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Fuzzy Solutions for Hyperbolic Partial Differential Equations

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Abstract

In this paper, the Banach fixed point theorem is used to investigate the existence of fuzzy solutions for ordinary and functional hyperbolic partial differential equations with local and nonlocal initial conditions.

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AMS (MOS) Subject Classifications: 34A99, 35L15.

1. Introduction

This paper is concerned with the existence of fuzzy solutions for hyperbolic partial differential equations. In the first part of Section 3 we consider the fuzzy hyperbolic differential equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = f(x,y,u(x,y)), \quad t \in J_a \times J_b := [0,a] \times [0,b] \tag{1}$$

$$u(0,0) = u_0, \quad u(x,0) = \eta_1(x), \quad u(0,y) = \eta_2(y), \; (x,y) \in [0,a] \times [0,b], \tag{2}$$

where we let E^n is the set of all upper semi-continuous, convex, normal fuzzy numbers with bounded $\alpha-$level and $f : J_a \times J_b \times E^n \to E^n$, $\eta_1 \in C(J_a, E^n), \eta_2 \in C(J_b, E^n)$ are given functions and $u_0 \in E^n$. In the second part of this section we consider the equation of the form

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = (p(x,y)u(x,y))_y + f(x,y,u(x,y)), \; (x,y) \in J_a \times J_b := [0,a] \times [0,b] \tag{3}$$

$$u(0,0) = u_0, \quad u(x,0) = \eta_1(x), \quad u(0,y) = \eta_2(y), \; (x,y) \in [0,a] \times [0,b], \qquad (4)$$

where f, u_0, η_1, η_2, are as in the problem (1)–(2) and $p : J_a \times J_b \to \mathbb{R}$. Finally extensions to nonlocal hyperbolic partial differential equations are also indicated at the end of this section. In Section 4 we consider fuzzy hyperbolic functional differential equations

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = f(x, y, u_{(x,y)}), \qquad (x,y) \in J_a \times J_b := [0,a] \times [0,b] \qquad (5)$$

$$u(x,y) = \varphi(x,y), \quad (x,y) \in [-r,0] \times [-r,0], \qquad (6)$$

and

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial (p(x,y) u_{(x,y)})}{\partial y} + f(x, y, u_{(x,y)}), \quad (x,y) \in J_a \times J_b := [0,a] \times [0,b] \quad (7)$$

$$u(x,y) = \varphi(x,y), \; (x,y) \in [-r,0] \times [-r,0], \qquad (8)$$

where $f : J_a \times J_b \times C([-r,0] \times [-r,0], E^n) \to E^n$, $\varphi : [-r,0] \times [-r,0] \to E^n$, $p : J_a \times J_b \to \mathbb{R}$ are given functions, and $u_{(x,y)}(s,t)$ is defined by

$$u_{(x,y)}(s,t) = u(x+s, y+t); \quad (s,t) \in [-r,0] \times [-r,0].$$

Possible extensions to nonlocal problems are indicated at the end of this section.

During the last three decades several papers have been devoted to the study of hyperbolic partial differential equations with local and nonlocal initial conditions; see for instance [1, 2, 3, 4, 5, 6, 15, 16, 20, 21] and the references cited therein. For more recent results on hyperbolic partial differential equations we refer the reader to the books of Kamont [12] and Pachpatte [18].

The theory of fuzzy sets, fuzzy valued functions and necessary calculus of fuzzy functions has been investigated in [7, 8, 9, 10, 13, 17]. The fuzzy differential equation has also been developed and basic properties of solutions of fuzzy differential equations is available in [11].

