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Further Generalizations of Inequalities and Monotonicity for the Ratio of Gamma Function[1]

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Abstract

In this article, using stirling's formula, the series-expansion of digamma functions and other techniques, some inequalities and monotonicity concerning the ratio of gamma functions are obtained, several inequalities involving the geometric mean of natural numbers are deduced.

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1. Introduction

In [1], Dr. H. Alzer proved that the inequalities

$$\frac{n+2\sqrt{2}-1}{n+1} \le \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+2}{n+1} \tag{1}$$

hold for all integers $n \ge 1$. The lower and upper bounds in (1) are the best possible. He also proved in [2] that the inequality.

$$\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}} < \frac{x+2}{x+1} \tag{2}$$

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holds for $x \geq 2$. Since $\Gamma(n+1) = n!$, the right hand side in (1) can be deduced from inequality (2) only if we let$x = n \geq 2$. Moreover,the right hand side in (1) refines the inequality

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+1}{n} \tag{3}$$

Which was obtained in[14] by H. Mine and L. Sathre. Recently, in [19] and [23] , obtained the following.

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < \sqrt{\frac{n+m}{n+m+k}} \tag{4}$$

for positive integers n and m and nonnegative integer k. In [37], obtained the following.

$$\frac{n+k+1}{n+m+k+1} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} \tag{5}$$

for positive integers n and m and nonnegative integer k. The inequality (3) was refined by H. Alzer in [3]: Let $n \in z$, then, for any $r > 0$, we have

$$\frac{n}{n+1} \leq \left(\frac{1}{n} \sum_{i=1}^{n} i^r \bigg/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n]{(n+1)!}} \tag{6}$$

The lower and upper bounds are the best possible.

Many new and simple proofs of the inequalities in (6) and some generalizations were given in [5–8, 12, 13, 16, 18, 23, 25, 31, 32, 35, 36]. The left hand side of inequality (6) was generalized in [17]: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n} \sum_{i=k+1}^{n+k} i^r \bigg/ \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r \right)^{\frac{1}{r}} \tag{7}$$

Where r is any given positive real number. The lower bound is the best possible. The integer analogue of (7) was presented in [9] and [16]: Let$b > a > 0$ and $\delta > 0$ be real numbers, then, for any given positive $r \in R$, we have

$$\frac{b}{b+\delta} < \left(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1} - a^{r+1}}{(b+\delta)^{r+1} - a^{r+1}} \right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{b-a} \int_a^b x^r dx \Big/ \frac{1}{b+\delta-a} \int_a^{b+\delta} x^r dx \right)^{\frac{1}{r}} < \frac{(b^b/a^a)^{\frac{1}{b-a}}}{((b+\delta)^{b+\delta}/a^a)^{\frac{1}{b+\delta-a}}} \tag{8}$$

The lower and upper bounds in (8) are the best possible. The inequality (8) was generalized to an inequality for linear positive functionals in [8]. Recently, results related to those above in [20]. These results were generalizations for monotonic sequences involving convex functions as follows:

- For $a > 1$, let $n \in R$ and $r > 0$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} a^{ir} \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} a^{ir}\right) > \frac{1}{a} \tag{9}$$

- For $n, m \in N$, $k \in N \cup \{0\}$ and $r > 0$, we have

$$\frac{1}{a^m} < \left(\frac{1}{a^n}\sum_{i=k+1}^{n+k} a^{ir} \Bigg/ \frac{1}{a^{n+m}}\sum_{i=k+1}^{n+m+1} a^{ir}\right)^{\frac{1}{r}} \tag{10}$$

that is,

$$\frac{1}{a^{m(r+1)}} \leq \sum_{i=k+1}^{n+k} a^{ir} \Bigg/ \sum_{i=k+1}^{n+m+k} a^{ir} \tag{11}$$

where $a > 1$ is a positive real number.

- If $\{a_i\}_{i\in N}$ is an increasing, positive sequence such that $\{i(\frac{a_{i+1}}{a_i} - 1)\}_{i\in N}$ increases, then we have

$$\frac{a_n}{a_{n+1}} \leq \frac{\sqrt[n]{\prod_{i=1}^{n}(a_i + a_n)}}{\sqrt[n+1]{\prod_{i=1}^{n+1}(a_i + a_{n+1})}} \leq \frac{\sqrt[n]{\prod_{i=1}^{n} a_i}}{\sqrt[n+1]{\prod_{i=1}^{n+1} a_i}} \tag{12}$$

- If φ is an increasing, convex, positive function defined on $(0, \infty)$ such that $\{\varphi(i)[\frac{\varphi(i)}{\varphi(i+1)} - 1]\}_{i\in N}$ decreases, then

$$\frac{[\varphi(n)]^{\frac{n}{\varphi(n)}}}{[\varphi(n+1)]^{\frac{n+1}{\varphi(n+1)}}} \leq \frac{\sqrt[\varphi(n)]{\prod_{i=1}^{n}[\varphi(i) + \varphi(n)]}}{\sqrt[\varphi(n+1)]{\prod_{i=1}^{n+1}[\varphi(i) + \varphi(n+1)]}} \tag{13}$$

These inequalities generalize those obtained in [11], [18] and [23].

