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On the Use of Renewal Theory in Machine Replacement Models[1]

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Abstract

In this paper, we give examples on how the renewal theory can be applied in modeling machine replacement problem. We begin with a deterministic model to illustrate the concept of a machine cycle, then follow by a stochastic model with a general cost. We then compare two popular replacement policies: the quantity-based replacement policy and time-based replacement policy for a single machine replacement problem. We also prove an interesting result that the optimal costs of both policies are the same under certain assumptions.

AMS subject classification:
Keywords: Renewal Theory, Machine Replacement, Average Running Cost.

1. Introduction

Machine replacement problem has been studied by a lot of researchers and is also an important topic in operations research and management science [3, 4, 5]. Renewal theory is a useful tool in modeling a lot of systems [1, 2]. In this paper, we give examples on how the renewal theory can be applied in modeling machine replacement problem. In particular, we compare the quantity-based replacement policy and time-based replacement policy for a single machine problem. These two kinds of policies have been applied to inventory management problem [2]. In a quantity-based replacement policy, a machine is replaced when an accumulated product of size q is produced. In this model, one has to determine the optimal production size q. While in a time-based replacement policy, a machine is replaced in every period of T. For this model, one

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has to determine the optimal replacement period T in each production cycle. The time-based policy is more preferable than the quantity-based dispatch policy for satisfying timely customer service. Especially, time-based shipment consolidation have became a part of the transportation contract among the members of a supply chain. To our best knowledge, the comparison of the performance of these two dispatching policies for machine replacement problem has not been addressed in the literature. Here we develop two analytic models and compare their average long-run performance. We derive the average long-run costs for both models by using renewal theory. The costs here include both the cost of a new machine and the machine maintenance cost. We will argue that the performance (in terms of the average long-run cost) of the optimal time-based dispatch policy is the same as the optimal quantity-based dispatch policy.

The rest of the paper is organized as follows. In Section 2, we discuss a deterministic machine replacement model to illustrate the concept of a cycle. In Section 3, we then discuss a stochastic model with general costs. The models presented in Sections 2 and 3 can be found in textbooks such as [5] except that we consider a more general cost function and more general results on optimal policies are obtained. These models are discussed so as to give more ideas and motivations to the readers. In Section 4, we compare the quantity-based replacement policy and time-based replacement policy for a single machine problem. Finally, concluding remarks are given in Section 5.

2. A Deterministic Machine Replacement Model

In this section, we consider a deterministic machine replacement model [5]. The machine system consists of a single machine and the machine is assumed to operate continuously. The downtime of the machine for repairing is assumed to be negligible. Every new machines for replacement are assumed to be identical. In this model, we only consider the maintenance and the replacement costs. The cost rate of maintaining a machine of age t is $p(t)$ and the replacement cost of a machine is K. Here $p(t)$ is assumed to be smooth and strictly increasing with respect to t. Therefore it is natural to have a replacement policy, to replace the machine by a new one for every period of T. We seek for optimal policy for machine replacement when the planning horizon is infinite.

The objective here is to minimize the average long-run cost due to replacement and maintenance in the system. We note that whenever an old machine is replaced, a new cycle starts. Since all the machines are identical, all the cycles are the same. Thus one needs only to consider the average running cost in a cycle. The replacement cost per cycle (a period of T) is of course K and the maintenance cost per cycle is given by

$$M(T) = \int_0^T p(t)dt.$$

Therefore the total average cost is

$$C(T) = \frac{K + M(T)}{T} = \frac{K}{T} + \frac{M(T)}{T}.$$

It is not difficult to show the following proposition.

Proposition 2.1: If $p(t)$ is strictly convex, i.e. $p''(t) > 0$ then the optimal value of T is the zero of the equation

$$K + \int_0^T p(t)dt - Tp(T) = 0.$$

Proof. We note that

$$C'(T) = \frac{-K}{T^2} + \frac{p(T)}{T} - \frac{M(T)}{T^2} = \frac{-K + Tp(T) - M(T)}{T^2}$$

and

$$C''(T) = \frac{2K}{T^3} + \frac{p'(T)}{T} - \frac{2p(T)}{T^2} + \frac{2M(T)}{T^3} = \frac{2K + T^2p'(T) - 2Tp(T) + 2M(T)}{T^3}.$$

Define

$$H(T) = T^2p'(T) - 2Tp(T) + 2M(T)$$

and we note that

$$H(0) = 0 \quad \text{and} \quad H'(T) = T^2p''(T).$$

Since $p''(T) > 0$, $H(T)$ is an increasing function in T, hence $C''(T) > 0$ for all $T > 0$.

