

## A higher-order nonlocal boundary value problem

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### Abstract

We study positive solutions of the multi-point higher order ( $m \geq 2$ ) nonlinear boundary-value problem

$$\begin{aligned}(-1)^m u^{(2m)}(t) &= a(t)f(t, u(t), u'(t), \dots, u^{(2m-1)}(t)), & 0 < t < 1, \\ u^{(2i+1)}(0) &= 0, \quad \alpha_i u^{(2i)}(\eta_i) = u^{(2i)}(1), & i = 0, \dots, m-1,\end{aligned}$$

where  $0 < \alpha_i, \eta_i < 1$ ,  $i = 0, \dots, m-1$ . As an example, we give sufficient conditions for the existence of triple solutions for  $m = 2$ , in which case our problem is interpreted as a nonlocal elastic beam problem.

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### 1. Introduction

We consider first the nonlinear multi-point boundary value problem

$$(-1)^m u^{(2m)}(t) = a(t)f(t, u(t), u'(t), \dots, u^{(2m-1)}(t)), \quad 0 < t < 1, \quad (1)$$

$$u^{(2i+1)}(0) = 0, \quad \alpha_i u^{(2i)}(\eta_i) = u^{(2i)}(1), \quad i = 0, \dots, m-1, \quad (2)$$

where  $0 < \alpha_i, \eta_i < 1$ ,  $i = 0, \dots, m-1$ . We provide sufficient conditions for the existence of triple positive solutions. Subsequently, the special case of (1), (2) corresponding to  $m = 2$  receives our attention.

One of the most important equations in material science is the linear elastic beam equation (Euler-Bernoulli equation)

$$\frac{d^2}{dt^2} \left( EI \frac{d^2 u(t)}{dt^2} \right) = f(t), \quad t \in (0, L),$$

where  $u(t)$  is the deformation (deflection) function,  $L$  is the length of the beam,  $f(t)$  is the load density,  $E$  is the Young's modulus of elasticity, and  $I$  is the moment of inertia of the cross-section of the beam. The physical meaning of the derivatives of the displacement function  $u(t)$  is as follows:  $\frac{d^4 u(t)}{dt^4}$  is the load density stiffness,  $\frac{d^3 u(t)}{dt^3}$  is the shear force stiffness,  $\frac{d^2 u(t)}{dt^2}$  is the bending moment stiffness, and  $\frac{du(t)}{dt}$  is the slope. If we assume the payload depends on higher order derivatives we obtain our nonlinear model equation (1) for  $m = 2$ . The boundary value problem (1), (2) with  $m = 2$  is of practical interest since it can serve as a nonlocal model of an elastic beam. The elastic beam problems have been analyzed by many authors (see, e.g., [10, 11, 13, 16, 17, 18]).

The fixed point theorem of Avery and Peterson [4] was used by many authors (see, e.g., [2, 3, 4, 5, 7, 9, 12]) to obtain triplets of positive solutions for a variety of boundary value problems. This theorem is based on the degree theory for a cone in a Banach space and generalizes a well-known theorem due to Leggett and Williams [14]. For some applications of the latter we refer the reader to [13] and the references therein.

The Avery-Peterson theorem was applied by Bai *et al* [7] to the conjugate and focal second order boundary value problems and by Kosmatov [12] to a symmetric second order multi-point problem. In the above mentioned works, the authors considered the inhomogeneous term with the dependence on the first order derivative. The results of [7] were extended by Kaufmann and Kosmatov in [11], where the fourth order conjugate type problem was studied with the inhomogeneous term that depended on the derivatives up to the third order. Both [7] and [11] dealt with two-point boundary value problems.

Positive solutions of higher order problems have been studied by many authors (see, e. g., [1, 6, 8, 11]). Recently, Liu *et al* [15] considered the  $2m$ -th order multi-point boundary value problem

$$\begin{aligned} u^{(2m)}(t) &= f(t, u(t), u''(t), \dots, u^{(2m-2)}(t)), \quad 0 < t < 1, \\ u^{(2i+1)}(0) &= 0, \quad \sum_{j=1}^{l-2} k_{ij} u^{(2i)}(\xi_j) = u^{(2i)}(1), \quad i = 0, \dots, m-1, \end{aligned}$$

where  $k_{ij} > 0$  for all  $i = 1, \dots, m-1$  and  $j = 1, \dots, l-2$ ,  $0 < \xi_1 < \dots < \xi_{l-2} < 1$  and  $\sum_{j=1}^{l-2} k_{ij} < 1$ . The authors of [15] established the existence of triple positive solutions using the Leggett-Williams fixed point theorem. In our paper we generalize the results of [15] (for  $l = 3$ ) by considering the inhomogeneous term that depends on the derivatives of all orders up to  $2m - 1$ .

