

On a Coupled Parabolic-Parabolic Problem with a Small Parameter

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Abstract

In this paper we study a model for convection-diffusion-reaction processes in which there is a small parameter $\varepsilon > 0$ representing the diffusion coefficient in a subdomain of the spatial domain. Specifically, the problem under investigation is (S_ε) , (IC_ε) , (BC_ε) , (TC_ε) formulated below. Under some appropriate smoothness and compatibility conditions on the data, we prove that the solution of this (perturbed) model approximates the one of the corresponding reduced model with respect to the uniform convergence norm. This is shown by some sharp estimates (see Theorem 3.1 in Section 3).

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1. Introduction

Let $a, b, c \in \mathbb{R}$, $a < b < c$, and let $T > 0$ be a fixed time instant. In $D_T = (a, c) \times (0, T)$ we consider the following coupled parabolic-parabolic problem, which will be called P_ε :

$$\begin{cases} u_t - \varepsilon u_{xx} + \alpha(x)u_x + \beta(x)u = f(x, t) & \text{in } D_1 = (a, b) \times (0, T), \\ v_t - (\mu v_x)_x + \alpha(x)v_x + \beta(x)v = g(x, t) & \text{in } D_2 = (b, c) \times (0, T), \end{cases} \quad (S_\varepsilon)$$

with the initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & a \leq x \leq b, \\ v(x, 0) = v_0(x), & b \leq x \leq c, \end{cases} \quad (IC_\varepsilon)$$

the boundary conditions of the Neumann and Dirichlet type

$$\begin{cases} u_x(a, t) = 0, \\ v(c, t) = 0, & 0 \leq t \leq T, \end{cases} \quad (BC_\varepsilon)$$

and the transmission conditions

$$\begin{cases} u(b, t) = v(b, t), \\ \varepsilon u_x(b, t) = (\mu v_x)(b, t), & 0 \leq t \leq T, \end{cases} \quad (TC_\varepsilon)$$

where ε is a positive small parameter.

The following assumptions will be required throughout this paper:

(i) $\alpha \in H^1(a, c)$, $\beta \in L^\infty(a, c)$, $\mu \in H^1(b, c)$;

(ii) $\alpha(x) \leq \alpha_0 < 0$ in $[a, c]$, $\mu(x) \geq \mu_0 > 0$ in $[b, c]$, $\beta - \frac{\alpha'}{2} \geq 0$ a.e. in (a, c) .

In (ii) we can assume the equivalent conditions $\alpha(x) < 0$ in $[a, c]$ and $\mu(x) > 0$ in $[b, c]$. The constants α_0 and μ_0 are introduced since we need them later.

In this paper we study the behavior of the solution of P_ε , denoted by $(u(x, t, \varepsilon), v(x, t, \varepsilon))$, as the parameter ε approaches zero.

Problem P_ε is a model for diffusion-convection-reaction processes. In some physical problems, the flux of some material may have both a diffusive component and a convective component. The latter is due to the flow velocity. Here, the corresponding spatial domain is assumed to be one-dimensional, being represented by the line segment $[a, c]$. In a subdomain $[a, b]$ the diffusion is considered to be negligible and so we set the diffusion coefficient equal to ε for $a \leq x \leq b$. The unknown functions, u and v , represent the density of the material in $[a, b]$ and $[b, c]$, respectively.

A problem similar to P_ε , but in which both the boundary conditions are of the Dirichlet type, has been discussed in [1] and [3]. There the convection coefficient α was assumed to be positive. This made the corresponding problem singularly perturbed with respect to the uniform convergence norm, with an internal transition layer at the coupling point $x = b$. Here, on the contrary, P_ε is regularly perturbed with respect to the same norm (under some conditions on the data, including the above assumptions). The key condition which makes P_ε regularly perturbed is the negativity of α . We show that the reduced problem, denoted by P_0 , can be obtained by setting $\varepsilon = 0$ in the original model P_ε , but without any boundary condition at $x = a$.

The next section contains results on the existence, uniqueness and high smoothness of the solutions of the problems P_ε and P_0 . These results are then used in Section 3 (the last one) to prove some sharp estimates for the difference of the solutions of the two

problems, under appropriate smoothness and compatibility conditions for the data. In fact, we establish a zeroth order asymptotic expansion for the solution of P_ε with respect to the uniform convergence norm.

