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Statistical Properties of a Correlated Bimodal Field Interacting with an Effective Two-Level System

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Abstract In this paper we consider a two-mode field interacting with a two-level atom. The initial field state is taken to be a correlated pair-coherent state. Some statistical properties such as the atomic inversion, the photon number operator, the squeezing phenomenon, the Glauber second order correlation function are studied. Also the Phase distribution is considered.

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1 Introduction

The Jaynes-Cummings model [1] for the interaction of a single two-level atom and the radiation field [1], however simple, is very rich with quantum features.Despite being simple enough to be analytically soluble in the rotating wave approximation (RWA), this model has been a source of insight into the nuances of the interaction between light and matter. It has been employed to discuss many different phenomena for example the collapse-revival phenomenon [2], sub-Poissonian statistics, antibunching [3], squeezing [4], chaos and trapping states [5,6]. In fact the complex dynamical evolution and the fully quantum nature of the model have turned it into a laboratory for theorists and the basis of many more elaborate models. A number of generalized JCM have been investigated. Generalizations include for example multimode as well as multiphoton processes [7]. Multiphoton process is certainly

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one of the most important among the many laser spectroscopic techniques that have revolutionized the optical spectroscopy field. It allows one to probe excited states that cannot be reached by one photon transitions. An interesting aspect of the multiphoton spectroscopy applied to gas systems is its ability to yield Doppler free spectra [8]. Furthermore multiphoton absorption processes are of great practical importance. Discussion of multimode field effects have been considered [9]. Stark shift contribution to the JCM has been taken into account [10] as well as the non-linear field effects (e.g Kerr like medium) on some statistical aspects have been considered in some state representation [11].

In this paper we consider one of the generalized JCM where the non-linear interaction of two modes of the radiation field with a two-level atom is presented and the multiphoton process is considered. Some statistical quantities are computed for the correlated coherent state representation of the field to discuss the revival and collapse phenomenon in the evolution of the atomic inversion and the photon number operator is obtained when the atom is prepared in its excited state. Also the squeezing, second order correlation function and the Phase distribution are discussed.

2 Model and solution

In the following we consider a system that consists of a two-mode field interacting with a two-level atom. We consider the non-degenerate case, in which photons with two different frequencies are involved in this interaction. The Hamiltonian that describes such a system in the rotating wave approximation can be written as ($\hbar = 1$),

$$\hat{H} = \Omega_1 \hat{S}_{11} + \Omega_2 \hat{S}_{22} + \omega_1 \hat{n}_1 + \omega_2 \hat{n}_2 + \lambda(\hat{S}_{12} \hat{a}_1^{k_1} \hat{a}_2^{\dagger k_2} + \hat{a}_1^{\dagger k_1} \hat{a}_2^{k_2} \hat{S}_{21}) \tag{1}$$

where ω_j and Ω_j are the field frequencies and the atomic energies respectively where $\Omega_1 > \Omega_2$, while $\hat{a}_j$ and $\hat{a}_j^{\dagger}$, are annihilation and creation operators for the $j^{\underline{th}}$ mode of the cavity field and satisfy the following commutation relation $[\hat{a}_i, \hat{a}_j^{\dagger}] = \delta_{ij}$, $\hat{n}_i = \hat{a}_i^{\dagger} \hat{a}_i$ the photon number operator of the $i^{\underline{th}}$ mode and the operators $\hat{S}_{ij}$ are the generators of the group $U(2)$ they satisfy the following commutation relation [12].

$$[\hat{S}_{ij}, \hat{S}_{lm}] = \hat{S}_{im} \delta_{jl} - \hat{S}_{lj} \delta_{mi} \tag{2}$$

and λ is the effective coupling constant. The Hamiltonian (1) can be used to discuss hyper Raman processes [9], or trapped ions with the center of mass of the particle has to move in two dimensions [10].