## 2. Preliminaries

Consider the linear boundary value problem

$$-u''(t) = g(t), \quad 0 < t < 1, \quad (3)$$

$$u'(0) = 0, \quad \alpha u(\eta) = u(1), \quad (4)$$

$0 < \alpha, \eta < 1$ . The following lemma is easy to show.

**Lemma 2.1.** *If  $g \in C[0, 1]$  and  $g(t) \geq 0$  on  $[0, 1]$ , then*

$$u(t) = - \int_0^t (t-s)g(s)ds + \frac{1}{1-\alpha} \int_0^1 (1-s)g(s)ds - \frac{\alpha}{1-\alpha} \int_0^\eta (\eta-s)g(s)ds$$

is the unique nonnegative solution on  $[0, 1]$  of the problem (3), (4).

The proof of the following lemma is based on a generalized concavity argument.

**Lemma 2.2.** *Let the function  $v \in C^{(2m)}[0, 1]$  satisfy  $(-1)^m v^{(2m)}(t) \geq 0$  and (2), then  $v(t)$  is nonnegative,*

$$\max_{t \in [0,1]} |v^{(2i+1)}(t)| \geq \kappa_i \max_{t \in [0,1]} |v^{(2i)}(t)|, \quad i = 0, \dots, m-1, \quad (5)$$

where

$$\kappa_i = \frac{1 - \alpha_i}{1 - \alpha_i \eta_i}$$

and

$$\max_{t \in [0,1]} |v^{(2i)}(t)| \geq \max_{t \in [0,1]} |v^{(2i-1)}(t)|, \quad i = 1, \dots, m-1. \quad (6)$$

Finally,

$$\min_{t \in [0,1]} |v^{(2i)}(t)| \geq \rho_i \max_{t \in [0,1]} |v^{(2i)}(t)|, \quad i = 0, \dots, m-1, \quad (7)$$

where

$$\rho_i = \frac{\alpha_i(1 - \eta_i)}{1 - \alpha_i \eta_i}.$$

**Proof:** First, let  $m$  be odd. In Lemma 2.1 replace  $u(t)$  with  $v^{(2m-2)}(t)$ . Then the function  $v^{(2m-2)}(t)$  is nonnegative, nonincreasing and concave, so that  $\max_{t \in [0,1]} |v^{(2m-1)}(t)| = -v^{(2m-1)}(1)$ ,  $\min_{t \in [0,1]} v^{(2m-2)}(t) = v^{(2m-2)}(1)$ , and  $\max_{t \in [0,1]} v^{(2m-2)}(t) = v^{(2m-2)}(0)$ .

Observe that

$$\frac{v^{(2m-2)}(1) - v^{(2m-2)}(\eta_{m-1})}{1 - \eta_{m-1}} \leq \frac{v^{(2m-2)}(1) - v^{(2m-2)}(0)}{1 - 0},$$

which, since  $\alpha_{m-1} v^{(2m-2)}(\eta_{m-1}) = v^{(2m-2)}(1)$ , brings us to

$$v^{(2m-2)}(1) \geq \frac{\alpha_{m-1}(1 - \eta_{m-1})}{1 - \alpha_{m-1} \eta_{m-1}} v^{(2m-2)}(0)$$

and thus (7) is verified for  $i = m - 1$ .

Using the inequality above, we obtain

$$v^{(2m-1)}(1) \leq \frac{v^{(2m-2)}(1) - v^{(2m-2)}(\eta_{m-1})}{1 - \eta_{m-1}} = -\frac{1 - \alpha_{m-1}}{\alpha_{m-1}(1 - \eta_{m-1})} v^{(2m-2)}(1) \leq -\frac{1 - \alpha_{m-1}}{1 - \alpha_{m-1} \eta_{m-1}} v^{(2m-2)}(0),$$

that is,

$$-v^{(2m-1)}(1) \geq \frac{1 - \alpha_{m-1}}{1 - \alpha_{m-1} \eta_{m-1}} v^{(2m-2)}(0),$$

which proves the inequality (5) for  $i = m - 1$ .

