

On a New Extension of Hilbert's Double Series Theorem and Applications¹

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Abstract

In this paper, by using the Euler-Maclaurin's summation formula, and estimating the weight coefficient, we give a new extension of Hilbert's double series theorem with (p, q) -parameter and $\lambda \in (0, 4]$, which involves the β function as a best constant factor. As its applications, we consider the equivalent form and some particular results.

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1. Introduction

The Hilbert's double series theorem is said (cf. Hardy et al. [1]): If $\{a_n\}, \{b_n\}$ are real sequences such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{1/2}, \quad (1.1)$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's inequality, which is important in analysis and its applications (cf. Mitrinovic et al. [2]). And the equivalent form is

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2, \quad (1.2)$$

where the constant factor π^2 is still the best possible.

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In 1925, Hardy-Riesz gave some classical extended results on (1.1) and (1.2), by introducing (p, q)-parameter as follows (see [3, 1]):

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \{a_n\}, \{b_n\}$ are non-negative real sequences such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}; \tag{1.3}$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^p < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} a_n^p, \tag{1.4}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)} \right]^p$ are all the best possible. Inequality (1.3) is named Hardy-Hilbert's inequality, which is equivalent to (1.4). For $p = q = 2$, inequality (1.3) reduces to (1.1), and (1.4) reduces to (1.2).

In 1992, Gao [4] gave a strengthened version of (1.1). In 1997–1998, by using Euler-Maclaurin's summation formula, Yang et al. [5, 6] gave a strengthened version of (1.3) as:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{\frac{1}{p}}} \right] a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{\frac{1}{q}}} \right] b_n^q \right\}^{\frac{1}{q}}, \tag{1.5}$$

where γ is Euler constant, and $1 - \gamma = 0.42278433^+$ is the best value.

In 1998, by introducing a parameter $\lambda \in (0, 1)$ and the β function, Yang [7] gave an extension of the integral form of (1.1). Following the way of [7], Yang [8] gave some extensions of (1.1) and (1.2) as: If $\lambda \in (0, 4]$, such that $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 \right\}^{1/2}; \tag{1.6}$$

$$\sum_{n=1}^{\infty} n^{\lambda-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda} \right]^2 < \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^2 \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2, \tag{1.7}$$

where the constant factors $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ and $\left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^2$ are all the best possible; $B(u, v)$ is the β function, and (cf. Wang et al. [9])

$$B(u, v) := \int_0^\infty \frac{1}{(1+x)^{u+v}} x^{-1+u} dx = B(v, u) \quad (u, v > 0). \tag{1.8}$$

In 2003, Yang et al. [10] provided an extensive account of the above results.

In this paper, by using the β function and the Euler-Maclaurin's summation formula, we obtain the weight coefficient and give a new extension of (1.1) with some parameters and a best constant factor, such that both (1.1) and (1.6) are its particular results. As applications, we also consider the equivalent form and some particular results.

2. Some Lemmas

We need the following Euler-Maclaurin's summation formula (cf. [11]): If $f \in C^1[1, \infty)$, both $\sum_{k=1}^{\infty} f(k)$ and $\int_1^{\infty} f(x)dx$ are convergence, then

$$\sum_{k=1}^{\infty} f(k) = \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) + \int_1^{\infty} \rho_1(x) f'(x)dx, \quad (2.1)$$

where $\rho_1(x) = x - [x] - 1/2$ is the first order Bernoulli's function; If $g \in C^4[1, \infty)$, $(-1)^r g^{(r)} > 0$ and $g^{(r)}(\infty) = 0$ ($r = 0, 1, 2, 3, 4$), then

$$-\frac{1}{12}g(1) < \int_1^{\infty} \rho_1(x)g(x)dx < -\frac{1}{12}g(1) + \frac{1}{720}g''(1) < 0. \quad (2.2)$$

Lemma 2.1: If $\lambda \in (0, 4]$ and $n \in N$, the function $f_{\lambda,n}(t)$ is defined by

$$f_{\lambda,n}(t) := \frac{t^{-1+\lambda/2}}{(t+n)^\lambda}, \quad t \in (0, \infty),$$

then we have

$$Q_\lambda(n) := \int_0^1 f_{\lambda,n}(t)dt - \frac{1}{2}f_{\lambda,n}(1) - \int_1^{\infty} \rho_1(t)f'_{\lambda,n}(t)dt > 0. \quad (2.3)$$