The equations of motion for the $\hat{n}_j$ and $\hat{S}_{jj}$ operators are given by

$$\begin{aligned} i\dot{\hat{n}}_1 &= \lambda k_1(-\hat{S}_{12}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2} + \hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}\hat{S}_{21}) \\ i\dot{\hat{n}}_2 &= \lambda k_2(\hat{S}_{12}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2} - \hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}\hat{S}_{21}) \\ i\dot{\hat{S}}_{11} &= \lambda(\hat{S}_{12}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2} - \hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}\hat{S}_{21}) \\ i\dot{\hat{S}}_{22} &= \lambda(-\hat{S}_{12}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2} + \hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}\hat{S}_{21}) \end{aligned}$$

From these equations we can show that the following operators $\hat{N}_1$ and $\hat{N}_2$ defined by

$$\begin{aligned} \hat{N}_1 &= \hat{n}_1 + \frac{k_1}{2}(\hat{S}_{11} - \hat{S}_{22}), \\ \hat{N}_2 &= \hat{n}_2 - \frac{k_2}{2}(\hat{S}_{11} - \hat{S}_{22}) \end{aligned} \qquad (3)$$

are constants of motion.

Thus the Hamiltonian (1) can be cast in the following form

$$\hat{H} = \frac{1}{2}(\Omega_1 + \Omega_2)\hat{I} + \hat{N} + \hat{C} \qquad (4)$$

where $\hat{I}$ is the identity operator while $\hat{N}$ and $\hat{C}$ take the form

$$\hat{N} = \sum_{i=1}^{2} \omega_i \hat{N}_i \qquad (5)$$

and

$$\hat{C} = \frac{\Delta}{2}(\hat{S}_{11} - \hat{S}_{22}) + \lambda(\hat{S}_{12}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2} + \hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}\hat{S}_{21}) \qquad (6)$$

The quantity Δ is the detuning parameter defined by

$$\Delta = \Omega_1 - \Omega_2 + k_2\omega_2 - k_1\omega_1 \qquad (7)$$

It is easy to show that the operators $\hat{N}$ and $\hat{C}$ commute, and hence each of them commute with $\hat{H}$. This means that the operators $\hat{N}$ and $\hat{C}$ are constants of motion. thus the time evolution operator $U(t)$ given by

$$U(t) = \exp[-i\hat{H}t], \qquad (8)$$

can be written in the form

$$U(t) = \exp[-\frac{i}{2}(\Omega_1+\Omega_2)t]\exp[-i\hat{N}t]\exp[-i\hat{C}t] \tag{9}$$

where

$$\exp[-i\hat{N}t] = \begin{bmatrix} \exp[-i\hat{Z}_1(\hat{n}_1,\hat{n}_2)t] & 0 \\ 0 & \exp[-i\hat{Z}_2(\hat{n}_1,\hat{n}_2)t] \end{bmatrix} \tag{10}$$

with

$$\begin{aligned} \hat{Z}_1(\hat{n}_1,\hat{n}_2) &= \omega_1(\hat{n}_1+\frac{k_1}{2})+\omega_2(\hat{n}_2-\frac{k_2}{2}) \\ \hat{Z}_2(\hat{n}_1,\hat{n}_2) &= \hat{Z}_1(\hat{n}_1-k_1,\hat{n}_2+k_2) \end{aligned}$$

and

$$\exp[-i\hat{C}t] = \begin{bmatrix} \left(\cos\hat{\mu}_1 t - \frac{i\Delta}{2\mu_1}\sin\hat{\mu}_1 t\right) & -i\lambda\frac{\sin\hat{\mu}_1 t}{\mu_1}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2} \\ -i\lambda\hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}\frac{\sin\hat{\mu}_1 t}{\mu_1} & \left(\cos\hat{\mu}_2 t + \frac{i\Delta}{2\mu_2}\sin\hat{\mu}_2 t\right) \end{bmatrix} \tag{11}$$

with

$$\begin{aligned} \mu_j^2(\hat{n}_1,\hat{n}_2) &= \frac{\Delta^2}{4}+\nu_j, \quad j=1,2 \\ \hat{\nu}_1(\hat{n}_1,\hat{n}_2) &= \lambda^2\frac{(\hat{n}_1+k_1)!}{(\hat{n}_1)!}\frac{(\hat{n}_2)!}{(\hat{n}_2-k_2)!}, \\ \hat{\nu}_2(\hat{n}_1,\hat{n}_2) &= \lambda^2\frac{(\hat{n}_1)!}{(\hat{n}_1-k_1)!}\frac{(\hat{n}_2+k_2)!}{(\hat{n}_2)!} \end{aligned}$$

.