Since  $v^{(2m-3)}(0) = 0$  and since the function  $v^{(2m-2)}(t)$  is nonnegative, nonincreasing and concave,

$$-v^{(2m-3)}(1) = -\int_0^1 v^{(2m-2)}(t) dt \geq -v^{(2m-2)}(0)$$

and, so

$$|v^{(2m-3)}(1)| \leq |v^{(2m-2)}(0)|,$$

which shows (6) for  $i = m - 1$ . Applying the above arguments repeatedly we obtain (5), (6), and (7) for all  $i$ . The case of even  $m$  receives the same treatment to complete the proof.

□

Setting  $g \equiv 1$  in Lemma 2.1 yields

$$\rho' = \frac{\min_{t \in [0,1]} u(t)}{\max_{t \in [0,1]} u(t)} = \frac{\alpha(1 - \eta^2)}{1 - \alpha\eta^2}.$$

Observe also that comparing  $\rho_i$  from Lemma 2.2 to  $\rho'_i$  (corresponding to  $\alpha_i$  and  $\eta_i$ ), we have  $\rho_i < \rho'_i$ . This shows that the estimate (7) is not sharp.

The Green's function of  $-u''(t) = 0$  with (4) is given by

$$G(t, s; \alpha, \eta) = \frac{1 - s}{1 - \alpha} - \begin{cases} t - s, & s \leq t \\ 0, & s > t \end{cases} - \frac{\alpha}{1 - \alpha} \begin{cases} \eta - s, & s \leq \eta \\ 0, & s > \eta. \end{cases}$$

For  $i = 0, \dots, m - 1$ , we denote  $G(t, s; \alpha_i, \eta_i)$  by  $G_i(t, s)$ . Define recursively the integral kernels  $G^i(t, s)$ ,  $i = 1, \dots, m - 1$ , by

$$G^1(t, s) = G_{m-1}(t, s), \quad G^{i+1}(t, s) = \int_0^1 G_{m-1-i}(t, \tau) G^i(\tau, s) d\tau. \tag{8}$$

Consider the equation

$$(-1)^m u^{(2m)}(t) = g(t) \tag{9}$$

along with the boundary conditions (2). The following lemma is immediate.

**Lemma 2.3.** *If  $g \in C[0, 1]$  and  $g(t) \geq 0$  on  $[0, 1]$ , then*

$$u_m(t) = \int_0^1 G^m(t, s)g(s) ds$$

*is the unique nonnegative solution of the problem (9), (2).*

The following are the standing assumptions on the inhomogeneous term of (1):

(A1)  $f \in C([0, 1] \times \mathbb{R}^{2m}, [0, \infty))$ ;

(A2)  $a(t) \geq 0$  and  $a(t) > 0$  a.e. on  $[0, 1]$ ;

(A3)  $a(t) \in L[0, 1]$  is singular.

Let  $\mathcal{B}$  be a Banach space and let  $\mathcal{C} \subset \mathcal{B}$  be closed and nonempty. Then  $\mathcal{C}$  is said to be a cone if

1.  $\alpha u + \beta v \in \mathcal{C}$  for all  $u, v \in \mathcal{C}$  and for all  $\alpha, \beta \geq 0$ , and
2.  $u, -u \in \mathcal{C}$  implies  $u \equiv 0$ .

Let the space  $\mathcal{B} = C^{2m-1}[0, 1]$  endowed with the norm  $\|u\| = \max_{1 \leq k \leq 2m-1} \|u^{(k)}\|_0$ , where  $\|u\|_0 = \max_{[0,1]} |u(t)|$ , be our Banach space.

We define the cone  $\mathcal{K} \subset \mathcal{B}$  by

$$\mathcal{K} = \{u \in \mathcal{B} : (-1)^m u^{(2m)}(t) \geq 0 \text{ on } [0, 1] \text{ and satisfies (2)}\}.$$

We define the integral operator  $T : \mathcal{B} \rightarrow \mathcal{B}$  associated with (1), (2) by

$$Tu(t) = \int_0^1 G^m(t, s)a(s)f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) ds.$$

By Arzela-Ascoli theorem, the operator  $T$  is completely continuous. In view of (A1) and (A2), fixed points of  $T$  are positive solutions of (1), (2).