In particular, such approximation results are useful for the numerical solution, since the original problem can be replaced by the simpler reduced model in designing the corresponding numerical schemes.

Although our P_ε is regularly perturbed it is essentially different from the cases treated in [1-4] and requires an even deeper analysis. This is due to the Neumann boundary condition.

This paper is inspired by [6], where a similar stationary boundary value problem is analyzed, as a first step toward the analysis of the coupling between the Navier-Stokes system and the simpler Euler system (which corresponds to a negligible viscosity in a large subdomain of the flow field).

Note that higher order asymptotic expansions can also be associated with the solution of P_ε , but in this case higher order boundary layers may occur in a vicinity of the point $x = a$. This case will be investigated elsewhere.

2. Existence, uniqueness and smoothness of the solutions

We shall first investigate the problem P_ε . This will be written as a Cauchy problem for an evolution equation in the real Hilbert space $H := L^2(a, b) \times L^2(b, c)$, equipped with the usual scalar product, denoted $\langle \cdot, \cdot \rangle$, and the norm induced by it, denoted $\| \cdot \|$. To this purpose, define the operator $\mathcal{L}_\varepsilon : D(\mathcal{L}_\varepsilon) \subset H \rightarrow H$, where

$$D(\mathcal{L}_\varepsilon) := \left\{ (p, q) \in H^2(a, b) \times H^2(b, c) : p'(a) = q(c) = 0, p(b) = q(b), \right. \\ \left. \varepsilon p'(b) = (\mu q')(b) \right\},$$

$$\mathcal{L}_\varepsilon(p, q) := (-\varepsilon p'' + \alpha p' + \beta p, -(\mu q')' + \alpha q' + \beta q).$$

Thus P_ε can be expressed as the following Cauchy problem in H

$$\begin{cases} U'_\varepsilon(t) + \mathcal{L}_\varepsilon U_\varepsilon(t) = F(t), & 0 < t < T, \\ U_\varepsilon(0) = U_0, \end{cases} \tag{2.1}$$

where

$$U_\varepsilon(t) := (u(\cdot, t, \varepsilon), v(\cdot, t, \varepsilon)), \quad U_0 := (u_0, v_0), \quad F(t) := (f(\cdot, t), g(\cdot, t)).$$

Concerning the operator \mathcal{L}_ε , the following result holds true:

LEMMA 2.1. *If assumptions (i), (ii) are fulfilled, then $\forall \varepsilon$ \mathcal{L}_ε is densely defined, linear and maximal monotone.*

Proof. Obviously, \mathcal{L}_ε is densely defined and linear. As for its monotonicity (positiveness), we see that $\forall (p, q) \in D(\mathcal{L}_\varepsilon)$

$$\begin{aligned} \langle \mathcal{L}_\varepsilon(p, q), (p, q) \rangle &\geq 0 \iff \\ -\varepsilon \int_a^b p p'' + \int_a^b \alpha p' p + \int_a^b \beta p^2 - \int_b^c (\mu q')' q + \int_b^c \alpha q' q + \int_b^c \beta q^2 &\geq 0 \iff \\ \varepsilon \int_a^b (p')^2 + \int_b^c \mu (q')^2 + \int_a^b \left(\beta - \frac{\alpha'}{2}\right) p^2 + \int_b^c \left(\beta - \frac{\alpha'}{2}\right) q^2 - \frac{\alpha(a)}{2} p^2(a) &\geq 0, \end{aligned}$$

which is a consequence of (ii). It remains to prove that \mathcal{L}_ε is maximal monotone or, equivalently (see [5, p. 23] or [8, p. 19]), that $\forall (f_1, f_2) \in H$, $\exists (p, q) \in D(\mathcal{L}_\varepsilon)$ such that

$$(p, q) + \mathcal{L}_\varepsilon((p, q)) = (f_1, f_2),$$

i.e., the following transmission problem

$$\begin{cases} -\varepsilon p'' + \alpha p' + (\beta + 1)p = f_1 & \text{in } L^2(a, b), \\ -(\mu q')' + \alpha q' + (\beta + 1)q = f_2 & \text{in } L^2(b, c), \\ p'(a) = q(c) = 0, \\ p(b) = q(b), \\ \varepsilon p'(b) = (\mu q')(b), \end{cases} \quad (2.2)$$