Next we define

$$R(T) = K + \int_0^T p(t)dt - Tp(T).$$

Since

$$R'(T) = p(T) - p(T) - Tp'(T) = -Tp'(T) < 0 \quad \text{for } T > 0,$$

$R(T)$ is strictly decreasing for $T > 0$. Moreover, we have

$$\int_0^T p(t)dt - Tp(T) = \int_0^T [p(t) - p(T)]dt = -\int_0^T tp'(t)dt.$$

Therefore

$$R(0) = K > 0 \quad \text{and} \quad \lim_{T\to\infty} R(T) = -\infty.$$

This means that $C(T)$ has a unique minimum in $(0, \infty)$ and from $C'(T) = 0$, it is the root of the equation

$$K + \int_0^T p(t)dt - Tp(T) = 0.$$

The root can be obtained by using bisection method. ■

We end this section by the following example.

Example 2.1. Suppose that

$$p(t) = \alpha t^{\beta}$$

where $\beta > 2$ and $\alpha > 0$ then $p''(t) > 0$ and

$$C'(T) = \frac{1}{T^2}\left(-K + \alpha T^{\beta+1} - \int_0^T \alpha t^{\beta} dt\right) = -KT^{-2} + \frac{\alpha\beta}{\beta+1}T^{\beta-1}.$$

The optimal planned replacement period is $\sqrt[\beta+1]{\frac{(\beta+1)K}{\alpha\beta}}$.

3. A Stochastic Machine Replacement Model

In this section, we consider the problem of planned machine replacement under uncertainty [5]. A single continuous operating machine whose lifetime is a continuous random variable t with known probability function $f(t)$ and cumulative distribution function $F(t)$. Suppose that it costs c to replace the machine when it fails and $p(t)(\leq c)$ to replace the machine prior to failure at age t. (let us say one may sell the old machine for $c - p(t)$). Here we assume that

$$p(0) = a \quad \text{and} \quad \lim_{t\to\infty} p(t) = c > a.$$

Similar to the replacement problem in Section 1, we are looking for optimal replacement policy for minimizing the average long-run cost. Let the planned replacement be made exactly T units of time after the last replacement. Our objective here is to find the optimal value of T to minimize the average cost per unit time of the planned replacement. Again we consider the expected cost per cycle in this problem. We define the expected cost per unit time as follows:

$$C(T) = \frac{\text{E(cost per cycle)}}{\text{E(length of a cycle)}}.$$

Our objective is then to find the value of T which minimizes $C(T)$.

We note that

$$\begin{aligned} \text{E(cost per cycle)} &= cP(\text{the replacement is a result of failure}) \\ &+ p(T)P(\text{the replacement is planned}) \end{aligned}$$

and therefore

$$\text{E(cost per cycle)} = cF(T) + p(T)(1 - F(T)).$$

Clearly the next replacement occurs at $\min(t, T)$ and we have

$$\text{E(length of cycle)} = \int_0^{\infty} \min(t, T) f(t)dt = \int_0^T tf(t)dt + T\int_T^{\infty} f(t)dt.$$

Hence the average long-run cost per cycle is given by

$$C(T) = \frac{cF(T) + p(T)(1 - F(T))}{\int_0^T tf(t)dt + T\int_T^{\infty} f(t)dt} = \frac{cF(T) + p(T)(1 - F(T))}{\int_0^T tf(t)dt + T(1 - F(T))}. \quad (3.1)$$

The optimal replacement time T can then be solved numerically. In the following we present an interesting result on the condition for the uselessness of preventive maintenance (early replacement).

Proposition 3.1: There is no advantage to plan replacement when the lifetime distribution is exponentially distributed, i.e.

$$f(t) = \lambda e^{-\lambda t}$$

and the replacement cost satisfies

$$p''(t) \leq \lambda p'(t).$$

Proof: We note that

$$\text{E(length of cycle)} = \int_0^T t\lambda e^{-\lambda t} dt + Te^{-\lambda T} = \frac{1}{\lambda}(1 - e^{-\lambda T})$$

and

$$\text{E(cost per cycle)} = c(1 - e^{-\lambda T}) + p(T)e^{-\lambda T}.$$

Thus we have the expected cost per unit time

$$C(T) = \frac{\lambda \left(c - (c - p(T))e^{-\lambda T}\right)}{1 - e^{-\lambda T}}.$$

We get

$$C'(T) = \frac{\lambda e^{-\lambda T} \left(p'(T) - \lambda p(T) - p'(T)e^{-\lambda T}\right)}{(1 - e^{-\lambda T})^2}.$$

Now we let

$$G(T) = p'(T) - \lambda p(T) - p'(T)e^{-\lambda T}.$$

We observe that

$$G(0) = -a\lambda \leq 0$$

and

$$G'(T) = \left(p''(T) - \lambda p'(T)\right)(1 - e^{-\lambda T}) \leq 0.$$

Thus $G(T) \leq 0$ for any $T > 0$ and this means that $C(T)$ is decreasing. It implies that there is no need to plan for replacement. ■

Example 3.2: Suppose that

$$p(t) = c + (a - c)e^{-\gamma t} \quad \text{with } \gamma > 0,$$

then we have

$$\lambda p'(t) = \lambda(c - a)\gamma e^{-\gamma t} \quad \text{and} \quad p''(t) = -(c - a)\gamma^2 e^{-\gamma t}.$$

The condition $p''(t) \leq \lambda p'(t)$ holds.

