

A higher-order nonlocal boundary value problem

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Abstract

We study positive solutions of the multi-point higher order ($m \geq 2$) nonlinear boundary-value problem

$$\begin{aligned}(-1)^m u^{(2m)}(t) &= a(t)f(t, u(t), u'(t), \dots, u^{(2m-1)}(t)), \quad 0 < t < 1, \\ u^{(2i+1)}(0) &= 0, \quad \alpha_i u^{(2i)}(\eta_i) = u^{(2i)}(1), \quad i = 0, \dots, m-1,\end{aligned}$$

where $0 < \alpha_i, \eta_i < 1$, $i = 0, \dots, m-1$. As an example, we give sufficient conditions for the existence of triple solutions for $m = 2$, in which case our problem is interpreted as a nonlocal elastic beam problem.

2000 Mathematics Subject Classifications: 34B10, 34B16, 34B18.

Key words: Green's function, fixed point theorem, positive solutions, multi-point boundary value problem.

1. Introduction

We consider first the nonlinear multi-point boundary value problem

$$(-1)^m u^{(2m)}(t) = a(t)f(t, u(t), u'(t), \dots, u^{(2m-1)}(t)), \quad 0 < t < 1, \quad (1)$$

$$u^{(2i+1)}(0) = 0, \quad \alpha_i u^{(2i)}(\eta_i) = u^{(2i)}(1), \quad i = 0, \dots, m-1, \quad (2)$$

where $0 < \alpha_i, \eta_i < 1$, $i = 0, \dots, m-1$. We provide sufficient conditions for the existence of triple positive solutions. Subsequently, the special case of (1), (2) corresponding to $m = 2$ receives our attention.

One of the most important equations in material science is the linear elastic beam equation (Euler-Bernoulli equation)

$$\frac{d^2}{dt^2} \left(EI \frac{d^2 u(t)}{dt^2} \right) = f(t), \quad t \in (0, L),$$

where $u(t)$ is the deformation (deflection) function, L is the length of the beam, $f(t)$ is the load density, E is the Young's modulus of elasticity, and I is the moment of inertia of the cross-section of the beam. The physical meaning of the derivatives of the displacement function $u(t)$ is as follows: $\frac{d^4 u(t)}{dt^4}$ is the load density stiffness, $\frac{d^3 u(t)}{dt^3}$ is the shear force stiffness, $\frac{d^2 u(t)}{dt^2}$ is the bending moment stiffness, and $\frac{du(t)}{dt}$ is the slope. If we assume the payload depends on higher order derivatives we obtain our nonlinear model equation (1) for $m = 2$. The boundary value problem (1), (2) with $m = 2$ is of practical interest since it can serve as a nonlocal model of an elastic beam. The elastic beam problems have been analyzed by many authors (see, e.g., [10, 11, 13, 16, 17, 18]).

The fixed point theorem of Avery and Peterson [4] was used by many authors (see, e.g., [2, 3, 4, 5, 7, 9, 12]) to obtain triplets of positive solutions for a variety of boundary value problems. This theorem is based on the degree theory for a cone in a Banach space and generalizes a well-known theorem due to Leggett and Williams [14]. For some applications of the latter we refer the reader to [13] and the references therein.

The Avery-Peterson theorem was applied by Bai *et al* [7] to the conjugate and focal second order boundary value problems and by Kosmatov [12] to a symmetric second order multi-point problem. In the above mentioned works, the authors considered the inhomogeneous term with the dependence on the first order derivative. The results of [7] were extended by Kaufmann and Kosmatov in [11], where the fourth order conjugate type problem was studied with the inhomogeneous term that depended on the derivatives up to the third order. Both [7] and [11] dealt with two-point boundary value problems.