has a unique solution $(p, q) \in H^2(a, b) \times H^2(b, c)$. Consider the space $W := \{\varphi \in H^1(a, c) : \varphi(c) = 0\}$, equipped with the usual H^1 -norm. Problem (2.2) has the following variational formulation: find a function $w_\varepsilon \in W$ such that

$$h_\varepsilon(w, \varphi) = \int_a^c l \varphi dx \quad \text{for all } \varphi \in W, \quad (2.3)$$

where

$$\begin{aligned} l|_{(a,b)} &= f_1, \quad l|_{(b,c)} = f_2, \\ h_\varepsilon : W \times W &\rightarrow \mathbb{R}, \quad h_\varepsilon(w, \varphi) := \int_a^c \mu_\varepsilon w' \varphi' dx + \int_a^c \alpha w' \varphi dx + \int_a^c (\beta + 1) w \varphi dx, \\ \mu_\varepsilon(x) &:= \begin{cases} \varepsilon, & x \in [a, b], \\ \mu(x), & x \in (b, c]. \end{cases} \end{aligned}$$

According to (i) and (ii), h_ε is bilinear, continuous and coercive. By the classical Lax-Milgram lemma there exists a unique solution $w_\varepsilon \in W$ of problem (2.3). It follows that $-(\mu_\varepsilon w')' + \alpha w' + (\beta + 1)w = l$ in (a, c) in the sense of distributions. From this equation we infer that $\mu_\varepsilon w' \in H^1(a, c)$, i.e.,

$$p \in H^2(a, b), \quad \mu q' \in H^1(b, c) \implies q \in H^2(b, c). \quad (2.4)$$

To conclude, we see that $p'(a) = 0$, $p(b) = q(b)$, and $\varepsilon p'(b) = (\mu q')(b)$.

Q.E.D

Theorem 2.1. *If assumptions (i) and (ii) hold and, in addition,*

$$F \in W^{1,1}(0, T; H), \tag{2.5}$$

$$U_0 \in D(\mathcal{L}_\varepsilon), \tag{2.6}$$

then, for every $\varepsilon > 0$, problem (2.1) has a unique strong solution

$$U_\varepsilon \in W^{1,\infty}(0, T; H) \cap W^{1,2}(0, T; H^1(a, b) \times H^1(b, c)) \\ \cap L^\infty(0, T; H^2(a, b) \times H^2(b, c)).$$

The proof is based on arguments similar to those used in [1], [3].

Remark 2.1. The conditions (2.5), (2.6) can be written equivalently

$$\begin{cases} f \in W^{1,1}(0, T; L^2(a, b)), & g \in W^{1,1}(0, T; L^2(b, c)), \\ u_0 \in H^2(a, b), & v_0 \in H^2(b, c), & u'_0(a) = v_0(c) = 0, \\ u_0(b) = v_0(b), & \varepsilon u'_0(b) = (\mu v'_0)(b). \end{cases} \tag{2.7}$$

Remark 2.2. Let the assumptions of Theorem 2.1 be satisfied. Suppose in addition that

$$F \in W^{2,1}(0, T; H), \quad F(0) - \mathcal{L}_\varepsilon U_0 \in D(\mathcal{L}_\varepsilon).$$

Then, we can show (see, e.g., [2]) that

$$U_\varepsilon := (u, v) \in W^{2,\infty}(0, T; H) \\ \cap W^{2,2}(0, T; H^1(a, b) \times H^1(b, c)) \cap W^{1,\infty}(0, T; H^2(a, b) \times H^2(b, c)).$$

Now, we can derive formally an asymptotic expansion of order zero for the solution of problem P_ε by using the same method as in [1-4] (which is actually the classic method developed in [10] and adapted to our transmission problems):

$$U_\varepsilon(x, t) := (u(x, t, \varepsilon), v(x, t, \varepsilon)) = (U(x, t), V(x, t)) + (r_\varepsilon(x, t), s_\varepsilon(x, t)), \tag{2.8}$$

where (U, V) is the first term of the regular series, and $(r_\varepsilon, s_\varepsilon)$ is the remainder of order zero. There is no boundary layer in this case. This will be justified rigorously in the next section. It is easily seen that formally (U, V) satisfies the following reduced problem, which we call P_0 :