Proof: We have $f_{\lambda,n}(1) = \frac{1}{(1+n)^\lambda}$. Integration by parts, since $\lambda \in (0, 4]$, $n \in N$, we find

$$\begin{aligned} \int_0^1 f_{\lambda,n}(t)dt &= \frac{2}{\lambda} \int_0^1 \frac{dt^{\lambda/2}}{(t+n)^\lambda} = \frac{2}{\lambda(1+n)^\lambda} + 2 \int_0^1 \frac{t^{\lambda/2}}{(t+n)^{\lambda+1}} dt \\ &= \frac{2}{\lambda(1+n)^\lambda} + \frac{2}{1+\lambda/2} \int_0^1 \frac{dt^{1+\lambda/2}}{(t+n)^{\lambda+1}} \\ &= \frac{2}{\lambda(1+n)^\lambda} + \frac{4}{(2+\lambda)(1+n)^{\lambda+1}} + \frac{4(\lambda+1)}{2+\lambda} \int_0^1 \frac{t^{1+\lambda/2}}{(t+n)^{\lambda+2}} dt \\ &= \frac{2}{\lambda(1+n)^\lambda} + \frac{4}{(2+\lambda)(1+n)^{\lambda+1}} + \frac{4(\lambda+1)}{2+\lambda/2} \int_0^1 \frac{dt^{2+\lambda/2}}{(t+n)^{\lambda+2}} \\ &= \frac{2}{\lambda(1+n)^\lambda} + \frac{4}{(2+\lambda)(1+n)^{\lambda+1}} + \frac{8(\lambda+1)}{(2+\lambda)(4+\lambda)(1+n)^{\lambda+2}} \\ &\quad + \frac{4(\lambda+1)}{2+\lambda/2} \int_0^1 \frac{t^{2+\lambda/2}}{(t+n)^{\lambda+3}} dt \end{aligned}$$

$$\begin{aligned}
 &> \frac{2}{\lambda(1+n)^\lambda} + \frac{4}{(2+\lambda)(1+n)^{\lambda+1}} + \frac{8(\lambda+1)}{(2+\lambda)(4+\lambda)(1+n)^{\lambda+2}} \\
 &> \frac{2}{\lambda(1+n)^\lambda} + \frac{2}{3(1+n)^{\lambda+1}} + \frac{1}{6(1+n)^{\lambda+2}}.
 \end{aligned}$$

Since we have

$$f'_{\lambda,n}(t) = \frac{-\lambda t^{-1+\frac{\lambda}{2}}}{(t+n)^{\lambda+1}} - \frac{(1-\frac{\lambda}{2})t^{-2+\frac{\lambda}{2}}}{(t+n)^\lambda} = -\frac{(1+\frac{\lambda}{2})t^{-2+\frac{\lambda}{2}}}{(t+n)^\lambda} + \frac{n\lambda t^{-2+\frac{\lambda}{2}}}{(t+n)^{\lambda+1}},$$

then in view of $\lambda \in (0, 4], n \in N$, and (2.2), we obtain

$$\begin{aligned}
 -\int_1^\infty \rho_1(t) f'_{\lambda,n}(t) dt &= \int_1^\infty \rho_1(t) \frac{(1+\frac{\lambda}{2})t^{-2+\frac{\lambda}{2}}}{(t+n)^\lambda} dt - \int_1^\infty \rho_1(t) \frac{n\lambda t^{-2+\frac{\lambda}{2}}}{(t+n)^{\lambda+1}} dt \\
 &> -\frac{(1+\frac{\lambda}{2})}{12(1+n)^\lambda} + \frac{n\lambda}{12(1+n)^{\lambda+1}} - \frac{n\lambda}{720} \left[\frac{(\lambda+1)(\lambda+2)}{(1+n)^{\lambda+3}} \right. \\
 &\quad \left. + \frac{(\lambda+1)(4-\lambda)}{(1+n)^{\lambda+2}} + \frac{(4-\lambda)(6-\lambda)}{4(1+n)^{\lambda+1}} \right] \\
 &> -\frac{(1+\lambda/2)}{12(1+n)^\lambda} + \frac{(1+n)\lambda - \lambda}{12(1+n)^{\lambda+1}} \\
 &\quad - \frac{4(1+n)}{720} \left[\frac{30}{(1+n)^{\lambda+3}} + \frac{7}{(1+n)^{\lambda+2}} + \frac{24}{4(1+n)^{\lambda+1}} \right] \\
 &> -\frac{(1+\lambda/2)}{12(1+n)^\lambda} + \frac{\lambda}{12(1+n)^\lambda} - \frac{4}{12(1+n)^{\lambda+1}} \\
 &\quad - \frac{1}{6(1+n)^{\lambda+2}} - \frac{1}{20(1+n)^{\lambda+1}} - \frac{1}{30(1+n)^\lambda} \\
 &= \left(-\frac{7}{60} + \frac{\lambda}{24} \right) \frac{1}{(1+n)^\lambda} - \frac{23}{60(1+n)^{\lambda+1}} - \frac{1}{6(1+n)^{\lambda+2}}.
 \end{aligned}$$