In this paper we study the existence of fuzzy solutions for hyperbolic differential equations. Our approach here is based on Banach fixed point theorem. The results of the present paper constitute a contribution to the study devoted to fuzzy partial differential equations.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. In the following $CC(\mathbb{R}^n)$ denotes the set of all nonempty compact, convex subsets of $\mathbb{R}^n$. Let

$$E^n = \{u : \mathbb{R}^n \to [0,1] \text{ such that satisfies } (i) \text{ to } (iv) \text{ mentioned below}\},$$

(i) u is normal, that is, there exists an $x_0 \in \mathrm{I\!R}^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is for $x, z \in \mathrm{I\!R}^n$ and $0 < \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)z) \geq \min[u(x), u(z)];$$

(iii) u is upper semi-continuous;

(iv) $[u]^0 = \overline{\{x \in \mathrm{I\!R}^n : u(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in \mathrm{I\!R}^n : u(x) \geq \alpha\}$. Then from (i) to (iv), it follows that the $\alpha-$level sets $[u]^\alpha \in CC(\mathrm{I\!R}^n)$. If $g : \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}^n$ is a function, then, according to Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the function defined by

$$g(u, \overline{u})(z) = \sup_{z = g(x, \bar{z})} \min\{u(x), \overline{u}(\bar{z})\}.$$

It is well known that

$$[g(u, \overline{u})]^\alpha = g([u]^\alpha, [\overline{u}]^\alpha)$$

for all $u, \overline{u} \in E^n$, $0 \leq \alpha \leq 1$ and g is continuous. Especially for addition and scalar multiplication, we have

$$[u + \overline{u}]^\alpha = [u]^\alpha + [\overline{u}]^\alpha, \quad [ky]^\alpha = k[u]^\alpha,$$

where $u, \overline{u} \in E^n, \ k \in \mathrm{I\!R}, \ \ 0 \leq \alpha \leq 1$.

Let A and B be two nonempty bounded subsets of $\mathrm{I\!R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A, B) = \max\left\{\sup_{a \in A} \|a - B\|, \sup_{b \in B} \|b - A\|\right\}$$

where $\|b - A\| = \inf_{a \in A} \|a - b\|, \ \|a - B\| = \inf_{b \in B} \|a - b\|$ and $\| \ . \ \|$ denotes the usual Euclidean norm in $\mathrm{I\!R}^n$. Then $(CC(\mathrm{I\!R}^n), H_d)$ is a metric space. The supermum metric d_∞ on E^n is defined by

$$d_\infty(u, \overline{u}) = \sup_{0 < \alpha \leq 1} H_d([u]^\alpha, [\overline{u}]^\alpha) \text{ for all } u, \overline{u} \in E^n.$$

(E^n, d_∞) is a complete metric space and is obviously metric on $C(J_a \times J_b, E^n)$. The supermum metric H_1 on $C(J_a \times J_b, E^n)$ is defined by

$$H_1(u, \overline{u}) = \sup_{(t,s) \in J_a \times J_b} d_\infty(u(t, s), \overline{u}(t, s)).$$

$(C(J_a \times J_b, E^n), H_1)$ is a complete metric space.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and is denoted by $x - y$.

Definition 2.1 *A map* $f : J_a \times J_b \to E^n$ *is called levelwise continuous at* $(t_0, s_0) \in J_a \times J_b$ *if the multi-valued map* $f_\alpha(t,s) = [f(t,s)]^\alpha$ *is continuous at* $(t,s) = (t_0, s_0)$ *with respect to the Hausdorff metric* d *for all* $\alpha \in [0,1]$. *A map* $f : J_a \times J_b \to E^n$ *is called integrably bounded if there exists an integrable function* $h \in L^1(J_a \times J_b, \mathbb{R}^n)$ *such that* $\|y\| \leq h(t,s)$ *for all* $y \in f_0(t,s)$.