$$\begin{cases} U_t + \alpha U_x + \beta U = f & \text{in } (a, b) \times (0, T), \\ V_t - (\mu V_x)_x + \alpha V_x + \beta V = g & \text{in } (b, c) \times (0, T), \end{cases} \tag{S_0}$$

with the initial conditions

$$\begin{cases} U(x, 0) = u_0(x), & x \in [a, b], \\ V(x, 0) = v_0(x), & x \in [b, c], \end{cases} \quad (IC_0)$$

the Dirichlet boundary condition

$$V(c, t) = 0, \quad 0 \leq t \leq T, \quad (BC_0)$$

and the transmission conditions

$$\begin{cases} U(b, t) = V(b, t), \\ V_x(b, t) = 0, \quad 0 \leq t \leq T. \end{cases} \quad (TC_0)$$

Obviously, the Neumann condition at $x = a$ must be eliminated, since $(S_0)_1$ is a first order equation and there are enough conditions for P_0 .

Now we are going to investigate the existence and regularity of the solution of problem P_0 . To this purpose, we define the operator $\mathcal{B} : D(\mathcal{B}) \subset H \rightarrow H$,

$$\begin{aligned} D(\mathcal{B}) &:= \left\{ (p, q) \in H, p \in H^1(a, b), q \in H^2(b, c), \right. \\ &\quad \left. p(b) = q(b), q'(b) = q(c) = 0 \right\}, \\ \mathcal{B}(p, q) &:= \text{col}(\alpha p' + \beta p, -(\mu q')' + \alpha q' + \beta q). \end{aligned}$$

So, P_0 can be written as the following Cauchy problem in H :

$$\begin{cases} z'(t) + \mathcal{B}z(t) = F(t), & 0 < t < T, \\ z(0) = U_0, \end{cases} \quad (2.9)$$

where $z(t) := (U(\cdot, t), V(\cdot, t))$ and $F(t) := (f(\cdot, t), g(\cdot, t))$.

LEMMA 2.2. *If assumptions (i) and (ii) hold, then the operator \mathcal{B} is densely defined, linear and maximal monotone.*

Proof. It is easily seen that \mathcal{B} is densely defined, linear and monotone (positive). It remains to show that \mathcal{B} is maximal monotone, which means that $\forall (f_1, f_2) \in H \exists (p, q) \in D(\mathcal{B})$ satisfying the equation

$$(p, q) + \mathcal{B}((p, q)) = (f_1, f_2),$$

or, equivalently, that the problem

$$\begin{cases} \alpha p' + (\beta + 1)p = f_1 & \text{in } L^2(a, b), \\ -(\mu q')' + \alpha q' + (\beta + 1)q = f_2 & \text{in } L^2(b, c), \\ p(b) = q(b), \\ q'(b) = q(c) = 0, \end{cases} \quad (2.10)$$

has a solution $(p, q) \in H^1(a, b) \times H^2(b, c)$. Indeed, by the Lax-Milgram lemma, there is a unique $q \in H^2(b, c)$ satisfying (2.10)_{2,4} (see also the proof of Lemma 2.1). It is also evident that Eq. (2.10)₁ with the condition (2.10)₃ has a unique solution $p \in H^1(a, b)$.

Q.E.D

Using this lemma we can prove the following result:

Theorem 2.2. *If (i) and (ii) are fulfilled and, in addition, the following conditions hold*

$$F \in W^{2,1}(0, T; H), U_0 \in D(\mathcal{B}), F(0) - \mathcal{B}U_0 \in D(\mathcal{B}), \tag{2.11}$$

then, problem (2.9) has a unique solution $z \in W^{2,\infty}(0, T; H)$. Moreover,

$$U \in W^{1,\infty}(0, T; H^1(a, b)), V \in W^{1,\infty}(0, T; H^2(b, c)).$$

Proof. Since \mathcal{B} is a maximal monotone operator (see Lemma 2.2), we can use the classical existence theory (see [5], [8]) to infer that problem (2.9) has a unique strong solution $z \in W^{1,\infty}(0, T; H)$. From $(S_0)_1$ we see that $\alpha U_x \in L^\infty(0, T; L^2(a, b)) \Rightarrow U \in L^\infty(0, T; H^1(a, b))$. Since