Where μ_j are the generalized Rabi frequencies in this case.

Now let us consider the atomic coherent state $|\theta,\phi\rangle$ which acquires both excited state $|e\rangle$ and ground state $|g\rangle$ for the two-level atoms in the following form

$$|\theta,\phi\rangle = \cos(\theta/2)|e\rangle+\sin(\theta/2)\exp(-i\phi)|g\rangle, \tag{12}$$

where θ is the coherence angle and ϕ is the relative phase of the two atomic levels. To reach the excited state we have to take $\theta\to 0$ while to make the wave function describes the particle in the ground state we have to let $\theta\to\pi$. If the initial state of the two-mode optical field is prepared in the pair-coherent state $|\zeta,q\rangle$ defined as the eigenstate of pair annihilation operators $\hat{a}_1\hat{a}_2$ for the two modes, and the photon number difference between the two modes i.e.[13]

$$\begin{aligned} \hat{a}_1\hat{a}_2|\zeta,q\rangle &= \zeta|\zeta,q\rangle \\ (\hat{a}_1^{\dagger}\hat{a}_1-\hat{a}_2^{\dagger}\hat{a}_2)|\zeta,q\rangle &= q|\zeta,q\rangle \end{aligned} \tag{13}$$

then, it can be seen that the pair-coherent state takes the form,

$$|\zeta,q\rangle = N_q\sum_{n=0}^{\infty}\frac{\zeta^n}{\sqrt{n!(n+q)!}}|n+q,n\rangle \tag{14}$$

with

$$N_q = \frac{1}{\sqrt{\sum_{n=0}^{\infty}\frac{|\zeta|^{2n}}{n!(n+q)!}}} = [|\zeta|^{-q}I_q(2|\zeta|)]^{-\frac{1}{2}}, \tag{15}$$

where N_q is the normalization constant (I_q is the modified Bessel function of order q). Assuming that at time $t=0$ the bimodal field-particle system is in the state defined by the wave function $|\psi(0)\rangle = |\theta,\phi\rangle\otimes|\zeta,q\rangle$. Therefore, after some calculations the wave function for the system at any time $t>0$ takes the form

$$\begin{aligned} |\psi(t)\rangle &= \Big\{\exp\{-i\hat{Z}_1(\hat{n}_1,\hat{n}_2)t\}\left(\cos\hat{\mu}_1 t-\frac{i\Delta}{2\mu_1}\sin\hat{\mu}_1 t\right)cos\frac{\theta}{2}-i\lambda\exp\{-i\hat{Z}_1(\hat{n}_1,\hat{n}_2)t\}\frac{\sin\hat{\mu}_1 t}{\mu_1} \\ &\quad \hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2}\exp\{-i\phi\}sin\frac{\theta}{2}\Big\}|\zeta,q,e\rangle+\Big\{\exp\{-i\hat{Z}_2(\hat{n}_1,\hat{n}_2)t\}\left(\cos\hat{\mu}_2 t+\frac{i\Delta}{2\mu_2}\sin\hat{\mu}_2 t\right) \\ &\quad \times\exp\{-i\phi\}sin\frac{\theta}{2}-i\lambda\exp\{-i\hat{Z}_2(\hat{n}_1,\hat{n}_2)t\}\frac{\sin\hat{\mu}_2 t}{\mu_2}\hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}cos\frac{\theta}{2}\Big\}|\zeta,q,g\rangle, \end{aligned} \tag{16}$$

(for briefly we will drop the arguments $(\hat{n}_1,\hat{n}_2)$ in $\hat{Z}_i$)

