = j+1$ and for $k = 2, 3, ..., j+1$, while (18) holds for $j = j+1$ and $k = 1, 2, ..., j+1$.
To prove that (17) and (18) hold for $j = j+1$ we need to show that they hold for $j = j+1$ and $k = j+2, j+3, ...2j+3$, $k = j+2, j+3, ...2j+2$ respectively.
In view of (21) and (22 with $k = j - m$, $m = 0, 1, ...j-1$ we get

$$x_{4j-4m+8} \geq x_{4j+4m+11}, \ x_{4j+4m+13} \geq x_{4j-4m+6} \ m = 0, 1, ...j-1 \ . \tag{32}$$

We claim that for $t = 0, 1, ...j-1$, we have

$$x_{4j-4t+9} \geq x_{4j+4t+14}, \ x_{4j+4t+16} \geq x_{4j-4t+7} \ . \tag{33}$$

For $t = 0$, the above inequalities become $x_{4j+9} \geq x_{4j+14}$ and $x_{4j+16} \geq x_{4j+7}$ respectively.
From (14) with $j = j+1$, we have $x_{4j+9} \geq x_{4j+10}$. Also from (18) with $j = j+1$ and $k = j+1$ we have $x_{4j+13} \geq x_{4j+10}$. Therefore

$$x_{4j+12} = \frac{\alpha + x_{4j+9}}{x_{4j+10}} \geq \frac{\alpha + x_{4j+10}}{x_{4j+13}} = x_{4j+11}$$

In addition from (21) with $j = j$ and with $k = j$, we get $x_{4j+8} \geq x_{4j+11}$, and from (19) for $j = j+1$ and $k = j$, we have $x_{4j+13} \geq x_{4j+6}$. Hence

$$x_{4j+9} = \frac{\alpha + x_{4j+8}}{x_{4j+11}} \geq \frac{\alpha + x_{4j+11}}{x_{4j+12}} = x_{4j+14}$$

$$x_{4j+16} = \frac{\alpha + x_{4j+13}}{x_{4j+14}} \geq \frac{\alpha + x_{4j+6}}{x_{4j+9}} = x_{4j+7}$$

and so the claim is true for $t = 0$.
Assume that the claim is true for arbitrary , $0 \leq t \leq j-2$. Then in view of (32) with $m = t+1$, we have

$$x_{4j-4t+4} \geq x_{4j+4t+15},\ x_{4j+4t+17} \geq x_{4j-4t+2}\ . \tag{34}$$

In view of (32) and (34) we have

$$x_{4j-4(t+1)+9} = \frac{\alpha + x_{4j-4t+4}}{x_{4j-4t+7}} \geq \frac{\alpha + x_{4j+4t+15}}{x_{4j+4t+16}} = x_{4j+4(t+1)+14}$$

and

$$x_{4j+4(t+1)+16} = \frac{\alpha + x_{4j+4t+17}}{x_{4j+4t+18}} \geq \frac{\alpha + x_{4j-4t+2}}{x_{4j-4t+5}} = x_{4j-4(t+1)+7}$$

and so the claim is true.
In view of (33) we have

$$x_{8(j+1)-4(j+t+2)+9} \geq x_{4(j+t+2)+6},\ x_{4(j+t+3)+4} \geq x_{8(j+1)-4(j+t+3)+11}\ .$$

We substitute $k = j+t+2$ and $k = j+t+3$ respectively to get

$$x_{8(j+1)-4k+9} \geq x_{4k+6},\ k = j+2, j+3, ...2j+1 \tag{35}$$

and

$$x_{4k+4} \geq x_{8(j+1)-4k+11},\ k = j+3, j+3, ...2j+2 \tag{36}$$

respectively. Note that (36) with $k = j+2$ becomes $x_{4j+12} \geq x_{4j+11}$ which we have previously shown that it is true. Therefore in view of (35) and (36), (17) holds for $j = j+1$ and $k = 2, 3, ...2j+2$, while (18) holds for $j = j+1$ and $k = 1, 2, ...2j+1$. To show that (17) and (18) hold for $j = j+1$ we need to show that they hold for $j = j+1$ and $k = 2j+3$, $k = 2j+2$ respectively.
In addition from (21) with $j = j$ and $k = 1$, we have $x_{12} \geq x_{8j+7}$. Also from (36) with $k = 2j+1$, we have $x_{8j+8} \geq x_{15}$. Therefore

$$x_{13} = \frac{\alpha + x_{12}}{x_{15}} \geq \frac{\alpha + x_{8j+7}}{x_{8j+8}} = x_{8j+10}\ .$$

In view of (22) with $j = j$ and $k = 1$, we have $x_{8j+9} \geq x_{10}$. Hence

$$x_{8j+12} = \frac{\alpha + x_{8j+9}}{x_{8j+10}} \geq \frac{\alpha + x_{10}}{x_{13}} = x_{11}\ .$$