$$\mu_0 \int_b^c V_x(x, t)^2 dx \leq \langle \mathcal{B}z(t), z(t) \rangle = \langle F(t) - z'(t), z(t) \rangle \text{ for a.a. } t \in (0, T),$$

we have $V_x \in L^\infty(0, T; L^2(b, c))$. By $(S_0)_2$ it follows that

$$(\mu V_x)_x \in L^\infty(0, T; L^2(b, c)),$$

which implies $V \in L^\infty(0, T; H^2(b, c))$. Now, by the Lumer-Phillips theorem (see, e.g., [9, p. 13]), the operator $-\mathcal{B}$ is the infinitesimal generator of a C_0 -semigroup $\{S(t), t \geq 0\}$. Therefore, we can write (see [7, p. 25] or [9, p. 109])

$$z(t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds, \quad 0 \leq t \leq T,$$

$$z'(t) = S(t)(F(0) - \mathcal{B}U_0) + \int_0^t S(t-s)F'(s)ds, \quad 0 \leq t \leq T.$$

This means that $\bar{z}(t) := z'(t)$ is a mild solution of the problem

$$\begin{cases} \bar{z}'(t) + \mathcal{B}\bar{z}(t) = F'(t), & 0 \leq t \leq T, \\ \bar{z}(0) = F(0) - \mathcal{B}U_0. \end{cases} \tag{2.12}$$

But under our assumptions, $\bar{z} = z'$ is actually a strong solution of problem (2.12), with the properties derived before for z . This concludes the proof of Theorem 2.2.

Q.E.D.

Remark 2.3. If in addition to the assumptions of Theorem 2.2 we require

$$\alpha, \beta \in W^{1,\infty}(a,b), f \in L^\infty(0,T;H^1(a,b)),$$

then from $(S_0)_1$ we can infer that $U \in L^\infty(0,T;H^2(a,b))$.

Remark 2.4. The above conditions (2.11) are fulfilled if

$$\left\{ \begin{array}{l} f \in W^{2,1}(0,T;L^2(a,b)), \quad g \in W^{2,1}(0,T;L^2(b,c)), \\ f(\cdot,0) \in H^1(a,b), \quad g(\cdot,0) \in H^2(b,c), \\ u_0 \in H^2(a,b), \quad v_0 \in H^4(b,c), \quad \mu \in H^3(b,c) \\ \alpha|_{[b,c]} \in H^2(b,c), \quad \beta|_{[a,b]} \in H^1(a,b), \quad \beta|_{[b,c]} \in H^2(b,c), \\ u_0(b) = v_0(b), \quad v'_0(b) = v_0(c) = 0, \\ f(b,0) - \alpha(b)u'_0(b) - \beta(b^-)u_0(b) \\ = g(b,0) + (\mu v'_0)'(b) - \beta(b^+)v_0(b), \\ g(c,0) + (\mu v'_0)'(c) - \alpha(c)v'_0(c) = 0, \\ g_x(b,0) + (\mu v'_0)''(b) - \alpha(b)v''_0(b) - \beta'(b^+)v_0(b) = 0. \end{array} \right. \quad (2.13)$$

We have assumed that (i) is fulfilled, in particular $\alpha \in H^1(a,c)$, so that $\alpha(b)$ makes sense.

Now, summarizing the above results (see Theorems 2.1, 2.2 and Remarks 2.1 - 2.4) we can formulate the following all-inclusive result:

Corollary 2.1. *Let (i), (ii) be satisfied. In addition, the following conditions are required*

$$\begin{aligned} & f \in W^{2,1}(0,T;L^2(a,b)), \quad g \in W^{2,1}(0,T;L^2(b,c)), \\ & f(\cdot,0) \in H^1(a,b), \quad g(\cdot,0) \in H^2(b,c), \quad f_x \in L^\infty(0,T;L^2(a,b)), \\ & u_0 \in H^2(a,b), \quad v_0 \in H^4(b,c), \quad \alpha|_{[a,b]} \in W^{1,\infty}(a,b), \quad \beta|_{[a,b]} \in W^{1,\infty}(a,b), \\ & \mu \in H^3(b,c), \quad \alpha|_{[b,c]} \in H^2(b,c), \quad \beta|_{[b,c]} \in H^2(b,c), \\ & u'_0(a) = v_0(c) = 0, \quad u_0(b) = v_0(b), \quad u'_0(b) = v'_0(b) = 0. \end{aligned}$$