We say that the map  $\beta$  is a nonnegative continuous concave (convex) functional on a cone  $\mathcal{K}$  of a real Banach space  $\mathcal{B}$  provided that  $\beta : \mathcal{K} \rightarrow [0, \infty)$  is continuous and

$$\beta(tu + (1-t)v) \geq (\leq)t\beta(u) + (1-t)\beta(v)$$

for all  $u, v \in \mathcal{K}$  and  $0 \leq t \leq 1$ .

Let  $\gamma$  and  $\theta$  be nonnegative continuous convex functionals on  $\mathcal{K}$ ,  $\alpha$  be a nonnegative continuous concave functional on  $\mathcal{K}$ , and  $\psi$  be a nonnegative continuous functional on  $\mathcal{K}$ . Then for positive real numbers  $a, b, c$ , and  $d$  we define the following convex sets:

$$P(\gamma, d) = \{u \in \mathcal{K} : \gamma(u) < d\},$$

$$P(\gamma, \alpha, b, d) = \{u \in \mathcal{K} : b \leq \alpha(u), \gamma(u) \leq d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{u \in \mathcal{K} : b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{u \in \mathcal{K} : a \leq \psi(u), \gamma(u) \leq d\}.$$

Now we state the fixed-point theorem due to Avery and Peterson.

**Theorem 2.4.** *Let  $\mathcal{K}$  be a cone in a real Banach space  $\mathcal{B}$ . Let  $\gamma$  and  $\theta$  be nonnegative continuous convex functionals on  $\mathcal{K}$ ,  $\alpha$  be a nonnegative continuous concave functional on  $\mathcal{K}$ , and  $\psi$  be a nonnegative continuous functional on  $\mathcal{K}$  satisfying  $\psi(\lambda u) \leq \lambda \psi(u)$  for  $0 \leq \lambda \leq 1$ , such that for some positive numbers  $M$  and  $d$ ,*

$$\alpha(u) \leq \psi(u) \text{ and } \|u\| \leq M\gamma(u),$$

for all  $u \in \overline{P(\gamma, d)}$ . Suppose  $T: \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)}$  is completely continuous and there exist positive numbers  $a, b$ , and  $c$  with  $a < b$  such that

- (S1)  $\{u \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(u) > b\} \neq \emptyset$  and  $\alpha(Tu) > b$  for  $u \in P(\gamma, \theta, \alpha, b, c, d)$ ;
- (S2)  $\alpha(Tu) > b$  for  $u \in P(\gamma, \alpha, b, d)$  with  $\theta(Tu) > c$ ;
- (S3)  $0 \notin R(\gamma, \psi, a, d)$  and  $\psi(Tu) < a$  for  $u \in R(\gamma, \psi, a, d)$  with  $\psi(u) = a$ .

Then  $T$  has at least three fixed points  $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$  such that  $b < \alpha(u_1)$ ,  $a < \psi(u_2)$  with  $\alpha(u_2) < b$ , and  $\psi(u_3) < a$ .

### 3. Positive solutions

We define the following nonnegative continuous functionals on the cone  $\mathcal{K}$ :  $\alpha(u) = \min_{t \in [0,1]} |u^{(2j)}(t)|$ ,  $\theta(u) = \psi(u) = \max_{t \in [0,1]} |u^{(2j)}(t)|$ , and  $\gamma(u) = \max_{t \in [0,1]} |u^{(2m-1)}(t)|$ , for each  $j = 0, \dots, m-1$ . It is obvious that the functionals  $\psi$  and  $\gamma$  are convex and the functional  $\alpha$  is concave. Also note that  $\alpha(u) \leq \psi(u)$ . Furthermore, if we set

$$M = \prod_{i=1}^m \frac{1}{\kappa_i},$$

then, by Lemma 2.2,  $\|u\| = \max_{1 \leq k \leq 2m-1} \|u^{(k)}\|_0 \leq M \|u^{(2m-1)}\|_0 = M\gamma(u)$  for all  $u \in \mathcal{K}$ . To this end, the standing assumptions of Theorem 2.4 are satisfied.