Definition 2.2 *Let* $f : J_a \times J_b \to E^n$. *The integral of* f *over* $J_a \times J_b$, *denoted* $\int_0^a \int_0^b f(t,s)dsdt$, *is defined by*

$$\begin{aligned}\left(\int_0^a \int_0^b f(t,s)dtds\right)^\alpha &= \int_0^a \int_0^b f_\alpha(t,s)dsdt \\ &= \Big\{\int_0^a \int_0^b v(t,s)dsdt \mid v : J_a \times J_b \to R^n \text{ is a measurable} \\ &\quad \text{selection for } f_\alpha\Big\}\end{aligned}$$

for all $\alpha \in (0,1]$. *A strongly measurable and integrably bounded map* $f : J_a \times J_b \to E^n$ *is said to be* integrable *over* $J_a \times J_b$, *if* $\int_0^a \int_0^b f(t,s)dsdt \in E^n$.

If $f : J_a \times J_b \to E^n$ is measurable and integrably bounded, then f is integrable.

Definition 2.3 *A map* $f : J_a \times J_b \times E^n \to E^n$ *is called levelwise continuous at point* $(t_0, s_0, x_0) \in J_a \times J_b \times E^n$ *provided, for any fixed* $\alpha \in [0,1]$ *and arbitrary* $\epsilon > 0$, *there exists a* $\delta(\epsilon, \alpha) > 0$ *such that*

$$H_d\left([f(t,s,x)]^\alpha, [f(t_0,s_0,x_0)]^\alpha\right) < \epsilon$$

whenever $\max(|t - t_0|, |s - s_0|) < \delta(\epsilon, \alpha)$ *and* $H_d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ *for all* $(t,s,x) \in J_a \times J_b \times E^n$.

Definition 2.4 *Let* $f : J_a \times J_b \to E^n$. *The fuzzy partial derivative of* f *with respect to* x *at the point* $(t_0, s_0) \in J_a \times J_b$ *is the fuzzy set* $\frac{\partial f(t_0,s_0)}{\partial x} \in E^n$ *defined by*

$$\frac{\partial f(t_0,s_0)}{\partial x} := \lim_{h \to 0} \frac{f(t_0 + h, s_0) - f(t_0, s_0)}{h}.$$

Here the limit is taken in the metric space (E^n, H_d). *The fuzzy partial derivative of* f *with respect to* y *at the point* $(t_0, s_0) \in J_a \times J_b$ *is defined similarly.*

For the concepts of fuzzy measurability and fuzzy continuity we refer to [13].

3. Hyperbolic Partial Differential Equations

In the first part of this section we consider the existence of fuzzy solutions for problem (1)–(2).

Definition 3.1 *By a solution of (1)–(2) we mean a function* $u(.,.) \in C(J_a \times J_b, E^n)$ *such that* $\dfrac{\partial^2 u(x,y)}{\partial x \partial y} = f(x,y,u(x,y))$ *on* $J_a \times J_b$ *and satisfies the conditions* $u(0,0) = u_0, \quad u(x,0) = \eta_1(x), \quad u(0,y) = \eta_2(y), \ (x,y) \in J_a \times J_b$.

Theorem 3.2 *Assume that the condition*

(H1) there exists a constant K *such that*

$$H_d([f(t,s,u)]^\alpha, [f(t,s,\overline{u})]^\alpha) \leq K H_d([u]^\alpha, [\overline{u}]^\alpha),$$

for all $(t,s) \in J_a \times J_b$ *and all* $u, \overline{u} \in E^n$,

holds. If $Kab < 1$, *then the problem (1)-(2) has a unique fuzzy solution.*

Proof. Transform the problem (1)-(2) into a fixed point problem. It is clear that the solutions of the problem (1)-(2) are fixed points of the operator $N : C(J_a \times J_b, E^n) \to C(J_a \times J_b, E^n)$ defined by:

$$N(u)(x,y) := \eta_1(x) + \eta_2(y) - u_0 + \int_0^x \int_0^y f(t,s,u(t,s))dsdt.$$

We shall show that N is a contraction operator. Indeed, consider $u, \overline{u} \in C(J_a \times J_b, E^n)$ and $\alpha \in (0,1]$, then

$$N(u)(x,y) = \eta_1(x) + \eta_2(y) - u_0 + \int_0^x \int_0^y f(t,s,u(t,s))dsdt,$$

and

$$N(\overline{u})(x,y) = \eta_1(x) + \eta_2(y) - u_0 + \int_0^x \int_0^y f(t,s,\overline{u}(t,s))dsdt.$$