Also since $x_8 \geq x_{8j+11}$, we get

$$\frac{\alpha + x_{8j+11}}{x_{8j+14}} = x_{8j+12} \geq x_{11} \frac{\alpha + x_8}{x_9} \geq \frac{\alpha + x_{8j+11}}{x_9}$$

which implies $x_9 \geq x_{8j+14}$. Also

$$x_{8j+16} = \frac{\alpha + x_{8j+13}}{x_{8j+14}} \geq \frac{\alpha + x_6}{x_9} = x_7\ .$$

and so (17), (18) hold for $j = j+1$.
In addition it holds that $x_4 \geq x_{4j+19}$ and so

$$\frac{\alpha + x_1}{x_2} = x_4 \geq x_{4j+19} = \frac{\alpha + x_{8j+16}}{x_{8j+17}} \geq \frac{\alpha + x_7}{x_{8j+17}} \frac{\alpha + x_1}{x_{8j+17}}$$

and so

$$x_{8j+17} \geq x_2 \ .$$

Also

$$x_5 = \frac{\alpha + x_4}{x_7} \geq \frac{\alpha + x_{8j+15}}{x_{8j+16}} = x_{8j+18}$$

and

$$x_{8j+20} = \frac{\alpha + x_{8j+17}}{x_{8j+18}} \geq \frac{\alpha + x_2}{x_5} = x_3 \ .$$

In addition it holds that $x_4 \geq x_{8j+23}$ and so

$$\frac{\alpha + x_3}{x_6} = x_4 \geq x_{8j+23} = \frac{\alpha + x_{8j+20}}{x_{8j+21}} \geq \frac{\alpha + x_3}{x_{8j+21}}$$

and so

$$x_{8j+21} \geq x_6$$

which implies

$$x_8 = \frac{\alpha + x_5}{x_6} \geq \frac{\alpha + x_{8j+18}}{x_{8j+21}} = x_{8j+19}$$

and so (20) holds for $j = j + 1$.
To finish the proof we need to show that (21) and (22) hold for $j = j + 1$.

In view of (17) with $j = j + 1$, we have for $k = 2, 3, ...2j + 3$

$$x_{4k+4} \geq x_{8j-4k+19} \ .$$

Since $x_4 \geq x_{8j+23}$, $x_8 \geq x_{8j+19}$ the above inequality also holds for $k = 0, 1, ...2j + 3$. Let $k = 2j - m + 3$, $m = 0, 1, ...2j + 3$ in the inequality above to get

$$x_{8j-4m+16} \geq x_{4m+7}, \ m = 0, 1, ...2j + 3 \ . \tag{37}$$

In view of (18) with $j = j + 1$, we have for $k = 1, 2, ...2j + 2$

$$x_{8j-4k+17} \geq x_{4k+6} \ .$$

Let $k = 2j - m + 2$, $m = 0, 1, ...2j + 1$ in the inequality above and we get

$$x_{4m+9} \geq x_{8j-4m+14}, \ m = 0, 2, ...2j + 1 \ . \tag{38}$$

At this point we claim that (21) and (22) hold for $j = j + 1$. In other words we claim that

$$x_{4k+8} \geq x_{8j-4k+19}, \ x_{8j-4k+21} \geq x_{4k+6}, \ k = 1, 2, ...j + 1 \ .$$

For $k = 1$, the inequalities above become $x_{8j+17} \geq x_{10}$ and $x_{12} \geq x_{8j+15}$.
It holds that $x_8 \geq x_{8j+19}$. Also in view of (17) with $j = j + 1$, $k = 2j + 3$ and (18) with $j = j + 1$, $k = 2j + 2$, we have $x_{8j+16} \geq x_7$ and $x_9 \geq x_{8j+14}$. Therefore

$$\frac{\alpha + x_7}{x_{10}} = x_8 \geq x_{8j+19} = \frac{\alpha + x_{8j+16}}{x_{8j+17}} \geq \frac{\alpha + x_7}{x_{8j+17}}$$

which implies

$$x_{8j+17} \geq x_{10}$$

and

$$x_{12} = \frac{\alpha + x_9}{x_{10}} \geq \frac{\alpha + x_{8j+14}}{x_{8j+17}} = x_{8j+15} \ .$$

Assume that the claim is true for arbitrary k, $1 \leq k \leq j$. Then

$$x_{4k+8} \geq x_{8j-4k+19},\ x_{8j-4k+21} \geq x_{4k+6}\ .$$

In view of (37) It holds that

$$\frac{\alpha + x_{4k+7}}{x_{4(k+1)+6}} = x_{4k+8} \geq x_{8j-4k+19} = \frac{\alpha + x_{8j-4k+16}}{x_{8j-4k+17}} \geq \frac{\alpha + x_{4k+7}}{x_{8j-4(k+1)+21}}$$

which implies that $x_{8j-4(k+1)+21} \geq x_{4(k+1)+6}$. Also in view of (38) we have

$$x_{4(k+1)+8} \geq \frac{\alpha + x_{4k+9}}{x_{4k+10}} \geq \frac{\alpha + x_{8j-4k+14}}{x_{8j-4k+17}} = x_{8j-4(k+1)+15}\ .$$

The proof is complete.