For  $a, b$  with  $a < b$  and  $n \geq 0$ , by  $(-1)^n[a, b]$  we understand the interval  $[-b, -a]$ , if  $n$  is odd and  $[a, b]$ , if  $n$  is even.

**Theorem 3.1** ( $j = 0, \dots, m-1$ ). *Suppose (A1)-(A3) hold. Let  $d_i, i = 0, \dots, 2m-1$ , be constants such that  $d_{2i} \leq \frac{d_{2i+1}}{\kappa_i}, i = 0, \dots, m-1, d_{2i-1} \leq d_{2i}, i = 1, \dots, m-1$ , and let  $a$  and  $b$  be constants such that  $a < b \leq \frac{\rho_j}{\kappa_j} d_{2j+1}$ .*

Assume that the following hypotheses are satisfied

(H1)  $f(t, z_0, z_1, \dots, z_{2m-1}) \leq \frac{d_{2m-1}}{D}$  for all  $t \in [0, 1]$ ,  $z_i \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$ ,  $i = 0, 1, \dots, 2m-1$ , where

$$D = \int_0^1 a(t) dt,$$

(H2)  $f(t, z_0, z_1, \dots, z_{2m-1}) > \frac{b}{B}$  for all  $t \in [0, 1]$ ,  $z_i \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$ ,  $i = 0, \dots, 2j-1, 2j+1, \dots, 2m-1$ ,  $|z_{2j}| \in [b, \frac{b}{\rho_j}]$ , where

$$B = \int_0^1 G^{m-j}(1, t) a(t) dt$$

(H3)  $f(t, z_0, z_1, \dots, z_{2m-1}) < \frac{a}{A}$  for  $t \in [0, 1]$ ,  $z_i \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$ ,  $i = 0, \dots, 2j-1, 2j+1, \dots, 2m-1$ ,  $|z_{2j}| \in [0, a]$ , where

$$A = \int_0^1 G^{m-j}(0, t) a(t) dt.$$

Then the boundary value problem (1), (2) has at least three positive solutions  $u_1, u_2, u_3 \in \mathcal{K}$  satisfying  $b < |u_1^{(2j)}(1)|$ ,  $a < |u_2^{(2j)}(0)|$  with  $|u_2^{(2j)}(1)| < b$ ,  $|u_3^{(2j)}(0)| < a$ , and  $|u_i^{(2m-1)}(1)| \leq d_{2m-1}$ ,  $i = 1, 2, 3$ .

**Proof:** Let  $u \in \overline{P(\gamma, d_{2m-1})}$ , that is,  $0 \leq |u^{(2m-1)}(s)| \leq \gamma(u) = \max_{s \in [0, 1]} |u^{(2m-1)}(s)| = |u^{(2m-1)}(1)| \leq d_{2m-1}$  for all  $s \in [0, 1]$ . Then, by Lemma 2.2,  $u^{(i)}(s) \in (-1)^{\frac{i(i+1)}{2}} [0, d_i]$ ,  $i = 0, \dots, 2m-1$ , for all  $s \in [0, 1]$ . Hence, by (H1),  $f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) \leq \frac{d_{2m-1}}{D}$  for all  $s \in [0, 1]$  and we have

$$\begin{aligned} \gamma(Tu) &= \max_{t \in [0, 1]} |(Tu)^{(2m-1)}(t)| \\ &= |(Tu)^{(2m-1)}(1)| \\ &= \int_0^1 a(s) f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) ds \\ &\leq \int_0^1 a(s) ds \frac{d_{2m-1}}{D} \\ &\leq d_{2m-1}, \end{aligned}$$

that is,  $T : \overline{P(\gamma, d_{2m-1})} \rightarrow \overline{P(\gamma, d_{2m-1})}$ .