Hence, by (2.3) and the above result, we have

$$\begin{aligned}
 Q_\lambda(n) &> \left(\frac{2}{\lambda} - \frac{37}{60} + \frac{\lambda}{24} \right) \frac{1}{(1+n)^\lambda} + \left(\frac{2}{3} - \frac{23}{60} \right) \frac{1}{(1+n)^{\lambda+1}} + \left(\frac{1}{6} - \frac{1}{6} \right) \frac{1}{(1+n)^{\lambda+2}} \\
 &> (5\lambda^2 - 74\lambda + 240) \frac{1}{120\lambda(1+n)^\lambda} \\
 &\geq (5 \times 4^2 - 74 \times 4 + 240) \frac{1}{120\lambda(1+n)^\lambda} = \frac{1}{5\lambda(1+n)^\lambda} > 0.
 \end{aligned}$$

The lemma is proved. ■

Lemma 2.2: If $\lambda \in (0, 4]$ and $n \in N$, define the weight coefficient $\omega_\lambda(n)$ as

$$\omega_\lambda(n) := \sum_{m=1}^{\infty} \frac{1}{(m+n)^\lambda} \left(\frac{1}{m}\right)^{1-\frac{\lambda}{2}}. \tag{2.4}$$

Then we have

$$\omega_\lambda(n) < n^{-\frac{\lambda}{2}} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right), \quad n \in N. \tag{2.5}$$

Proof: Since by (2.1) and (2.3), we have

$$\omega_\lambda(n) = \sum_{m=1}^{\infty} f_{\lambda,n}(m) = \int_0^{\infty} f_{\lambda,n}(t)dt - Q_\lambda(n) < \int_0^{\infty} f_{\lambda,n}(t)dt. \tag{2.6}$$

Setting $y=t/n$, by (1.5), we have

$$\int_0^{\infty} f_{\lambda,n}(t)dt = n^{-\frac{\lambda}{2}} \int_0^{\infty} \frac{y^{-1+\lambda/2}}{(1+y)^\lambda} dy = n^{-\frac{\lambda}{2}} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right).$$

Hence by (2.6), we have (2.5). The lemma is proved. ■

Lemma 2.3: If $\lambda \in (0, 4]$, $n \in N$ and $0 < \varepsilon < \frac{p\lambda}{2}$, define the function $h_n(t)$ as

$$h_n(t) := \frac{t^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^\lambda}, \quad t \in (0, \infty).$$

Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{q}} \sum_{m=1}^{\infty} h_n(m) &> B\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}, \frac{\lambda}{2} + \frac{\varepsilon}{p}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \\ &- \left[\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}\right)^{-1} + \frac{\lambda}{12} \right] \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda}{2}+\frac{\varepsilon}{q}}}. \end{aligned} \tag{2.7}$$

Proof: Setting $y = t/n$, we find

$$\begin{aligned} \int_0^{\infty} h_n(t)dt &= n^{-\frac{\lambda}{2}-\frac{\varepsilon}{p}} B\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}, \frac{\lambda}{2} + \frac{\varepsilon}{p}\right); \\ \int_0^1 h_n(t)dt &= n^{-\frac{\lambda}{2}-\frac{\varepsilon}{p}} \int_0^{1/n} \frac{y^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(1+y)^\lambda} dy \\ &< n^{-\frac{\lambda}{2}-\frac{\varepsilon}{p}} \int_0^{1/n} y^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{p}} dy < \left(\frac{\lambda}{2} - \frac{\varepsilon}{p}\right)^{-1} \frac{1}{n^\lambda}. \end{aligned}$$