Positive solutions of higher order problems have been studied by many authors (see, e. g., [1, 6, 8, 11]). Recently, Liu *et al* [15] considered the $2m$ -th order multi-point boundary value problem

$$\begin{aligned} u^{(2m)}(t) &= f(t, u(t), u''(t), \dots, u^{(2m-2)}(t)), \quad 0 < t < 1, \\ u^{(2i+1)}(0) &= 0, \quad \sum_{j=1}^{l-2} k_{ij} u^{(2i)}(\xi_j) = u^{(2i)}(1), \quad i = 0, \dots, m-1, \end{aligned}$$

where $k_{ij} > 0$ for all $i = 1, \dots, m-1$ and $j = 1, \dots, l-2$, $0 < \xi_1 < \dots < \xi_{l-2} < 1$ and $\sum_{j=1}^{l-2} k_{ij} < 1$. The authors of [15] established the existence of triple positive solutions using the Leggett-Williams fixed point theorem. In our paper we generalize the results of [15] (for $l = 3$) by considering the inhomogeneous term that depends on the derivatives of all orders up to $2m - 1$.

2. Preliminaries

Consider the linear boundary value problem

$$-u''(t) = g(t), \quad 0 < t < 1, \quad (3)$$

$$u'(0) = 0, \quad \alpha u(\eta) = u(1), \quad (4)$$

$0 < \alpha, \eta < 1$. The following lemma is easy to show.

Lemma 2.1. *If $g \in C[0, 1]$ and $g(t) \geq 0$ on $[0, 1]$, then*

$$u(t) = - \int_0^t (t-s)g(s)ds + \frac{1}{1-\alpha} \int_0^1 (1-s)g(s)ds - \frac{\alpha}{1-\alpha} \int_0^\eta (\eta-s)g(s)ds$$

is the unique nonnegative solution on $[0, 1]$ of the problem (3), (4).

The proof of the following lemma is based on a generalized concavity argument.

Lemma 2.2. *Let the function $v \in C^{(2m)}[0, 1]$ satisfy $(-1)^m v^{(2m)}(t) \geq 0$ and (2), then $v(t)$ is nonnegative,*

$$\max_{t \in [0,1]} |v^{(2i+1)}(t)| \geq \kappa_i \max_{t \in [0,1]} |v^{(2i)}(t)|, \quad i = 0, \dots, m-1, \quad (5)$$

where

$$\kappa_i = \frac{1 - \alpha_i}{1 - \alpha_i \eta_i}$$

and

$$\max_{t \in [0,1]} |v^{(2i)}(t)| \geq \max_{t \in [0,1]} |v^{(2i-1)}(t)|, \quad i = 1, \dots, m-1. \quad (6)$$

Finally,

$$\min_{t \in [0,1]} |v^{(2i)}(t)| \geq \rho_i \max_{t \in [0,1]} |v^{(2i)}(t)|, \quad i = 0, \dots, m-1, \quad (7)$$

where

$$\rho_i = \frac{\alpha_i(1 - \eta_i)}{1 - \alpha_i \eta_i}.$$

Proof: First, let m be odd. In Lemma 2.1 replace $u(t)$ with $v^{(2m-2)}(t)$. Then the function $v^{(2m-2)}(t)$ is nonnegative, nonincreasing and concave, so that $\max_{t \in [0,1]} |v^{(2m-1)}(t)| = -v^{(2m-1)}(1)$, $\min_{t \in [0,1]} v^{(2m-2)}(t) = v^{(2m-2)}(1)$, and $\max_{t \in [0,1]} v^{(2m-2)}(t) = v^{(2m-2)}(0)$.

Observe that

$$\frac{v^{(2m-2)}(1) - v^{(2m-2)}(\eta_{m-1})}{1 - \eta_{m-1}} \leq \frac{v^{(2m-2)}(1) - v^{(2m-2)}(0)}{1 - 0},$$

which, since $\alpha_{m-1} v^{(2m-2)}(\eta_{m-1}) = v^{(2m-2)}(1)$, brings us to

$$v^{(2m-2)}(1) \geq \frac{\alpha_{m-1}(1 - \eta_{m-1})}{1 - \alpha_{m-1} \eta_{m-1}} v^{(2m-2)}(0)$$

and thus (7) is verified for $i = m - 1$.