The reduced density matrix for the field is given by $\rho_f\,(t) = Tr_{atom}|\psi(t)\rangle\langle\psi(t)|$, such that

$$\rho_f\,(t) = |D(t)\rangle\langle D(t)| + |T(t)\rangle\langle T(t)|, \tag{17}$$

where

$$\begin{aligned} |D(t)\rangle &= \Big\{\exp\{-i\hat{Z}_1)t\}\left(\cos\hat{\mu}_1(\hat{n}_1,\hat{n}_2)t-\frac{i\Delta}{2\mu_1(\hat{n}_1,\hat{n}_2)}\sin\hat{\mu}_1(\hat{n}_1,\hat{n}_2)t\right)cos\frac{\theta}{2} \\ &\quad -i\lambda\exp\{-i\hat{Z}_1 t\}\frac{\sin\hat{\mu}_1(\hat{n}_1,\hat{n}_2)t}{\mu_1(\hat{n}_1,\hat{n}_2)}\hat{a}_1^{k_1}\hat{a}_2^{\dagger k_2}\exp\{-i\phi\}sin\frac{\theta}{2}\Big\}|\zeta,q\rangle, \end{aligned} \tag{18}$$

$$\begin{aligned} |T(t)\rangle &= \Big\{\exp\{-i\hat{Z}_2 t\}\left(\cos\hat{\mu}_2(\hat{n}_1,\hat{n}_2)t+\frac{i\Delta}{2\mu_2(\hat{n}_1,\hat{n}_2)}\sin\hat{\mu}_2(\hat{n}_1,\hat{n}_2)t\right)\exp\{-i\phi\}sin\frac{\theta}{2} \\ &\quad -i\lambda\exp\{-i\hat{Z}_2 t\}\frac{\sin\hat{\mu}_2(\hat{n}_1,\hat{n}_2)t}{\mu_2(\hat{n}_1,\hat{n}_2)}\hat{a}_1^{\dagger k_1}\hat{a}_2^{k_2}cos\frac{\theta}{2}\Big\}|\zeta,q\rangle. \end{aligned} \tag{19}$$

By employing the reduced field density operator given by equation (17) and the wave function $|\psi(t)\rangle$ of equation (16), we shall investigate some statistical aspects of the present system. This will be done in the next sections.

3 Atomic inversion

In this section we discuss the revival-collapse phenomenon in the evolution of the atomic population inversion. This phenomenon is a purely quantum mechanical effect and has its origin in the regular structure of the photon number distribution of the initial field, and can be considered as the simplest important quantity in the JCM. It is defined as the difference between the probabilities of finding the atom in the exited state and in the ground state. When we assume that the atom initially starts from the excited state i.e. ($\theta = 0$) and the field state is given by eqn. (14). The atomic inversion is given by,

$$W(t) = \frac{|N_q|^2}{2} \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{n!(n+q)!} \left\{ |F_1(n,t)|^2 - [E_1(n,t)]^2 \nu_1(n+q,n) \right\}, \tag{20}$$

with

$$\begin{aligned} F_1(n,t) &= \left(\cos \mu_1(n+q,n)t - \frac{i\Delta}{2\mu_1(n+q,n)} \sin \mu_1(n+q,n)t \right), \\ E_1(n,t) &= \frac{\sin \mu_1(n+q,n)t}{\mu_1(n+q,n)} \end{aligned} \tag{21}$$

In the numerical investigations we examine the influence of the correlated coherent states $|\zeta, q\rangle$ on the atomic inversion by plotting $W(t)$ of eqn. (20) against time $\tau = \lambda t$ taking into consideration the atom initially in the excited state ($\theta = 0$). We have taken the correlated pair-coherent parameters $\zeta = 9$ and $q = 3$. First we look at the two-photon case ($k_1 = k_2 = 1$), in fig.(1a) we find in the absence of the detuning parameter (resonance case $\Delta = 0$) that the function fluctuates around $W(t) = 0$, and the collapses and revivals phenomenon starts to appear. It is periodical with period (2π). It reflects the same behavior as the case of one mode and two-photon JCM with the field prepared in a coherent state. It is interesting to see here the periodic oscillations of the atomic inversion and the atom flipping from the upper to lower state occurring periodically. Since the revival times can be estimated [2,14-16], therefore the revival times for a coherent state can be written as $t_R = \pi$. Similarly, we show the off-resonance case when ($\Delta = 20$). We can see an increase in the oscillations. However the collapses for the same parameters start to disappear after the 3rd collapse. It is to be noted that the function values are shifted upwards, see fig.(1b). The case of the three-photon process ($k_1 = 2,\ k_2 = 1$), is considered next in absence of the detuning (exact resonance case see fig. 1c). We note that the first collapse appears after a short time and the function $W(t)$ shows a revival followed by another collapse for a short period the picture is almost like a standard JCM, for a one-photon coherent state but the revival time is far more shorter. However as time goes on the revivals interfere and the collapses almost disappear during the course of the interaction, see figs.(1c). When the detuning parameter takes place ($\Delta = 20$) we can see an increase in the oscillations and the first two collapses and revivals are distinctly apparent, and interference sets up earlier than in the exact resonance case. However the function value is shifted upwards. It is to be remarked that energy is stored in the atomic system in the case of detuning more than in the case of resonance (compare figs. 1c and 1d, as well as 1a and 1b).