Consider the auxiliary equation

$$(-1)^{j+1} v^{(2j+2)}(s) = 1$$

with the boundary conditions (2) for  $i = 0, \dots, j$ , whose unique positive solution (by Lemma 2.3 with  $j+1$  in place of  $m$ ) is

$$v(s) = \int_0^1 G^{j+1}(s, \tau) d\tau,$$

so that

$$v^{(2j)}(s) = \int_0^1 G_j(s, \tau) d\tau.$$

Set

$$u(s) = \frac{b}{\rho_j \|v^{(2j)}\|_0} v(s).$$

Clearly,  $u \in \mathcal{K}$  and

$$\theta(u) = \max_{s \in [0,1]} |u^{(2j)}(s)| = \frac{b}{\rho_j \|v^{(2j)}\|_0} \max_{s \in [0,1]} |v^{(2j)}(s)| = \frac{b}{\rho_j}. \quad (10)$$

By remark following the proof of Lemma 2.2

$$\begin{aligned} \alpha(u) &= \frac{b}{\rho_j \|v^{(2j)}\|_0} \min_{s \in [0,1]} \left| \int_0^1 G_j(s, \tau) d\tau \right| \\ &= \frac{b}{\rho_j \|v^{(2j)}\|_0} \rho'_j \|v^{(2j)}\|_0 \\ &= b \frac{\rho'_j}{\rho_j} \\ &> b. \end{aligned} \quad (11)$$

Note that since  $u(s)$  is a polynomial of degree  $2j + 2$ , then

$$\gamma(u) = \max_{s \in [0,1]} |u^{(2m-1)}(s)| = 0$$

for all  $j = 0, \dots, m - 2$ . If  $j = m - 1$ , then

$$\gamma(u) = \max_{s \in [0,1]} |u^{(2m-1)}(s)| = \max_{s \in [0,1]} \left| (-1)^m \frac{b}{\rho_{m-1} \|v^{(2m-2)}\|_0} s \right| = \frac{b}{\rho_{m-1} \|v^{(2m-2)}\|_0} \leq d_{2m-1}.$$

We obtain, for all  $j = 0, \dots, m - 1$ , that  $\gamma(u) \leq d_{2m-1}$ , which, together with (10) and (11), shows that

$$\{u \in P(\gamma, \theta, \alpha, b, \frac{b}{\rho_j}, d_{2m-1}) : \alpha(u) > b\} \neq \emptyset.$$

If  $u \in P(\gamma, \theta, \alpha, b, \frac{1}{\rho_j} b, d_{2m-1})$ , then  $b \leq |u^{(2j)}(s)| \leq \frac{1}{\rho_j} b$  and  $|u^{(2m-1)}(s)| \leq d_{2m-1}$  for all  $s \in [0, 1]$ . Then, by (H2),

$$\begin{aligned} \alpha(Tu) &= \min_{t \in [0,1]} |(Tu)^{(2j)}(t)| \\ &= |(Tu)^{(2j)}(1)| \\ &= \int_0^1 G^{m-j}(1, s) a(s) f(s, u(s), u'(s), \dots, u^{(2m-1)}(s)) ds \\ &> \int_0^1 G^{m-j}(1, s) a(s) ds \frac{b}{B} \\ &= b, \end{aligned}$$

which shows that the condition (S1) of Theorem 2.4 is fulfilled since  $\alpha(Tu) > b$  for all  $u \in P(\gamma, \theta, \alpha, b, \frac{b}{\rho_j}, d_{2m-1})$ .

Let  $u \in P(\gamma, \alpha, b, d_{2m-1})$  with  $\theta(Tu) > \frac{b}{\rho_j}$ . Since the operator  $T$  is cone preserving, it follows from Lemma 2.2 that  $\alpha(Tu) \geq \rho_j \theta(Tu) > \rho_j \frac{b}{\rho_j} = b$ . So, the condition (S2) of Theorem 2.4 holds.

Finally, letting  $u \in R(\gamma, \psi, a, d_{2m-1})$  with  $\psi(u) = a$  (note that  $0 \notin R(\gamma, \psi, a, d_{2m-1})$  since  $\psi(0) = 0$ ). By (H3), we obtain that

$$\begin{aligned} \psi(Tu) &= \max_{t \in [0,1]} |(Tu)^{(2j)}(t)| \\ &= |(Tu)^{(2j)}(0)| \\ &= \int_0^1 G^{m-j}(0,s)a(s)f(s,u(s),u'(s),\dots,u^{(2m-1)}(s)) ds \\ &< \int_0^1 G^{m-j}(0,s)a(s) ds \frac{a}{A} \\ &= a. \end{aligned}$$

The condition (S3) of Theorem 2.4 is now satisfied.