Using the inequality above, we obtain

$$v^{(2m-1)}(1) \leq \frac{v^{(2m-2)}(1) - v^{(2m-2)}(\eta_{m-1})}{1 - \eta_{m-1}} = -\frac{1 - \alpha_{m-1}}{\alpha_{m-1}(1 - \eta_{m-1})} v^{(2m-2)}(1) \leq -\frac{1 - \alpha_{m-1}}{1 - \alpha_{m-1} \eta_{m-1}} v^{(2m-2)}(0),$$

that is,

$$-v^{(2m-1)}(1) \geq \frac{1 - \alpha_{m-1}}{1 - \alpha_{m-1} \eta_{m-1}} v^{(2m-2)}(0),$$

which proves the inequality (5) for $i = m - 1$.

Since $v^{(2m-3)}(0) = 0$ and since the function $v^{(2m-2)}(t)$ is nonnegative, nonincreasing and concave,

$$-v^{(2m-3)}(1) = -\int_0^1 v^{(2m-2)}(t) dt \geq -v^{(2m-2)}(0)$$

and, so

$$|v^{(2m-3)}(1)| \leq |v^{(2m-2)}(0)|,$$

which shows (6) for $i = m - 1$. Applying the above arguments repeatedly we obtain (5), (6), and (7) for all i . The case of even m receives the same treatment to complete the proof.

□

Setting $g \equiv 1$ in Lemma 2.1 yields

$$\rho' = \frac{\min_{t \in [0,1]} u(t)}{\max_{t \in [0,1]} u(t)} = \frac{\alpha(1 - \eta^2)}{1 - \alpha\eta^2}.$$

Observe also that comparing ρ_i from Lemma 2.2 to ρ'_i (corresponding to α_i and η_i), we have $\rho_i < \rho'_i$. This shows that the estimate (7) is not sharp.

The Green's function of $-u''(t) = 0$ with (4) is given by

$$G(t, s; \alpha, \eta) = \frac{1-s}{1-\alpha} - \begin{cases} t-s, & s \leq t \\ 0, & s > t \end{cases} - \frac{\alpha}{1-\alpha} \begin{cases} \eta-s, & s \leq \eta \\ 0, & s > \eta. \end{cases}$$

For $i = 0, \dots, m - 1$, we denote $G(t, s; \alpha_i, \eta_i)$ by $G_i(t, s)$. Define recursively the integral kernels $G^i(t, s)$, $i = 1, \dots, m - 1$, by

$$G^1(t, s) = G_{m-1}(t, s), \quad G^{i+1}(t, s) = \int_0^1 G_{m-1-i}(t, \tau) G^i(\tau, s) d\tau. \tag{8}$$

Consider the equation

$$(-1)^m u^{(2m)}(t) = g(t) \tag{9}$$

along with the boundary conditions (2). The following lemma is immediate.

Lemma 2.3. *If $g \in C[0, 1]$ and $g(t) \geq 0$ on $[0, 1]$, then*

$$u_m(t) = \int_0^1 G^m(t, s)g(s) ds$$

is the unique nonnegative solution of the problem (9), (2).

The following are the standing assumptions on the inhomogeneous term of (1):

(A1) $f \in C([0, 1] \times \mathbb{R}^{2m}, [0, \infty))$;

(A2) $a(t) \geq 0$ and $a(t) > 0$ a.e. on $[0, 1]$;

(A3) $a(t) \in L[0, 1]$ is singular.

Let \mathcal{B} be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be closed and nonempty. Then \mathcal{C} is said to be a cone if

1. $\alpha u + \beta v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$ and for all $\alpha, \beta \geq 0$, and
2. $u, -u \in \mathcal{C}$ implies $u \equiv 0$.

Let the space $\mathcal{B} = C^{2m-1}[0, 1]$ endowed with the norm $\|u\| = \max_{1 \leq k \leq 2m-1} \|u^{(k)}\|_0$, where $\|u\|_0 = \max_{[0,1]} |u(t)|$, be our Banach space.