4 Photon number

In this section we discuss the evolution of the photon number for the present system. When the atom starts from its excited state ($\theta = 0$) we can write the mean number of photons for each mode according to the following formulae

$$
\begin{aligned}
< \ \hat{n}_1(t) >&= |N_q|^2 \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{n!(n+q)!} \left\{(n+q) + k_1[E_1(n,t)]^2 \nu_1(n+q,n)\right\} \\
< \ \hat{n}_2(t) >&= |N_q|^2 \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{n!(n+q)!} \left\{n - k_2[E_1(n,t)]^2 \nu_1(n+q,n)\right\}
\end{aligned} \qquad (22)
$$

It can be verified in a straightforward way that $k_2 < n_1 > +k_1 < n_2 >$ is constant at all times.

Once the photon number is obtained in eqn. (22), therefore we use this equation to plot the time evolution of $\langle \hat{n}_1(t) \rangle$ against the scaled time t in two different cases. First we consider the two-photon case ($k_1 = k_2 = 1$) in this case the Rabi frequency is proportional to $\sqrt{(n+q+1)n}$, which is similar to that of the two-photon simple mode JCM case, and we take the parameters $\zeta = 9$ and $q = 4$. With the atom starting from the excited state. It is shown by using the constants of motion that in the absence of detuning effect (exact resonance case $\Delta = 0$), the function $< \hat{n}_1(t) >$ fluctuates around $\bar{n}_1 - 0.5$ where $\bar{n}_1 = |N_q|^2 \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{n!(n+q)!}(n+q) \simeq 10.9$ and the collapses and revivals phenomenon appears clearly and periodically which periodicity 2π. The curve take the same shape (but with a phase of π) as the atomic inversion see fig (2a). Also in the case off-resonance ($\Delta = 20$) it is show that collapses and revivals are apparent in the same manner as the atomic inversion of fig. (1b), but the function $< \hat{n}_1(t) >$ values are shifted downwards see fig.(2b). This result is in agreement with result of the constants of motion, since the sum of both the photon number operator $\hat{n}_1(t)$ and the atomic inversion is a constant, (see equation (2)). The same remarks go along for the other cases considered in the atomic inversion for the three-photon process in the preceding section.

5 Second-order correlation function

In this section we discuss another example of nonclassical effects of light. We introduce in this context the sub-Poissonian distribution of light which can be measured by photodetectors. Further, this light has several applications, e.g. in the gravitational wave detector and quantum nondemolition measurement, and can be generated in semiconductor lasers [17] and in the microwave region using masers operating in the microscopic regime [18]. A state (of a single mode for convenience) which displays sub-Poisson statistics is characterized by the fact that the variance of the photon number $\langle(\Delta\hat{n}_i(t))^2\rangle$ is less then the average photon number $\langle\hat{a}_i^\dagger(t)\hat{a}_i(t)\rangle = \langle\hat{n}_i(t)\rangle$. This can be expressed through the normalized second-order correlation function [19] as follows.