□

Lemma 2.10. *Let $\{x_n\}_{n=0}^{\infty}$ be an oscillatory solution of Eq.(1) for which (11) holds. Then for $j \geq 1$*

$$x_{4j+4} \geq x_{4j+8} \geq x_{4j+7} \geq x_{4j+3} \geq \bar{x} \geq x_{4j+5} \geq x_{4j+9} \geq x_{4j+6} \geq x_{4j+2}\ . \tag{39}$$

Proof. In view of (13), it holds for $j = 1, 2, ...$

$$x_{4j+4} \geq x_{4j+7} \geq x_{4j+3} \geq \bar{x}\ .$$

Also in view of (15) it holds

$$x_{4j+1} \leq \bar{x},\ j = 1, 2, \ldots\ .$$

Therefore

$$x_{4j+5} \leq \bar{x},\ j = 1, 2, \ldots\ .$$

Also in view of (17) it holds

$$x_{4j+9} \geq x_{4j+6},\ j = 1, 2, \ldots\ .$$

Also since $x_{4j+6} \leq \bar{x} \leq x_{4j+4}$, $j = 1, 2, ...$ in view of (10) we get that $x_{4j+9} \leq x_{4j+5}$ for $j = 1, 2, ...$ In addition since $x_{4j+3} \geq \bar{x} \geq x_{4j+1}$ for $j = 1, 2, ...$, in view of (10) it follows that $x_{4j+6} \geq x_{4j+2}$ for $j = 1, 2, ...$ Furthermore it holds that

$$x_{4j+5} \geq x_{4j+9} \geq x_{4j+6},\ j = 1, 2, ...$$

and since $x_{4j+3} \geq \bar{x} \geq x_{4j+5}$, $j = 1, 2, ...$ in view of (10), it holds that

$$x_{4j+4} \geq x_{4j+8},\ j = 1, 2, \ldots\ .$$

Finally

$$x_{4j+8} = \frac{\alpha + x_{4j+5}}{x_{4j+6}} \geq \frac{\alpha + x_{4j+6}}{x_{4j+9}} = x_{4j+7}\ .$$

The proof is complete. □

Lemma 2.11. *Let $\{x_n\}_{n=0}^{\infty}$ be an oscillatory solution of Eq.(1) for which there exists $N \geq 4$ such that*

$$x_N = \max x_i,\ i = 0, 1, \ldots\ .$$

Then $x_n = \bar{x}$ for all n.

Proof. Suppose that $x_N \geq x_n$ for all n. In view of Lemma 2.10 the solution contains four monotonic subsequences as described in (39) and so it converges to a periodic solution of period two, three or four. This is a contradiction unless the solution is trivial. □

At this point we will give the proof of Theorem 2.1

Proof. Assume that the solution is not unbounded and consider the limiting sequence $\{l_i\}$, $i = 0, 1, ...$ where

$$l_i = \lim_{j\to\infty} x_{n_j+i},\ i = 0, 1, ...$$

and $l_4 = \limsup_{n\to\infty} x_n$. Then in view of Lemma 2.7, the limiting sequence $\{l_i\}$, $i = 0, 1, ...$ is oscillatory and also in view of Lemma 2.11, $\limsup_{n\to\infty} x_n = \bar{x}$. In addition in view of Lemma 2.8, $\liminf_{n\to\infty} x_n = \bar{x}$. The proof is complete. □

Lemma 2.12. *Let* $\{x_n\}_{n=0}^{\infty}$ *be an oscillatory solution of Eq.(1) which is bounded. Then there exists,* $0 \leq N < 4$ *such that* $x_N = \sup x_n$.

Proof. Assume for the sake of contradiction that $\sup x_n > x_n$ for all n. Since x_n is a positive bounded oscillatory solution we have that $\infty > \sup x_n > \bar{x}$. In addition there exists a subsequence x_{n_j} such that $\lim_{j\to\infty} x_{n_j} = \sup x_n > \bar{x}$ which is a contradiction. □

At this point we will give the proof of Theorem 2.2.

Proof. Suppose that the solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(1) is bounded. Then in view of Lemma 2.12 there exists $0 \leq N < 4$ such that $\sup x_n = x_N$ which is a contradiction and the proof is complete. □

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