We define the cone $\mathcal{K} \subset \mathcal{B}$ by

$$\mathcal{K} = \{u \in \mathcal{B} : (-1)^m u^{(2m)}(t) \geq 0 \text{ on } [0, 1] \text{ and satisfies (2)}\}.$$

We define the integral operator $T : \mathcal{B} \rightarrow \mathcal{B}$ associated with (1), (2) by

$$Tu(t) = \int_0^1 G^m(t, s)a(s)f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) ds.$$

By Arzela-Ascoli theorem, the operator T is completely continuous. In view of (A1) and (A2), fixed points of T are positive solutions of (1), (2).

We say that the map β is a nonnegative continuous concave (convex) functional on a cone \mathcal{K} of a real Banach space \mathcal{B} provided that $\beta : \mathcal{K} \rightarrow [0, \infty)$ is continuous and

$$\beta(tu + (1-t)v) \geq (\leq) t\beta(u) + (1-t)\beta(v)$$

for all $u, v \in \mathcal{K}$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on \mathcal{K} , α be a nonnegative continuous concave functional on \mathcal{K} , and ψ be a nonnegative continuous functional on \mathcal{K} . Then for positive real numbers a, b, c , and d we define the following convex sets:

$$P(\gamma, d) = \{u \in \mathcal{K} : \gamma(u) < d\},$$

$$P(\gamma, \alpha, b, d) = \{u \in \mathcal{K} : b \leq \alpha(u), \gamma(u) \leq d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{u \in \mathcal{K} : b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{u \in \mathcal{K} : a \leq \psi(u), \gamma(u) \leq d\}.$$

Now we state the fixed-point theorem due to Avery and Peterson.

Theorem 2.4. *Let \mathcal{K} be a cone in a real Banach space \mathcal{B} . Let γ and θ be nonnegative continuous convex functionals on \mathcal{K} , α be a nonnegative continuous concave functional on \mathcal{K} , and ψ be a nonnegative continuous functional on \mathcal{K} satisfying $\psi(\lambda u) \leq \lambda \psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d ,*

$$\alpha(u) \leq \psi(u) \text{ and } \|u\| \leq M\gamma(u),$$

for all $u \in \overline{P(\gamma, d)}$. Suppose $T: \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b , and c with $a < b$ such that

- (S1) $\{u \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for $u \in P(\gamma, \theta, \alpha, b, c, d)$;
- (S2) $\alpha(Tu) > b$ for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$;
- (S3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tu) < a$ for $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$.

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$ such that $b < \alpha(u_1)$, $a < \psi(u_2)$ with $\alpha(u_2) < b$, and $\psi(u_3) < a$.

3. Positive solutions

We define the following nonnegative continuous functionals on the cone \mathcal{K} : $\alpha(u) = \min_{t \in [0,1]} |u^{(2j)}(t)|$, $\theta(u) = \psi(u) = \max_{t \in [0,1]} |u^{(2j)}(t)|$, and $\gamma(u) = \max_{t \in [0,1]} |u^{(2m-1)}(t)|$, for each $j = 0, \dots, m-1$. It is obvious that the functionals ψ and γ are convex and the functional α is concave. Also note that $\alpha(u) \leq \psi(u)$. Furthermore, if we set

$$M = \prod_{i=1}^m \frac{1}{\kappa_i},$$

then, by Lemma 2.2, $\|u\| = \max_{1 \leq k \leq 2m-1} \|u^{(k)}\|_0 \leq M \|u^{(2m-1)}\|_0 = M\gamma(u)$ for all $u \in \mathcal{K}$. To this end, the standing assumptions of Theorem 2.4 are satisfied.

For a, b with $a < b$ and $n \geq 0$, by $(-1)^n[a, b]$ we understand the interval $[-b, -a]$, if n is odd and $[a, b]$, if n is even.