From (H1) we get

$$\begin{aligned} H_d([N(u)(t,s)]^\alpha, [N(\overline{u})(t,s)]^\alpha) &= H_d \left([\eta_1(x) + \eta_2(y) - u_0]^\alpha \right. \\ &\quad + \left[\int_0^x \int_0^y f(t,s,u(t,s))dsdt \right]^\alpha, \\ &\quad [\eta_1(x) + \eta_2(y) - u_0]^\alpha \\ &\quad \left. + \left[\int_0^x \int_0^y f(t,s,\overline{u}(t,s))dtds \right]^\alpha \right) \\ &= H_d \Big([\eta_1(x) + \eta_2(y) - u_0]^\alpha \\ &\quad + \left[\int_0^x \int_0^y f(t,s,u(t,s))dsdt \right]^\alpha, \end{aligned}$$

$$
\begin{aligned}
& [u_0]^\alpha + [\eta_1(x)+\eta_2(y)-u_0]^\alpha \\
+ & \left[\int_0^x\int_0^y f(t,s,\overline{u}(t,s))dtds\right]^\alpha\Big) \\
\leq & \int_0^x\int_0^y H_d([f(t,s,u(t,s))]^\alpha,[f(t,s,\overline{u}(t,s)]^\alpha)dsdt \\
\leq & \int_0^x\int_0^y KH_d([u(t,s)]^\alpha,[\overline{u}(t,s))]^\alpha)dsdt \\
\leq & \int_0^a\int_0^b K\sup_{\alpha\in(0,1]} H_d([u(t,s)]^\alpha,[\overline{u}(t,s)]^\alpha)dsdt \\
\leq & K\int_0^a\int_0^b d_\infty(u(t,s),\overline{u}(t,s))dsdt \\
\leq & KabH_1(u,\overline{u}).
\end{aligned}
$$

Hence for each $(t,s)\in J_a\times J_b$

$$H_1(N(u),N(\overline{u}))\leq KabH_1(u,\overline{u}).$$

So, N is a contraction and thus, by Banach fixed point theorem, N has a unique fixed point, which is solution to (1)-(2).

In this second part we give an existence and uniqueness result for the hyperbolic problem (3)–(4).

Definition 3.3 *A function $u\in C(J_a\times J_b,E^n)$ is said to be a solution of (3)-(4) if u satisfies the equation $\dfrac{\partial^2 u(x,y)}{\partial x\partial y}=(p(x,y)u(x,y))_y+f(x,y,u(x,y))$ on $J_a\times J_b$ and the conditions $u(0,0)=u_0,\ \ u(x,0)=\eta_1(x),\ \ u(0,y)=\eta_2(y),\ (x,y)\in J_a\times J_b$.*

Theorem 3.4 *Assume (H1) is satisfied. If*

$$a\sup_{(t,s)\in J_a\times J_b}|p(t,s)|+Kab<1,$$

then the problem (3)-(4) has a unique fuzzy solution on $J_a\times J_b$.

Proof. Transform the problem (3)-(4) into a fixed point problem. Clearly (see [19]) the solutions of the problem (3)-(4) are fixed points of the operator $N_1 : C(J_a\times J_b,E^n)\to C(J_a\times J_b,E^n)$ defined by:

$$N(u)(x,y):=q(x,y)+\int_0^x p(s,y)u(s,y)ds+\int_0^x\int_0^y f(t,s,u(t,s))dsdt$$

where

$$q(x,y)=\eta_1(x)+\eta_2(y)-u_0+\int_0^x p(s,0)\eta_1(s)ds.$$

We shall show that N_1 is a contraction operator. Indeed, consider $u, \overline{u} \in C(J_a \times J_b, E^n)$ and $\alpha \in (0,1]$, then

$$N(u)(x,y) = q(x,y) + \int_0^x p(s,y)u(s,y)ds + \int_0^x \int_0^y f(t,s,u(t,s))dsdt,$$

and

$$N(\overline{u})(x,y) = q(x,y) + \int_0^x p(s,y)\overline{u}(s,y)ds + \int_0^x \int_0^y f(t,s,\overline{u}(t,s))dsdt.$$