If furthermore the compatibility conditions (2.13)₆₋₈ are fulfilled, then our problems P_ε , $\varepsilon > 0$, and P_0 have unique solutions,

$$\begin{aligned} U_\varepsilon & := (u, v) \in W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;H^1(a,b) \times H^1(b,c)) \\ & \quad \cap L^\infty(0,T;H^2(a,b) \times H^2(b,c)), \\ (U, V) & \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;H^1(a,b) \times H^2(b,c)), \\ U & \in L^\infty(0,T;H^2(a,b)). \end{aligned}$$

Note that all the assumptions above are independent of ε . These smoothness and compatibility assumptions are essential in deriving higher smoothness for the solutions of both P_ε and P_0 . The higher smoothness of the solutions is certainly interesting by itself, but we need it here for the estimates we are going to obtain in the next section.

3. Estimates

In order to establish some estimates for the remainder components

$$\begin{cases} r_\varepsilon(x, t) := u(x, t, \varepsilon) - U(x, t), & (x, t) \in (a, b) \times (0, T), \\ s_\varepsilon(x, t) := v(x, t, \varepsilon) - V(x, t), & (x, t) \in (b, c) \times (0, T), \end{cases} \quad (3.1)$$

we shall first prove an auxiliary result:

LEMMA 3.1. *If (i), (ii) and (2.7) are fulfilled, then the following estimates hold true for the solution $U_\varepsilon := (u, v)$ of P_ε*

$$\left\{ \begin{array}{l} \|u\|_{C([0, T]; L^2(a, b))} = O(1), \quad \|v\|_{C([0, T]; L^2(b, c))} = O(1), \\ \|u_t\|_{L^\infty(0, T; L^2(a, b))} = O(1), \quad \|v_t\|_{L^\infty(0, T; L^2(b, c))} = O(1), \\ \|u_x\|_{C([0, T]; L^2(a, b))} = O(\varepsilon^{-1/2}), \quad \|v_x\|_{C([0, T]; L^2(b, c))} = O(1), \\ \|v_{xx}\|_{L^\infty(0, T; L^2(b, c))} = O(1), \quad \|u(a, \cdot, \varepsilon)\|_{C[0, T]} = O(1), \\ \|u(b, \cdot, \varepsilon)\|_{C[0, T]} = O(1), \quad \|v(\cdot, \cdot, \varepsilon)\|_{C([b, c] \times [0, T])} = O(1), \\ \|u_x(b, \cdot, \varepsilon)\|_{L^\infty(0, T)} = O(\varepsilon^{-3/4}), \quad \|v_x(b, \cdot, \varepsilon)\|_{L^\infty(0, T)} = O(1). \end{array} \right. \quad (3.2)$$

Proof. Obviously, all the assumptions of Theorem 2.1 are fulfilled. Since U_ε is a strong solution of problem (2.1), we have the well-known estimate (see, e.g., [8, p. 48])

$$\|U'_\varepsilon(t)\| \leq \|F(0) - \mathcal{L}_\varepsilon U_0\| + \int_0^t \|F'(s)\| ds = O(1) \quad (3.3)$$

and hence the estimates (3.2)₂ are proved. From

$$\|U_\varepsilon(t)\| = \|U_0 + \int_0^t U'_\varepsilon(s) ds\| \leq \|U_0\| + \int_0^t \|U'_\varepsilon(s)\| ds = O(1), \quad (3.4)$$

it follows (3.2)₁. Now, using (3.3), (3.4) and the equality

$$\langle U'_\varepsilon(t), U_\varepsilon(t) - U_0 \rangle + \langle \mathcal{L}_\varepsilon(U_\varepsilon(t) - U_0), U_\varepsilon(t) - U_0 \rangle = \langle F(t) - \mathcal{L}_\varepsilon U_0, U_\varepsilon(t) - U_0 \rangle,$$

we obtain

$$\frac{1}{2} \alpha^2(a) u^2(a, t, \varepsilon) + \varepsilon \|u_x(\cdot, t, \varepsilon) - u'_0\|_{L^2(a, b)}^2 + \mu_0 \|v_x(\cdot, t, \varepsilon) - v'_0\|_{L^2(b, c)}^2$$

$$\leq \|U_\varepsilon(t) - U_0\|(\|F(t)\| + \|\mathcal{L}_\varepsilon U_0\| + \|U'_\varepsilon(t)\|) = O(1). \quad (3.5)$$