$$g_i^{(2)}(t) = \frac{\langle\hat{a}_i^{\dagger 2}(t)\hat{a}_i^2(t)\rangle}{\langle\hat{a}_i^\dagger(t)\hat{a}_i(t)\rangle^2} = 1 + \frac{\langle(\Delta\hat{n}_i(t))^2\rangle - \langle\hat{n}_i(t)\rangle}{\langle\hat{n}_i(t)\rangle^2}, \tag{23}$$

where the subscript i relates to the i^{th} mode. Then it holds that $g_i^{(2)}(t) < 1$ for sub-Poissonian distribution, $g_i^{(2)}(t) > 1$ for super-Poissonian distribution and when $g_i^{(2)}(t) = 1$ Poisson distribution of photons occurs. To discuss the temporal behavior of the correlation function related to the present system we calculate the quantities $\langle\hat{a}_j^{\dagger 2}\hat{a}_j^2\rangle, j = 1, 2$. For the first mode we have the expression

$$\begin{aligned}\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle &= |N_q|^2\sum_{n=0}^{\infty}\frac{|\zeta|^{2n}}{n!(n+q)!}\{\Big[(n+q)^2-(n+q)\Big]|F_1(n,t)|^2 - \\ &\Big[(n+q+k_1)^2-(n+q+k_1)\Big][E_1(n,t)]^2\nu_1(n+q,n)\}\end{aligned} \tag{24}$$

Similarly for the second mode we have the following

$$\langle\hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle = |N_q|^2\sum_{n=0}^{\infty}\frac{|\zeta|^{2n}}{n!(n+q)!}\{\Big[(n)^2-(n)\Big]|F_1(n,t)|^2 -$$

$$\left[(n-k_2)^2-(n-k_2)\right][E_1(n,t)]^2\nu_1(n+q,n)\} \qquad (25)$$

Numerical calculations for the cases considered in the proceeding sections show that the distribution is always sub-Poissonian and can never reach Poissonian distribution at any time [see fig.(3a,3b) for two-photon process and cases of resonance and off-resonance].

6 Squeezing phenomenon

The phenomenon of squeezing represents one of the most interesting phenomena in quantum optics. For this reason we devote this section to discuss the squeezing phenomenon related to the present system. In fact squeezed light is a radiation field without a classical analogue, one of the quadratures of the electric field has less fluctuations than those for vacuum at the expense of increased fluctuations in the other quadrature, such that the Heisenberg uncertainty relation is fulfilled. The usefulness of such light relates to several applications in optical communication networks [20], to interferometric techniques [21], and to optical waveguide trap [22]. Generation of squeezed light has been observed in many optical processes [23,24]. In order to develop our discussion for squeezing properties, we shall concentrate on discussing the two mode frequency sum squeezing [25]. Therefore to reach our goal we calculate the variances of $\hat{X}$ and $\hat{Y}$ where

$$\hat{X}=\frac{\hat{a}_1\hat{a}_2+\hat{a}_1^\dagger\hat{a}_2^\dagger}{2}, \qquad \hat{Y}=\frac{\hat{a}_1\hat{a}_2-\hat{a}_1^\dagger\hat{a}_2^\dagger}{2i} \qquad (26)$$

which satisfy the commutation relation

$$\begin{aligned}\left[\hat{X},\hat{Y}\right] &= i\hat{C},\\ \hat{C} &= \frac{1}{2}(\hat{n}_1+\hat{n}_2+1)\end{aligned} \qquad (27)$$

and the uncertainty relation:

$$(\Delta\hat{X})^2(\Delta\hat{Y})^2\geq\frac{1}{4}<\hat{C}>^2 \qquad (28)$$

The variance is given in terms of annihilation and creation operators expectation values by

$$\langle(\Delta\hat{X})^2\rangle=\frac{1}{4}+\frac{1}{4}(2Re<(\hat{a}_1^2\hat{a}_2^2)>+2<\hat{n}_1\hat{n}_2>+<\hat{n}_1+\hat{n}_2>)-(Re(<\hat{a}_1\hat{a}_2>))^2$$