Since we have

$$\begin{aligned} h'_n(t) &= \frac{-\lambda t^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^{\lambda+1}} + \frac{(-1+\frac{\lambda}{2}-\frac{\varepsilon}{p})t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^{\lambda+1}} \\ &= \frac{[-\lambda(t+n)+\lambda n]t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^{\lambda+1}} + \frac{(-1+\frac{\lambda}{2}-\frac{\varepsilon}{p})t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^{\lambda+1}} \\ &= \frac{(-1-\frac{\lambda}{2}-\frac{\varepsilon}{p})t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^\lambda} + \frac{\lambda n t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^{\lambda+1}}, \end{aligned}$$

in virtue of (2.2) and $0 < \lambda \leq 4$, we obtain

$$\begin{aligned} \int_1^\infty \rho_1(t)h'_n(t)dt &= -\int_1^\infty \rho_1(t)\frac{(1+\frac{\lambda}{2}+\frac{\varepsilon}{p})t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^\lambda}dt \\ &\quad + \lambda n \int_1^\infty \rho_1(t)\frac{t^{-2+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}{(t+n)^{\lambda+1}}dt > 0 - \frac{\lambda n}{12(1+n)^{\lambda+1}} > -\frac{\lambda}{12n^\lambda}. \end{aligned}$$

By (2.1), we have

$$\begin{aligned} \sum_{m=1}^\infty h_n(m) &= \int_0^\infty h_n(t)dt - \int_0^1 h_n(t)dt + \frac{1}{2}h_n(1) + \int_1^\infty \rho_1(t)h'_n(t)dt \\ &> n^{-\frac{\lambda}{2}-\frac{\varepsilon}{p}}B\left(\frac{\lambda}{2}-\frac{\varepsilon}{p}, \frac{\lambda}{2}+\frac{\varepsilon}{p}\right) - \left[\left(\frac{\lambda}{2}-\frac{\varepsilon}{p}\right)^{-1} + \frac{\lambda}{12}\right]\frac{1}{n^\lambda}. \end{aligned} \tag{2.8}$$

By simplification, we have (2.7). The lemma is proved.

3. Main Result and some Applications

Theorem 3.1: If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $0 < \lambda \leq 4$, such that $0 < \sum_{n=1}^\infty n^{p(1-\frac{\lambda}{2})-1}a_n^p < \infty$ and $0 < \sum_{n=1}^\infty n^{q(1-\frac{\lambda}{2})-1}b_n^q < \infty$, then we have

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=1}^\infty n^{p(1-\frac{\lambda}{2})-1}a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q(1-\frac{\lambda}{2})-1}b_n^q \right\}^{\frac{1}{q}}, \tag{3.1}$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible. In particular, for $\lambda = 1, 2, 3, 4$, we have

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^\infty n^{\frac{p}{2}-1}a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{q}{2}-1}b_n^q \right\}^{\frac{1}{q}}; \tag{3.2}$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^2} < \left\{ \sum_{n=1}^{\infty} \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} b_n^q \right\}^{\frac{1}{q}}; \tag{3.3}$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^3} < \frac{\pi}{8} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p}{2}+1}} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{\frac{q}{2}+1}} b_n^q \right\}^{\frac{1}{q}}; \tag{3.4}$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^4} < \frac{1}{6} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{q+1}} b_n^q \right\}^{\frac{1}{q}}, \tag{3.5}$$

where the above constant factors are all the best possible.

Proof: By Hölder’s inequality and (2.4), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{(m+n)^{\lambda/p}} \frac{m^{(1-\frac{\lambda}{2})/q}}{n^{(1-\frac{\lambda}{2})/p}} \right] \left[\frac{b_n}{(m+n)^{\lambda/q}} \frac{n^{(1-\frac{\lambda}{2})/p}}{m^{(1-\frac{\lambda}{2})/q}} \right] \\ &\leq \left\{ \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \frac{a_m^p}{(m+n)^{\lambda}} \frac{m^{p(1-\frac{\lambda}{2})/q}}{n^{(1-\frac{\lambda}{2})}} \right] \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{b_n^q}{(m+n)^{\lambda}} \frac{n^{q(1-\frac{\lambda}{2})/p}}{m^{(1-\frac{\lambda}{2})}} \right] \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{m=1}^{\infty} \omega_{\lambda}(m) m^{(p-1)(1-\frac{\lambda}{2})} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega_{\lambda}(n) n^{(q-1)(1-\frac{\lambda}{2})} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

Hence by (2.5), we have (3.1).