Therefore, $\|u(a, \cdot, \varepsilon)\|_{C[0,T]} = O(1)$ and (3.2)₃ hold true. Now, let us multiply $(S_\varepsilon)_1$ by u_x and then integrate the resulting equation over $[a, b]$:

$$\int_a^b u_t(x, t, \varepsilon) u_x(x, t, \varepsilon) dx - \frac{\varepsilon}{2} u_x^2(b, t, \varepsilon) + \int_a^b \alpha(x) u_x^2(x, t, \varepsilon) dx + \int_a^b \beta(x) u(x, t, \varepsilon) u_x(x, t, \varepsilon) dx = \int_a^b f(x, t) u_x(x, t, \varepsilon) dx.$$

Since $\alpha(x) \leq \alpha_0 < 0$, this implies that

$$\begin{aligned} \frac{\varepsilon}{2} u_x^2(b, t, \varepsilon) &\leq \|u_t\|_{L^\infty(0,T;L^2(a,b))} \|u_x\|_{C([0,T];L^2(a,b))} \\ &+ \|\beta\|_{L^\infty(a,b)} \|u\|_{C([0,T];L^2(a,b))} \|u_x\|_{C([0,T];L^2(a,b))} \\ &+ \|f\|_{C([0,T];L^2(a,b))} \|u_x\|_{C([0,T];L^2(a,b))} = O(\varepsilon^{-1/2}) \end{aligned}$$

for a.a. $t \in (0, T)$, which yields

$$\|u_x(b, \cdot, \varepsilon)\|_{L^\infty(0,T)} = O(\varepsilon^{-3/4}).$$

Making use of the equation $(S_\varepsilon)_2$ we get by virtue of the above estimates

$$\|v_{xx}(\cdot, \cdot, \varepsilon)\|_{L^\infty(0,T;L^2(b,c))} = O(1). \quad (3.6)$$

Since $H^1(b, c)$ is continuously embedded in $C[b, c]$, we can employ (3.2)₁ and (3.2)₃ to derive the following estimate

$$\|v(\cdot, \cdot, \varepsilon)\|_{C([b,c] \times [0,T])} = O(1),$$

which yields (see $(TC_\varepsilon)_1$)

$$\|u(b, \cdot, \varepsilon)\|_{C[0,T]} = O(1).$$

Finally, by (3.2)₃ and (3.6) there exists a constant M such that

$$\|v_x(\cdot, t, \varepsilon)\|_{C[b,c]} \leq M \quad \text{for a.a. } t \in (0, T),$$

which in particular implies that

$$\|v_x(b, \cdot, \varepsilon)\|_{L^\infty(0,T)} = O(1).$$

Q.E.D.

Theorem 3.1. *If all the assumptions of Corollary 2.1 are fulfilled, then the following estimates hold true*

$$\|r_\varepsilon\|_{C([a,b] \times [0,T])} = O(\varepsilon^{1/8}), \quad \|s_\varepsilon\|_{C([b,c] \times [0,T])} = O(\varepsilon^{3/8}). \quad (3.7)$$

Proof By Corollary 2.1 we have (see also (3.1))

$$R_\varepsilon(t) := (r_\varepsilon(\cdot, t), s_\varepsilon(\cdot, t)) \in W^{1,\infty}(0, T; H) \cap W^{1,2}(0, T; H^1(a, b) \times H^1(b, c)) \\ \cap L^\infty(0, T; H^2(a, b) \times H^2(b, c)).$$