Theorem 3.1 ($j = 0, \dots, m-1$). *Suppose (A1)-(A3) hold. Let $d_i, i = 0, \dots, 2m-1$, be constants such that $d_{2i} \leq \frac{d_{2i+1}}{\kappa_i}, i = 0, \dots, m-1, d_{2i-1} \leq d_{2i}, i = 1, \dots, m-1$, and let a and b be constants such that $a < b \leq \frac{\rho_j}{\kappa_j} d_{2j+1}$.*

Assume that the following hypotheses are satisfied

(H1) $f(t, z_0, z_1, \dots, z_{2m-1}) \leq \frac{d_{2m-1}}{D}$ for all $t \in [0, 1]$, $z_i \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$, $i = 0, 1, \dots, 2m-1$, where

$$D = \int_0^1 a(t) dt,$$

(H2) $f(t, z_0, z_1, \dots, z_{2m-1}) > \frac{b}{B}$ for all $t \in [0, 1]$, $z_i \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$, $i = 0, \dots, 2j-1, 2j+1, \dots, 2m-1$, $|z_{2j}| \in [b, \frac{b}{\rho_j}]$, where

$$B = \int_0^1 G^{m-j}(1, t) a(t) dt$$

(H3) $f(t, z_0, z_1, \dots, z_{2m-1}) < \frac{a}{A}$ for $t \in [0, 1]$, $z_i \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$, $i = 0, \dots, 2j-1, 2j+1, \dots, 2m-1$, $|z_{2j}| \in [0, a]$, where

$$A = \int_0^1 G^{m-j}(0, t) a(t) dt.$$

Then the boundary value problem (1), (2) has at least three positive solutions $u_1, u_2, u_3 \in \mathcal{K}$ satisfying $b < |u_1^{(2j)}(1)|$, $a < |u_2^{(2j)}(0)|$ with $|u_2^{(2j)}(1)| < b$, $|u_3^{(2j)}(0)| < a$, and $|u_i^{(2m-1)}(1)| \leq d_{2m-1}$, $i = 1, 2, 3$.

Proof: Let $u \in \overline{P(\gamma, d_{2m-1})}$, that is, $0 \leq |u^{(2m-1)}(s)| \leq \gamma(u) = \max_{s \in [0, 1]} |u^{(2m-1)}(s)| = |u^{(2m-1)}(1)| \leq d_{2m-1}$ for all $s \in [0, 1]$. Then, by Lemma 2.2, $u^{(i)}(s) \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$, $i = 0, \dots, 2m-1$, for all $s \in [0, 1]$. Hence, by (H1), $f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) \leq \frac{d_{2m-1}}{D}$ for all $s \in [0, 1]$ and we have

$$\begin{aligned} \gamma(Tu) &= \max_{t \in [0, 1]} |(Tu)^{(2m-1)}(t)| \\ &= |(Tu)^{(2m-1)}(1)| \\ &= \int_0^1 a(s) f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) ds \\ &\leq \int_0^1 a(s) ds \frac{d_{2m-1}}{D} \\ &\leq d_{2m-1}, \end{aligned}$$

that is, $T : \overline{P(\gamma, d_{2m-1})} \rightarrow \overline{P(\gamma, d_{2m-1})}$.

Consider the auxiliary equation

$$(-1)^{j+1} v^{(2j+2)}(s) = 1$$

with the boundary conditions (2) for $i = 0, \dots, j$, whose unique positive solution (by Lemma 2.3 with $j+1$ in place of m) is

$$v(s) = \int_0^1 G^{j+1}(s, \tau) d\tau,$$

so that

$$v^{(2j)}(s) = \int_0^1 G_j(s, \tau) d\tau.$$

Set

$$u(s) = \frac{b}{\rho_j \|v^{(2j)}\|_0} v(s).$$

Clearly, $u \in \mathcal{K}$ and

$$\theta(u) = \max_{s \in [0,1]} |u^{(2j)}(s)| = \frac{b}{\rho_j \|v^{(2j)}\|_0} \max_{s \in [0,1]} |v^{(2j)}(s)| = \frac{b}{\rho_j}. \quad (10)$$