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting \tilde{a}_n, \tilde{b}_n as:

$$\tilde{a}_n = n^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}, \quad \tilde{b}_n = n^{-1+\frac{\lambda}{2}-\frac{\varepsilon}{q}}, \quad \text{for } n \in N,$$

then we find

$$J := \left\{ \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} \tilde{a}_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} \tilde{b}_n^q \right\}^{1/q} = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}.$$

If the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ in (3.1) is not the best possible, then there exists a positive constant k (with $k < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$), such (3.1) is still valid that we replace $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ by k . In particular, by (2.5), we have

$$B\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}, \frac{\lambda}{2} + \frac{\varepsilon}{p}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} - \left[\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}\right)^{-1} + \frac{\lambda}{12} \right] \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda}{2}+\frac{\varepsilon}{q}}}$$

$$< \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(m+n)^\lambda} < kJ = k \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}},$$

and

$$B\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}, \frac{\lambda}{2} + \frac{\varepsilon}{p}\right) - \left[\left(\frac{\lambda}{2} - \frac{\varepsilon}{p}\right)^{-1} + \frac{\lambda}{12}\right] \left[\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}\right]^{-1} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda}{2}+\frac{\varepsilon}{q}}} < k.$$

For $\varepsilon \rightarrow 0^+$, it follows that $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \leq k$, which contradicts the fact that $k < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$. Hence the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ in (3.1) is the best possible. The theorem is proved. ■

Theorem 3.2: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $0 < \lambda \leq 4, \{a_n\}$ is a non-negative real sequence, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p < \infty$, then we have

$$\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda} \right]^p < \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^p \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p, \tag{3.7}$$

where the constant factor $\left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^p$ is the best possible; Inequality (3.7) is equivalent to (3.1). In particular, for $\lambda = 1, 2, 3, 4$, we have

$$\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right]^p < (\pi)^p \sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p; \tag{3.8}$$

$$\sum_{n=1}^{\infty} n^{p-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^2} \right]^p < \sum_{n=1}^{\infty} \frac{1}{n} a_n^p; \tag{3.9}$$

$$\sum_{n=1}^{\infty} n^{\frac{3p}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^3} \right]^p < \left(\frac{\pi}{8}\right)^p \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p}{2}+1}} a_n^p; \tag{3.10}$$

$$\sum_{n=1}^{\infty} n^{2p-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^4} \right]^p < \left(\frac{1}{6}\right)^p \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} a_n^p, \tag{3.11}$$

where the constant factors in the above inequalities are all the best possible.

Proof: Setting b_n as

$$b_n := n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda} \right]^{p-1} > 0, \quad n \in N,$$

then by (3.1), we find

$$\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q = \sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda} \right]^p = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda}$$

$$\leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right\}^{1/q}. \quad (3.12)$$

Hence we obtain

$$0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \leq \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^p \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p < \infty. \quad (3.13)$$

By (3.1), both (3.12) and (3.13) take the form of strict inequality, and we have (3.7).

On the other hand, suppose that (3.7) is valid. By Hölder's inequality, we find

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} &= \sum_{n=1}^{\infty} \left[n^{\frac{\lambda}{2}-\frac{1}{p}} \sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda} \right] [n^{-\frac{\lambda}{2}+\frac{1}{p}} b_n] \\ &\leq \left\{ \sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m+n)^\lambda} \right]^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (3.14)$$

Then by (3.7), we have (3.1). Hence (3.1) and (3.7) are equivalent.

If the constant factor $\left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^p$ in (3.7) is not the best possible, by (3.14), we may get a contradiction that the constant factor in (3.1) is not the best possible. Thus we complete the proof of the theorem. ■

Remark 3.3: (a) For $p = q = 2$, (3.1) and (3.7) reduce respectively to (1.6) and (1.7). It follows that (3.1) is a new extension of (1.1) and (1.6) with some parameters and the equivalent form (3.7) is a new extension of (1.2) and (1.7).

(b) It is interesting that (3.3) and (1.3) are deferent, although both of them are extensions of (1.1) with (p, q) -parameter and the best constant factor.

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