Then

$$\begin{aligned}
H_d([N_1(u)(t,s)]^\alpha, [N_1(\overline{u})(t,s)]^\alpha) &= H_d\Big([q(x,y)]^\alpha + \Big[\int_0^x p(s,y)u(s,y)ds)\Big]^\alpha \\
&+ \Big[\int_0^x \int_0^y f(t,s,u(t,s))dsdt\Big]^\alpha, \\
&\quad [q(x,y)]^\alpha + \Big[\int_0^x p(s,y)\overline{u}(s,y)ds\Big]^\alpha \\
&+ \Big[\int_0^x \int_0^y f(t,s,\overline{u}(t,s))dsdt\Big]^\alpha\Big) \\
&\leq H_d\Big(\Big[\int_0^x p(s,y)u(s,y)ds)\Big]^\alpha \\
&+ \Big[\int_0^x \int_0^y f(t,s,u(t,s))dsdt\Big]^\alpha, \\
&\quad \Big[\int_0^x p(s,y)\overline{u}(s,y)ds\Big]^\alpha \\
&+ \Big[\int_0^x \int_0^y f(t,s,\overline{u}(t,s))dsdt\Big]^\alpha\Big) \\
&\leq \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a H_d([u(t,s)]^\alpha, [\overline{u}(t,s)]^\alpha)dt \\
&+ \int_0^x \int_0^y H_d([f(t,s,u(t,s))]^\alpha, [f(t,s,\overline{u}(t,s)]^\alpha)dsdt \\
&\leq \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a H_d([u(t,s)]^\alpha, [\overline{u}(t,s)]^\alpha)dt \\
&+ \int_0^x \int_0^y KH_d([u(t,s)]^\alpha, [\overline{u}(t,s))]^\alpha)dsdt \\
&\leq \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a \sup_{\alpha\in[0,1]} H_d([u(t,s)]^\alpha, [\overline{u}(t,s)]^\alpha)dt \\
&+ \int_0^a \int_0^b K \sup_{\alpha\in(0,1]} H_d([u(t,s)]^\alpha, [\overline{u}(t,s)]^\alpha)dsdt \\
&\leq \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a d_\infty(u(t,s), \overline{u}(t,s))dt \\
&+ K \int_0^a \int_0^b d_\infty(u(t,s), \overline{u}(t,s))dsdt
\end{aligned}$$

$$\leq \left(a \sup_{(t,s)\in J_a\times J_b} |p(t,s)| + Kab\right) H_1(u,\overline{u}).$$

Hence for each $(t,s) \in J_a \times J_b$

$$H_1(N_1(u), N_1(\overline{u})) \leq \left(a \sup_{(t,s)\in J_a\times J_b} |p(t,s)| + Kab\right) H_1(u,\overline{u}).$$

So, N_1 is a contraction and thus, by Banach fixed point theorem, N_1 has a unique fixed point, which is fuzzy solution to (3)-(4).