By remark following the proof of Lemma 2.2

$$\begin{aligned} \alpha(u) &= \frac{b}{\rho_j \|v^{(2j)}\|_0} \min_{s \in [0,1]} \left| \int_0^1 G_j(s, \tau) d\tau \right| \\ &= \frac{b}{\rho_j \|v^{(2j)}\|_0} \rho'_j \|v^{(2j)}\|_0 \\ &= b \frac{\rho'_j}{\rho_j} \\ &> b. \end{aligned} \quad (11)$$

Note that since $u(s)$ is a polynomial of degree $2j + 2$, then

$$\gamma(u) = \max_{s \in [0,1]} |u^{(2m-1)}(s)| = 0$$

for all $j = 0, \dots, m - 2$. If $j = m - 1$, then

$$\gamma(u) = \max_{s \in [0,1]} |u^{(2m-1)}(s)| = \max_{s \in [0,1]} \left| (-1)^m \frac{b}{\rho_{m-1} \|v^{(2m-2)}\|_0} s \right| = \frac{b}{\rho_{m-1} \|v^{(2m-2)}\|_0} \leq d_{2m-1}.$$

We obtain, for all $j = 0, \dots, m - 1$, that $\gamma(u) \leq d_{2m-1}$, which, together with (10) and (11), shows that

$$\{u \in P(\gamma, \theta, \alpha, b, \frac{b}{\rho_j}, d_{2m-1}) : \alpha(u) > b\} \neq \emptyset.$$

If $u \in P(\gamma, \theta, \alpha, b, \frac{1}{\rho_j} b, d_{2m-1})$, then $b \leq |u^{(2j)}(s)| \leq \frac{1}{\rho_j} b$ and $|u^{(2m-1)}(s)| \leq d_{2m-1}$ for all $s \in [0, 1]$. Then, by (H2),

$$\begin{aligned} \alpha(Tu) &= \min_{t \in [0,1]} |(Tu)^{(2j)}(t)| \\ &= |(Tu)^{(2j)}(1)| \\ &= \int_0^1 G^{m-j}(1, s) a(s) f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) ds \\ &> \int_0^1 G^{m-j}(1, s) a(s) ds \frac{b}{B} \\ &= b, \end{aligned}$$

which shows that the condition (S1) of Theorem 2.4 is fulfilled since $\alpha(Tu) > b$ for all $u \in P(\gamma, \theta, \alpha, b, \frac{b}{\rho_j}, d_{2m-1})$.

Let $u \in P(\gamma, \alpha, b, d_{2m-1})$ with $\theta(Tu) > \frac{b}{\rho_j}$. Since the operator T is cone preserving, it follows from Lemma 2.2 that $\alpha(Tu) \geq \rho_j \theta(Tu) > \rho_j \frac{b}{\rho_j} = b$. So, the condition (S2) of Theorem 2.4 holds.

Finally, letting $u \in R(\gamma, \psi, a, d_{2m-1})$ with $\psi(u) = a$ (note that $0 \notin R(\gamma, \psi, a, d_{2m-1})$ since $\psi(0) = 0$). By (H3), we obtain that

$$\begin{aligned} \psi(Tu) &= \max_{t \in [0,1]} |(Tu)^{(2j)}(t)| \\ &= |(Tu)^{(2j)}(0)| \\ &= \int_0^1 G^{m-j}(0,s)a(s)f(s,u(s),u'(s),\dots,u^{(2m-1)}(s)) ds \\ &< \int_0^1 G^{m-j}(0,s)a(s) ds \frac{a}{A} \\ &= a. \end{aligned}$$

The condition (S3) of Theorem 2.4 is now satisfied.

The assumptions of Theorem 2.4 are verified and we assert the existence of at least three positive solutions $u_1, u_2, u_3 \in \mathcal{K}$ such that $b < |u_1^{(2j)}(1)|, a < |u_2^{(2j)}(0)|$ with $|u_2^{(2j)}(1)| < b, |u_3^{(2j)}(0)| < a$, and $|u_i^{(2m-1)}(1)| \leq d_{2m-1}, i = 1, 2, 3$.