Therefore, taking into account P_ε and P_0 , we can see that $(r_\varepsilon, s_\varepsilon)$ is a strong solution of the problem

$$\begin{cases} r_{\varepsilon t} - \varepsilon r_{\varepsilon xx} + \alpha r_{\varepsilon x} + \beta r_\varepsilon = \varepsilon U_{xx} & \text{in } (a, b) \times (0, T), \\ s_{\varepsilon t} - (\mu s_{\varepsilon x})_x + \alpha s_{\varepsilon x} + \beta s_\varepsilon = 0 & \text{in } (b, c) \times (0, T), \end{cases} \quad (3.8)$$

$$r_\varepsilon(x, 0) = 0, \quad a \leq x \leq b, \quad s_\varepsilon(x, 0) = 0, \quad b \leq x \leq c, \quad (3.9)$$

$$\begin{cases} r_{\varepsilon x}(a, t) = -U_x(a, t), \\ s_\varepsilon(c, t) = 0, \quad 0 \leq t \leq T, \end{cases} \quad (3.10)$$

$$\begin{cases} r_\varepsilon(b, t) = s_\varepsilon(b, t), \\ \varepsilon r_{\varepsilon x}(b, t) - (\mu s_{\varepsilon x})(b, t) = -\varepsilon U_x(b, t), \quad 0 \leq t \leq T. \end{cases} \quad (3.11)$$

Let us multiply in H the system (3.8) by $R_\varepsilon(t)$ and then integrate the resulting equation over $[0, t]$:

$$\begin{aligned} & (1/2) \|R_\varepsilon(t)\|^2 + \varepsilon \|r_{\varepsilon x}\|_{L^2((a,b) \times (0,t))}^2 + \mu_0 \|s_{\varepsilon x}\|_{L^2((b,c) \times (0,t))}^2 \\ & \leq \varepsilon \int_0^t |U_x(b, s) r_\varepsilon(b, s)| ds + \varepsilon \int_0^t \|U_{xx}(\cdot, s)\|_{L^2(a,b)} \|r_\varepsilon(\cdot, s)\|_{L^2(a,b)} ds \\ & \quad + \varepsilon \int_0^t |U(a, s) r_\varepsilon(a, s)| ds. \end{aligned} \quad (3.12)$$

By Lemma 3.1 (see also Corollary 2.1), we can derive from (3.12) the following inequality

$$(1/2) \|R_\varepsilon(t)\|^2 \leq M\varepsilon (1 + \int_0^t \|r_\varepsilon(\cdot, s)\|_{L^2(a,b)} ds),$$

which implies by Gronwall's lemma

$$\begin{cases} \|r_\varepsilon(\cdot, t)\|_{L^2(a,b)} = O(\varepsilon^{1/2}), \\ \|s_\varepsilon(\cdot, t)\|_{L^2(b,c)} = O(\varepsilon^{1/2}). \end{cases} \quad (3.13)$$

We multiply again the system (3.8) by $R_\varepsilon(t)$ and thus obtain by a simple computation

$$\begin{aligned} \varepsilon \|r_{\varepsilon x}(\cdot, t)\|_{L^2(a,b)}^2 + \mu_0 \|s_{\varepsilon x}(\cdot, t)\|_{L^2(b,c)}^2 &\leq \|R'_\varepsilon(\cdot, t)\| \cdot \|R_\varepsilon(\cdot, t)\| \\ + \varepsilon |U_x(a, t)r_\varepsilon(a, t)| + \|U_{xx}(\cdot, t)\|_{L^2(a,b)} \|r_\varepsilon(\cdot, t)\|_{L^2(a,b)} &+ \varepsilon |U_x(b, t)r_\varepsilon(b, t)|. \end{aligned}$$

By Lemma 3.1 and (3.13) this leads to the following estimates

$$\|r_{\varepsilon x}(\cdot, t)\|_{L^2(a,b)} = O(\varepsilon^{-1/4}), \quad \|s_{\varepsilon x}(\cdot, t)\|_{L^2(b,c)} = O(\varepsilon^{1/4}). \quad (3.14)$$

Using (3.13) and the mean value theorem, we can associate with each (t, ε) a number $m_{t\varepsilon} \in [a, b]$ such that $|r_\varepsilon(m_{t\varepsilon}, t)| = O(\varepsilon^{1/2})$. Now, taking into account the following formulas

$$\begin{aligned} r_\varepsilon^2(x, t) &= 2 \int_{m_{t\varepsilon}}^x r_\varepsilon(y, t)r_{\varepsilon y}(y, t)dy + r_\varepsilon^2(m_{t\varepsilon}, t), \\ s_\varepsilon^2(x, t) &= -2 \int_x^c s_\varepsilon(y, t)s_{\varepsilon y}(y, t)dy \end{aligned}$$

as well as the estimates (3.13) and (3.14), we get (3.7).

Q.E.D.

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