The model possesses $\hat{X}$-quadrature frequency sum squeezing if the S-factor defined by

$$S(t) = \frac{\langle(\Delta\hat{X})^2\rangle - 0.5 < \hat{C} >}{0.5 < \hat{C} >} \tag{29}$$

satisfies the inequality $-1 \leq S < 0$. Now we discuss frequency sum-squeezing through calculations of equation. (29) figs(4) show the numerical results of the two-mode frequency sum squeezing for the atom starting initially in its excited state and the field is prepared in the correlated pair-coherent state. We have taken the parameters $\zeta = 9$ and $q = 3$. First we consider the two-photon case ($k_1 = k_2 = 1$), in the absence of detuning (exact resonance case $\Delta = 0$) which is plotted in fig.(4a). In this case we can see a large amount of squeezing and squeezing persists until the 3rd revival for the atomic inversion (see fig. 1a) and we have squeezing for short period even after 15 revivals. However when the detuning parameter is taken ($\Delta = 10$) (off-resonance case) we observe a decrease in the number of the fluctuations with the decreasing in its maximum value and squeezing is washed out after 5^{th} revival see fig.(4b). This result is not expected since when the field is prepared in uncorrelated coherent state the squeezing in the off-resonance case is greater than the case of resonance and persists for longer times. Second we consider the case of three-photon ($k_1 = 2$, $k_2 = 1$), and in the absence of detuning (exact resonance case), we note that the squeezing is less than the case of two-photon as can be seen from fig.(4c). While in the case of off-resonance the squeezing decreases which is observed in fig.(4d), similar to intend the case of two-photon.

7 Phase distribution

We use the formalism suggested by Barnett and Pegg [26-29] to study the phase distribution of the present system in the correlated pair-coherent state. They used the fact that, in the state space one can define phase states rigorously. The phase operator is then defined as the projection operator on the particular phase state multiplied by the corresponding value of the phase. The Pegg-Barnett phase distribution $P(\eta_1, \eta_2, t)$ for the two-mode case is defined through [26]:

$$P(\eta_1, \eta_2, t) = \frac{1}{2\pi} \sum_{l_1, l_2, m_1, m_2} \rho^f_{l_1, m_1, l_2, m_2}(t) \exp[i(l_1 - m_1)\eta_1 + i(l_2 - m_2)\eta_2] \tag{30}$$

When we apply this definition to the case of the correlated pair-coherent state we fined that $l_1 = l_2 + q$ and $m_1 = m_2 + q$. Therefore, the phase distribution can be written as

$$\begin{aligned} P(\eta, t) &= \frac{|N_q|^2}{2\pi}\{|\sum_{l_2} \frac{\zeta^n}{\sqrt{l_2!(l_2+q)!}} F_1(l_2,t)\exp[il_2\eta]|^2 + \\ & |\sum_{l_2} \frac{\zeta^n}{\sqrt{l_2!(l_2+q)!}} \sqrt{\nu_1(l_2)} E_1(l_2,t)\exp[il_2\eta]|^2\} \end{aligned} \tag{31}$$

where $\eta = \eta_1 + \eta_2$ and $F_1(l_2, t)$, $E_1(l_2, t)$ are defined in equation (21). Here we note that the dependence is on a single phase angle (which is the sum of the two phase angles) instead of two angle in the uncorrelated coherent state.

In our computations, the field is initially in a correlated pair-coherent state as in the above sections and the atom in its excited state and we take the initial parameters $\zeta = 9$ and $q = 3$. In figures (5) the phase distribution $P(\eta, t)$ is plotted against η and time $\tau = \lambda t$.

We consider the of two-photon case in the absence of the detuning (exact resonance case), the dynamical phase distribution $P(\theta, t)$ for the initial condition. First at $\tau = 0$ we see a single peak appears at $\eta = 0$, which is symmetric about $\eta = 0$ so that the mean phase always remains equal to zero. As time τ increases the peak in the middle splits into two peaks diverging away from the middle towards the wings. The split peaks reach the wings at $\tau = \lambda t_R = \pi$ where two peaks at $\eta = -\pi$ and $\eta = \pi$. As time develops further they continue converging on their ways until the two peaks meet at their initial value $\eta = 0$ when $\tau = 2\lambda t_R$. This behavior is similar the case of one-mode and two-photon JCM [31] which the field prepared in a coherent state see fig.(5a). As it has been noted before that in the case of resonance the collapses and revivals are periodic with period π.