Now we extend the problem (1)-(2) by considering nonlocal hyperbolic problems. More precisely, we consider the following nonlocal problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = f(x,y,u(x,y)), \quad t \in J_a \times J_b := [0,a]\times[0,b] \tag{9}$$

$$u(x,0) + \sum_{i=1}^{p} g_i(x)u(x,b_i) = \eta_1(x), \quad x \in J_a, \ \ i = 1,\dots,p \tag{10}$$

$$u(0,y) + \sum_{j=1}^{r} h_j(y)u(a_j,y) = \eta_2(y), \quad y \in J_b, \ \ j = 1,\dots,r \tag{11}$$

where η_1, η_2 are as in problem (1)-(2), $\eta_1, \ g_i \in C(J_a, E^n), \ i = 1,\dots,p, \ \eta_2, \ h_j \in C(J_b, E^n), \ j = 1,\dots,r, \ 0 < a_1 < a_2 \dots < a_p \leq a, \ 0 < b_1 < b_2 \dots < b_r \leq b.$

Definition 3.5 *By a fuzzy solution of (9)–(11) we mean a function* $u(.,.) \in C(J_a \times J_b, E^n)$ *which satisfies (9)–(11).*

Theorem 3.6 *Assume (H1) is satisfied. If*

$$\sum_{i=1}^{p} \sup_{x\in J_a} |g_i(x)| + \sum_{j=1}^{r} \sup_{y\in J_b} |h_j(y)| + Kab < 1,$$

then the problem (9)-(11) has a unique fuzzy solution on $J_a \times J_b$.

Proof. Transform the problem (9)–(11) into a fixed point problem. Clearly the solutions of the problem (9)-(11) are fixed points of the operator $N_2 : C(J_a \times J_b, E^n) \to C(J_a \times J_b, E^n)$ defined by:

$$N_2(u)(x,y) := \overline{q}(x,y) - \sum_{i=1}^{p} g_i(x)u(x,b_i) - \sum_{j=1}^{r} h_j(y)u(a_j,y) + \int_0^x \int_0^y f(t,s,u(t,s))dsdt,$$

where

$$\overline{q}(x,y) = \eta_1(x) + \eta_2(y) - \eta_1(0), \quad (x,y) \in J_a \times J_b.$$

The reasoning used in the proof of Theorem 3.2 shows that N_2 is a contraction operator, and hence, it has a unique fixed point, which is a fuzzy solution of (9)-(11).

Remark 3.7 *The arguments used in Theorem 3.2 can be applied to obtain a uniqueness result for the following fuzzy hyperbolic problem with nonlocal condition*

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = (p(x,y)u(x,y))_y + f(x,y,u(x,y)), \quad t \in J_a \times J_b := [0,a] \times [0,b] \quad (12)$$

$$u(x,0) + \sum_{i=1}^{p} g_i(x)u(x,b_i) = \eta_1(x), \quad x \in J_a, \; i = 1,\dots,p, \quad (13)$$

$$u(0,y) + \sum_{j=1}^{r} h_j(y)u(a_j,y) = \eta_2(y), \quad y \in J_b, \; j = 1,\dots,r. \quad (14)$$

4. Hyperbolic Partial Functional Differential Equations

In the first part of this section we consider the existence of fuzzy solutions for problem (5)–(6).

Definition 4.1 *By a solution of (5)–(6) we mean a function* $u(.,.) \in C([-r,a] \times [-r,b], E^n)$ *such that* $\dfrac{\partial^2 u(x,y)}{\partial x \partial y} = f(x,y,u_{(x,y)})$ *on* $J_a \times J_b$ *and the conditions* $u(x,y) = \varphi(x,y), \quad (x,y) \in [-r,0] \times [-r,0]$.

Theorem 4.2 *Assume that the condition*

(H2) there exists a constant K' *such that*

$$H_d([f(t,s,u)]^\alpha, [f(t,s,\overline{u})]^\alpha) \le K' H_d([u(\theta)]^\alpha, [\overline{u}(\theta)]^\alpha), \text{ for all } (t,s) \in J_a \times J_b$$

$$\text{and all } u, \overline{u} \in C([-r,a], E^n) \times C([-r,b], E^n), \; \theta \in [-r,0]$$

holds. If $K'ab < 1$*, then the problem (5)-(6) has a unique fuzzy solution.*

Proof. It is clear that the solutions of the problem (5)–(6) are fixed points of the operator $N' : C([-r,a] \times [-r,b], E^n) \to C([-r,a] \times [-r,b], E^n)$ defined by:

$$N'(u)(x,y) := \begin{cases} \varphi(x,y), & \text{if } (x,y) \in [-r,0] \times [-r,0] \\ \varphi(0,0) + \displaystyle\int_0^x \int_0^y f(t,s,u_{(t,s)})dsdt, & \text{if } (x,y) \in J_a \times J_b. \end{cases}$$