□

As an example we give the existence conditions for $m = 2$.

In the sequel of the paper we assume for simplicity that $a(t) = 1$. Our boundary value problem takes shape of

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{12}$$

$$u^{(2i+1)}(0) = 0, \quad \alpha_i u^{(2i)}(\eta_i) = u^{(2i)}(1), \quad i = 0, 1, \tag{13}$$

where $\alpha_i = \eta_i = \frac{1}{2}, i = 0, 1$. The solution of the linearized problem to the problem (12), (13), given by Lemma 2.3 with $g \equiv 1$, is

$$\begin{aligned} u_2(t) &= \int_0^1 G^2(t,s) ds \\ &= \frac{1}{24}t^4 - \frac{1 - \alpha_1 \eta_1^2}{4(1 - \alpha_1)}t^2 + \frac{(1 - \alpha_0 \eta_0^2)(1 - \alpha_1 \eta_1^2)}{4(1 - \alpha_0)(1 - \alpha_1)} - \frac{(1 - \alpha_0 \eta_0^4)}{24(1 - \alpha_0)} \\ &= \frac{1}{24}t^4 - \frac{7}{16}t^2 + \frac{263}{384}. \end{aligned}$$

The constants from Theorem 3.1 corresponding to $m = 2, j = 1$ and $a(t) = 1$ are given by (recall (8))

$$A = \int_0^1 G^{m-j}(0,t) dt = \int_0^1 G_1(0,t) dt = \max_{t \in [0,1]} |u_2''(t)| = \frac{7}{8},$$

$$B = \int_0^1 G^{m-j}(1,t) dt = \int_0^1 G_1(1,t) dt = \min_{t \in [0,1]} |u_2''(t)| = \frac{3}{8},$$

and

$$D = \int_0^1 a(t) dt = 1.$$

Let

$$\phi(z) = \begin{cases} \frac{1}{5}z, & z \in [0, 1], \\ 8z - \frac{39}{5}, & z \in (1, \frac{3}{2}], \\ \frac{2}{7}z + \frac{132}{35}, & z \in (\frac{3}{2}, 12]. \end{cases}$$

Consider the inhomogeneous term in the form

$$f(t, z_0, z_1, z_2, z_3) = \frac{1}{5}t^2 + \frac{1}{1620}z_0^2 + \frac{1}{720}z_1^2 + \phi(|z_2|) + \frac{\sqrt{2|z_3|}}{20}$$

and let $a = 1$, $b = \frac{3}{2}$, $d_0 = 18$, $d_1 = 12$, $d_2 = 12$, and $d_3 = 8$. Then the function f satisfies the following inequalities:

1. $f(t, z_0, z_1, z_2, z_3) \leq 8$, $(t, z_0, z_1, z_2, z_3) \in [0, 1] \times [0, 18] \times [-12, 0] \times [-12, 0] \times [0, 8]$;
2. $f(t, z_0, z_1, z_2, z_3) > 4$, $(t, z_0, z_1, z_2, z_3) \in [0, 1] \times [0, 18] \times [-12, 0] \times [-\frac{9}{2}, -\frac{3}{2}] \times [0, 8]$;
3. $f(t, z_0, z_1, z_2, z_3) < \frac{8}{7}$, $(t, z_0, z_1, z_2, z_3) \in [0, 1] \times [0, 18] \times [-12, 0] \times [-1, 0] \times [0, 8]$.

The above inequalities are consistent with the hypotheses (H1)-(H3) of Theorem 3.1 and we obtain three positive solutions satisfying $u_1''(1) < -\frac{3}{2}$, $u_2''(0) < -1$ with $u_2''(1) > -\frac{3}{2}$, $u_3''(0) > -1$, and $u_i'''(1) \leq 8$, $i = 1, 2, 3$.

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