The assumptions of Theorem 2.4 are verified and we assert the existence of at least three positive solutions  $u_1, u_2, u_3 \in \mathcal{K}$  such that  $b < |u_1^{(2j)}(1)|, a < |u_2^{(2j)}(0)|$  with  $|u_2^{(2j)}(1)| < b, |u_3^{(2j)}(0)| < a$ , and  $|u_i^{(2m-1)}(1)| \leq d_{2m-1}, i = 1, 2, 3$ .

□

As an example we give the existence conditions for  $m = 2$ .

In the sequel of the paper we assume for simplicity that  $a(t) = 1$ . Our boundary value problem takes shape of

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{12}$$

$$u^{(2i+1)}(0) = 0, \quad \alpha_i u^{(2i)}(\eta_i) = u^{(2i)}(1), \quad i = 0, 1, \tag{13}$$

where  $\alpha_i = \eta_i = \frac{1}{2}, i = 0, 1$ . The solution of the linearized problem to the problem (12), (13), given by Lemma 2.3 with  $g \equiv 1$ , is

$$\begin{aligned} u_2(t) &= \int_0^1 G^2(t,s) ds \\ &= \frac{1}{24}t^4 - \frac{1 - \alpha_1 \eta_1^2}{4(1 - \alpha_1)}t^2 + \frac{(1 - \alpha_0 \eta_0^2)(1 - \alpha_1 \eta_1^2)}{4(1 - \alpha_0)(1 - \alpha_1)} - \frac{(1 - \alpha_0 \eta_0^4)}{24(1 - \alpha_0)} \\ &= \frac{1}{24}t^4 - \frac{7}{16}t^2 + \frac{263}{384}. \end{aligned}$$

The constants from Theorem 3.1 corresponding to  $m = 2, j = 1$  and  $a(t) = 1$  are given by (recall (8))

$$A = \int_0^1 G^{m-j}(0,t) dt = \int_0^1 G_1(0,t) dt = \max_{t \in [0,1]} |u_2''(t)| = \frac{7}{8},$$

$$B = \int_0^1 G^{m-j}(1,t) dt = \int_0^1 G_1(1,t) dt = \min_{t \in [0,1]} |u_2''(t)| = \frac{3}{8},$$

and

$$D = \int_0^1 a(t) dt = 1.$$

Let

$$\phi(z) = \begin{cases} \frac{1}{5}z, & z \in [0, 1], \\ 8z - \frac{39}{5}, & z \in (1, \frac{3}{2}], \\ \frac{2}{7}z + \frac{132}{35}, & z \in (\frac{3}{2}, 12]. \end{cases}$$

Consider the inhomogeneous term in the form

$$f(t, z_0, z_1, z_2, z_3) = \frac{1}{5}t^2 + \frac{1}{1620}z_0^2 + \frac{1}{720}z_1^2 + \phi(|z_2|) + \frac{\sqrt{2|z_3|}}{20}$$

and let  $a = 1$ ,  $b = \frac{3}{2}$ ,  $d_0 = 18$ ,  $d_1 = 12$ ,  $d_2 = 12$ , and  $d_3 = 8$ . Then the function  $f$  satisfies the following inequalities:

1.  $f(t, z_0, z_1, z_2, z_3) \leq 8$ ,  $(t, z_0, z_1, z_2, z_3) \in [0, 1] \times [0, 18] \times [-12, 0] \times [-12, 0] \times [0, 8]$ ;
2.  $f(t, z_0, z_1, z_2, z_3) > 4$ ,  $(t, z_0, z_1, z_2, z_3) \in [0, 1] \times [0, 18] \times [-12, 0] \times [-\frac{9}{2}, -\frac{3}{2}] \times [0, 8]$ ;
3.  $f(t, z_0, z_1, z_2, z_3) < \frac{8}{7}$ ,  $(t, z_0, z_1, z_2, z_3) \in [0, 1] \times [0, 18] \times [-12, 0] \times [-1, 0] \times [0, 8]$ .

The above inequalities are consistent with the hypotheses (H1)-(H3) of Theorem 3.1 and we obtain three positive solutions satisfying  $u_1''(1) < -\frac{3}{2}$ ,  $u_2''(0) < -1$  with  $u_2''(1) > -\frac{3}{2}$ ,  $u_3''(0) > -1$ , and  $u_i'''(1) \leq 8$ ,  $i = 1, 2, 3$ .

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