In this article, we will prove the following inequalities.

Theorem 1.1: For $m, n \in N$, $p < 0$ or $p = 1$, and nonnegative integer k, we have

$$\frac{n + k + \frac{1}{p}}{n + m + k + \frac{1}{p}} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} \tag{14}$$

Remark 1: Let $p = 1$, inequality (5) follow from (14).

Theorem 1.2: The function

$$\frac{[\Gamma(x+1)/\Gamma(y+1)]^{\frac{1}{x}}}{\sqrt{x+y}} \tag{15}$$

is increasing in $x > 0$ for fixed $y \geq \frac{2}{3}$. For positive real number x and y , we have

$$\frac{[\Gamma(x+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+2)/\Gamma(y+1)]^{\frac{1}{x+1}}} \leq \sqrt{\frac{x+y}{x+y+1}} \tag{16}$$

where Γ denotes the gamma function.

Corollary 1.3: If we take $x, y \in N$, we have

$$\frac{\sqrt[n]{n!/k!}}{\sqrt[n+m]{(n+m)!/k!}} < \sqrt{\frac{n+m}{n+m+k}} \tag{17}$$

2. Preliminaries

In this section, we present some useful formulas related to the derivatives of the logarithm of the gamma function. In [38, pp. 103–105], the following formula was given:

$$\frac{\Gamma'(z)}{\Gamma(z)} + \gamma = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt = \int_0^1 \frac{1 - t^{z-1}}{1 - t} dt \tag{18}$$

where $\gamma = 0.577215664901532860606 51 \cdots$ denotes the Euler's constants. See [38, p. 94]. Formula (18) can be used to calculate $\Gamma'(k)$ for $k \in N$. We call $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [4, p. 71].

In is well known that the Bernoulli numbers B_n are generally defined [38, p. 1] by

$$\frac{1}{e^t - 1} + \frac{1}{2} - \frac{1}{t} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n}}{(2n)!} B_n \tag{19}$$

In particular, we have the following

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \cdots$$

In [38, p. 45], the following summation formula is given

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = \frac{\pi^{2k+1} E_k}{2^{2k+2}(2k)!} \tag{20}$$

for nonnegative integer k, where E_k denotes Euler's number, which implies

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}} \quad n \in N \tag{21}$$

The formula (21) can also be found in [40, chapter 23] or in [41, p. 1237].

Lemma 2.1: For a real number $x > 0$ and natural number m, we have

$$ln\Gamma(x) = \frac{1}{2}ln(2\pi) + (x - \frac{1}{2})lnx - x + \sum_{n=1}^{m}(1)^{n-1} \cdot \frac{B_n}{2(2n-1)n} \cdot \frac{1}{x^{2n-1}}$$

$$+(-1)^m\theta_1\frac{B_{m+1}}{(2m+1)(2m+2)} \cdot \frac{1}{x^{2m+1}}, \quad 0 < \theta_1 < 1, \tag{22}$$

$$\frac{d}{dx}ln\Gamma(x) = lnx - \frac{1}{2x} + \sum_{n=1}^{m}(-1)^n\frac{B_n}{2n} \cdot \frac{1}{x^{2n}}$$

$$+(-1)^{m+1}\theta_2 \cdot \frac{B_{m+1}}{2m+2} \cdot \frac{1}{x^{2m+2}}, \quad 0 < \theta_2 < 1, \tag{23}$$

$$\frac{d^2}{dx}ln\Gamma(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} + (-1)^m\theta_3 \cdot \frac{B_m}{x^{2m+3}}, \quad 0 < \theta_3 < 1. \tag{24}$$

Remark 2.1: The formulas and their proofs in Lemma 2.1 are well-known and can be found in many textbooks on Analysis; See, for instance, [39, Section 54 and Section 541].

3. Proofs of Theorems

Proof of Theorem 1. Inequality (14) can be rearranged so that we have

$$\frac{n+k+\frac{1}{p}}{\sqrt[n]{(n+k)!/k!}} < \frac{n+m+k+\frac{1}{p}}{\sqrt[n+m]{(n+m+k)!/k!}}$$

which is equivalent to

$$\frac{n+k+\frac{1}{p}}{\sqrt[n]{(n+k)!/k!}} < \frac{n+k+1+\frac{1}{p}}{\sqrt[n+1]{(n+k+1)!/k!}} \tag{25}$$

when $k = 0, p = 1$, inequality (25) follow from the right inequality in (1). When $k \geq 1, p < 0$, the inequality (25) can be written as

$$(\frac{(n+k)!}{n!})^{\frac{1}{n}} > \frac{(n+k+\frac{1}{p})^{n+1} \cdot (n+k+1)}{(n+k+1+\frac{1}{p})^{n+1}} \tag{26}$$

In [14] and [11, P. 92], the following inequalities were given for $n \in N$.

$$\sqrt{2\pi n}(\frac{n}{e})^n < n! < \sqrt{2\pi n}(\frac{n}{e})^n exp\frac{1}{12n} \tag{27}$$

Inequality (27) is related to the stirling's formula.