4. A Comparison of Time-based and Quantity-based Replacement Policies

In this section, we compare the time-based and the quantity-based machine replacement policies. Under a time-based replacement policy, we replace the machine at age T. One needs to compute the optimal T which minimizes the total average cost. Under a quantity-based policy, we replace the machine after it produced Q items. One needs to compute the optimal Q which minimizes the total cost. We will compare the optimal costs of the two policies and show that they are equal.

To compare the two policies, we assume that the cost of a new machine is K and the production cost of Q items is $f(Q)$. We further assume that $f(Q)$ is smooth and strictly increasing. The expected cost per unit time is then given by

$$C(T) = \frac{\text{E(cost per cycle)}}{\text{E(length of a cycle)}}.$$

Under the quantity-based policy, we have

$$\text{E(cost per cycle)} = K + f(Q) \quad \text{and} \quad \text{E(length of cycle)} = E(T).$$

Under the time-based policy, we have

$$\text{E(cost per cycle)} = K + f(E(Q)) \quad \text{and} \quad \text{E(length of cycle)} = T.$$

Proposition 4.1: If f(Q) is strictly convex and the demand is random with mean rate λ, then both policies have the same optimal running cost.

Proof: Under the quantity-based policy,

$$E(T) = \frac{Q}{\lambda}.$$

Therefore the total average cost is

$$C_Q(Q) = \frac{K + f(Q)}{Q/\lambda} = \frac{\lambda K + \lambda f(Q)}{Q}.$$

Thus, we have

$$C'_Q(Q) = \frac{\lambda Q f'(Q) - \lambda K - \lambda f(Q)}{Q^2}.$$

Similar to the proof of Proposition (2), it can be shown that the optimal Q is the zero of the equation

$$K + f(Q) - Qf'(Q) = 0. \tag{4.1}$$

Denote the optimal value of Q be Q^*, this means the optimal cost is

$$\frac{\lambda K + \lambda f(Q^*)}{Q^*}.$$

Now under the time-based policy,

$$E(Q) = \lambda T.$$

Therefore the total average cost is

$$C_T(T) = \frac{K + f(\lambda T)}{T}.$$

Thus, we have

$$C'_T(T) = \frac{\lambda T f'(\lambda T) - K - f(\lambda T)}{T^2}.$$

Similar to the proof of Proposition (2), the optimal T is the zero of the equation

$$K + f(\lambda T) - \lambda T f'(\lambda T) = 0$$

Denote the optimal value of T be T^*, since (4.1) has a unique zero, we can conclude that

$$Q^* = \lambda T^*.$$

This means that the optimal cost is

$$C_T(T^*) = \frac{K + f(\lambda T^*)}{T^*} = \frac{\lambda K + \lambda f(Q^*)}{Q^*} = C_Q(Q^*).$$

The result follows. ■

Example 4.2: Suppose that $f(Q) = \alpha Q^\beta$ where $\alpha > 0$ and $\beta > 2$, then $f(Q)$ is strictly convex. Then under the quantity-based policy, we have

$$Q^* = \left(\frac{K}{\alpha(\beta - 1)}\right)^{\frac{1}{\beta}} \quad \text{and} \quad C_Q(Q^*) = \lambda\beta\alpha^{\frac{1}{\beta}}\left(\frac{K}{\beta - 1}\right)^{\frac{\beta-1}{\beta}}.$$

While under the time-based policy, we have

$$T^* = \frac{1}{\lambda}\left(\frac{K}{\alpha(\beta - 1)}\right)^{\frac{1}{\beta}} \quad \text{and} \quad C_T(T^*) = \lambda\beta\alpha^{\frac{1}{\beta}}\left(\frac{K}{\beta - 1}\right)^{\frac{\beta-1}{\beta}}.$$

5. Concluding Remarks

In this paper, we demonstrate how the renewal theory can be applied in modeling machine replacement problem. A deterministic model is given to illustrate the concept of cycle, then follow by stochastic models with general costs. In particular, we compare the quantity-based replacement policy and time-based replacement policy for a single machine problem. Analytic models are developed for both policies. We show that the optimal costs of both policies are the same. It is worth to notice that the result is independent of the distribution of the demand and the cost function $f(Q)$, as long as it is strictly convex. In addition, time-based policy is usually preferred because of the following reasons.

(i) The optimal value of Q is not necessary an integer, rounding it to an integer will lead to lost of optimality.

(ii) Using time-based policy, we can have regular plans of replacement. Therefore one can make early order of the machines and obtain a better deal with the vendor.

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