To visualize the off-resonance case in the phase distribution for one value of detuning parameter $\Delta = 4$ while all other parameters are the same as in Fig. 5a. The outcome is presented in Fig.(5b). In the case of resonance the figure is symmetric around $\eta = 0$ as can be seen from figs (5a,c) where the two-peak figure shows this symmetry. On the other hand this symmetry is broken in the case of off-resonance where one of the peaks is subdued while the second peak is raised. The reason is due to the amplitudes of the two peaks where one of them has higher value than the other.

Next we consider the three-photon case ($k_1 = 2, k_2 = 1$), this case has the same behavior as two-photon where the time revival in this case is $t_R = \frac{2\pi}{3\sqrt{\bar{n}}}$. This result shows that the time revival in three-photon case is less than the two-photon case as observed in figs.(5c,d), as well as in fig.(1c,d) for the atomic inversion. Similar remarks to the case of the two-photon process are applicable in this case.

8 conclusion

We have studied in this work some statistical aspects of the interaction between the two-level atom and a two-mode field with the state of the field initially papered in a correlated pair-coherent state. The dynamics of the atomic inversion, the mean photon number, the squeezing phenomenon, the Glauber second order correlation function and the phase distribution are studied. The atomic inversion and photon number reflect the same behavior as the case of one mode and two-photon JCM with the field prepared in coherent state. We show that the distribution is always sub-Poissonian and can never reach Poissonian distribution at any time. The squeezing phenomenon is examined, we find from the numerical calculation that the squeezing of the two-photon process is greater than the three-photon process. Furthermore, we have discussed the Pegg-Barnett phase distribution and plotted for some parameters which amounts to be a function of a single angle for the pair-coherent state. Notice that a similar bifurcation of phase distribution has been obtained.

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Captions to Figures

Fig. 1: Time evolution of the atomic inversion against scaled time $\tau = \lambda t$, the atom is initially in its excited state ($\theta = 0$ and $\phi = 0$) and the field is prepared in a correlated pair-coherent state with parameters ($q = 3, \zeta = 9$).
(a) $k_1 = k_2 = 1$ and $\Delta = 0$, (b) $k_1 = k_2 = 1$ and $\Delta = 20$
(c) $k_1 = 2$, $k_2 = 1$ and $\Delta = 0$, (d) $k_1 = 2$, $k_2 = 1$ and $\Delta = 20$

Fig. 2: Time evolution of the photon number operator against time $\tau = \lambda t$ the atom is initially in its excited state ($\theta = 0$ and $\phi = 0$) and the field is prepared in a correlated pair-coherent state with parameters ($q = 4, \zeta = 9$).
(a) $k_1 = k_2 = 1$ and $\Delta = 0$, (b) $k_1 = k_2 = 1$ and $\Delta = 20$

Fig. 3: Time evolution of the Correlation function $g_2^{(2)}(t)$against time $\tau = \lambda t$ the atom initially in excited state ($\theta = 0$and $\phi = 0$)and the field is prepared in

correlated pair-coherent state with parameters $(q = 3, \zeta = 9)$.
(a) $k_1 = k_2 = 1$ and $\Delta = 0$, (b) $k_1 = k_2 = 1$ and $\Delta = 20$

Fig. 4: Time evolution of the function $S(t)$ against time $\tau = \lambda t$ the atom is initially in its excited state ($\theta = 0$ and $\phi = 0$) and the field is prepared in a correlated pair-coherent state with parameters $(q = 3, \zeta = 9)$.
(a) $k_1 = k_2 = 1$ and $\Delta = 0$, (b)$k_1 = k_2 = 1$ and $\Delta = 10$
(c) $k_1 = 2$, $k_2 = 1$and $\Delta = 0$, (d) $k_1 = 2$, $k_2 = 1$ and $\Delta = 10$

Fig. 5: The time evolution of the phase distribution against (η, τ) the atom is initially in its excited state ($\theta = 0$and $\phi = 0$)and the field is prepared in a correlated pair-coherent state with parameters $(q = 3, \zeta = 9)$.
(a) $k_1 = k_2 = 1$ and $\Delta = 0$, (b) $k_1 = k_2 = 1$ and $\Delta = 4$
(c) $k_1 = 2$, $k_2 = 1$ and $\Delta = 0$, (d) $k_1 = 2$, $k_2 = 1$ and $\Delta = 20$