By substituting the inequalities in (27) into the left term of inequality (26), we see that it is sufficient to prove the following.

$$\left[\sqrt{2\pi(n+k)}\left(\frac{n+k}{e}\right)^{n+k}\right]^{\frac{1}{n}} > \frac{(n+k+\frac{1}{p})^{n+1}\cdot(n+k+1)}{(n+k+1+\frac{1}{p})^{n+1}} \cdot \left[\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\exp\frac{1}{12k}\right]^{\frac{1}{n}} \tag{28}$$

Taking logarithm on both sides of inequality (28), simplifying directly and using standard arguments, we obtain

$$\frac{2k+1}{2n}\ln\left(1+\frac{n}{k}\right)+(n+1)\ln\left(1+\frac{1}{n+k+\frac{1}{p}}\right)-\ln\left(1+\frac{1}{n+k}\right) - \frac{1}{12kn}-1>0 \tag{29}$$

In [10, pp. 367–368], [14, pp. 273–274] and [21], we have for $t>0$

$$ln(1+t) > \frac{2}{2t+1} \tag{30}$$

$$ln(1+t) < \frac{t(2+t)}{2(t+1)} \tag{31}$$

Thus, to get inequality (30) and (31), it suffices to show that

$$\frac{2(n+1)}{2(n+k+\frac{1}{p})+1}+\frac{2k+1}{2k+n}-\frac{2(n+k)+1}{2(n+k)\cdot(n+k+1)}-\frac{1}{12kn}-1>0 \tag{32}$$

which can be deduced from the following

$$\begin{aligned}&2n^4(6k-1)+3n^3(16k^3-1)+14n^3k+n^2(72nk^2-1)+6n^2k(7k-1)\\&\quad+15nk^2(k-1)+6k^3(14n^2-1)+2k^2(12n^4-1)+3nk(8nk^3-1)\\&\quad+4k^4(6n-1)+7nk^3-\frac{1}{p}[2n(n+1)(12k^3+24nk^2+n)+4kn^2(6n-1)\\&\quad+4k^2(k+1)+nk(10k+6)+24n^4k]>0\end{aligned}$$

Thus we complete the proof.

Remark 3.1: In [33], J. Sandor and L. Debnath proved a new form of the stirling's formula: For all positive $n\geq 2$, we have double inequality

$$\sqrt{2\pi}e^{-n}n^{(n+1)/2} < n! < \left(\frac{n}{n-1}\right)^{\frac{1}{2}}\sqrt{2\pi}e^{-n}n^{(n+1)/2} \tag{33}$$

Proof of Theorem 2.For a fixed real number $y \geq \frac{2}{3}$, define

$$f(x) = \frac{[\Gamma(x+1)/\Gamma(y+1)]^{\frac{1}{x}}}{\sqrt{x+y}} \quad x \in (0, \infty), \tag{34}$$

Taking the logarithm yields and a simple calculation yields

$$lnf(x) = \frac{1}{x} ln\Gamma(x+1) - \frac{1}{x} ln\Gamma(y+1) - \frac{1}{2} ln(x+y) \tag{35}$$

Differentiating with respect to x on both sides of (35) and rearranging leads to

$$x^2 \frac{f'(x)}{f(x)} = ln\Gamma(y+1) - ln\Gamma(x+1) - \frac{x^2}{2(x+y)} + x\Psi(x+1) \tag{36}$$

and, using (31),

$$\begin{aligned}(x^2 \frac{f'(x)}{f(x)})' &= x\Psi'(x+1) - \frac{x^2+2xy}{2(x+y)^2} > x\left[\frac{1}{x+1} + \frac{1}{2(x+1)^2}\right] - \frac{x^2+2xy}{2(x+y)^2} \\ &= \frac{(2x^2+3x)(x+y)^2 - (x^2+2xy)(x+1)^2}{2(x+1)^2(x+y)^2} \\ &= \frac{x^4 + (2y+1)x^3 + (2y^2+2y-1)x^2 + (3y^2-2y)x}{2(x+y+1)^2(x+y)^2} > 0 \end{aligned} \tag{37}$$

Therefore the function $\xi(x) = x^2 \frac{f'(x)}{f(x)}$ is strictly increasing in $(0, \infty)$, $\xi(x) > \xi(0) = 0$, and then $f'(x) > 0$, hence is strictly increasing in $(0, \infty)$. The proof of theorem 2 is complete.

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