We shall show that N' is a contraction operator. Indeed, consider $u, \overline{u} \in C([-r,a] \times [-r,b], E^n)$ and $\alpha \in (0,1]$, then for $(x,y) \in J_a \times J_b$

$$N'(u)(x,y) = \varphi(0,0) + \int_0^x \int_0^y f(t,s,u_{(t,s)})dsdt,$$

and

$$N'(\overline{u})(x,y) = \varphi(0,0) + \int_0^x \int_0^y f(t,s,\overline{u}_{(t,s)})dsdt.$$

From (H2) we get

$$\begin{aligned}
H_d([N'(u)(t,s)]^\alpha \quad , \quad & [N'(\overline{u})(t,s)]^\alpha) = H_d\left(\left[\varphi(0,0) + \int_0^x \int_0^y f(t,s,u_{(t,s)})dsdt\right]^\alpha,\right. \\
& \left.\left[\varphi(0,0) + \int_0^x \int_0^y f(t,s,\overline{u}_{(t,s)})dsdt\right]^\alpha\right) \\
= \; & H_d\left([\varphi(0,0)]^\alpha + \left[\int_0^x \int_0^y f(t,s,u(t+w,s+\overline{w}))dsdt\right]^\alpha,\right. \\
& \left.[\varphi(0,0)]^\alpha + \left[\int_0^x \int_0^y f(t,s,\overline{u}(t+w,s+\overline{w}))dsdt\right]^\alpha\right) \\
\leq \; & \int_0^x \int_0^y H_d([f(t,s,u(t+w,s+\overline{w}))]^\alpha, \\
& [f(t,s,\overline{u}(t+w,s+\overline{w}))]^\alpha)dsdt \\
\leq \; & \int_0^x \int_0^y K' H_d\left([u(t+w,s+\overline{w})]^\alpha, [\overline{u}(t+w,s+\overline{w})]^\alpha\right) dsdt \\
\leq \; & \int_0^a \int_0^b K' \sup_{\alpha\in(0,1]} H_d([u(t+w,s+\overline{w})]^\alpha, [\overline{u}(t+w,s+\overline{w})]^\alpha)dsdt \\
\leq \; & K' \int_0^a \int_0^b d_\infty(u(t+w,s+\overline{w}), \overline{u}(t+w,s+\overline{w}))dsdt \\
\leq \; & K'abH_1(u,\overline{u}).
\end{aligned}$$

Hence for each $(t,s) \in J_a \times J_b$

$$H_1(N'(u), N'(\overline{u})) \leq K'abH_1(u,\overline{u}).$$

So, N' is a contraction and thus, by Banach fixed point theorem, N' has a unique fixed point which is solution to (5)–(6).
If $(t,s) \in [-r,0] \times [-r,0]$ we have $N'(u)(t,s) = \varphi(t,s)$ and the previous inequality holds.

In this second part we give an existence and uniqueness result for the hyperbolic problem (7)–(8).

Definition 4.3 *A function* $u \in C([-r,a] \times [-r,b], E^n)$ *is said to be a solution of (7)–(8) if* u *satisfies the equation* $\dfrac{\partial^2 u(x,y)}{\partial x \partial y} = \dfrac{\partial(p(x,y)u_{(x,y)})}{\partial y} + f(x,y,u_{(x,y)})$ *on* $J_a \times J_b$ *and the condition* $u(x,y) = \varphi(x,y), \quad (x,y) \in [-r,0] \times [-r,0]$.

Theorem 4.4 *Assume (H2) is satisfied. If*

$$a \sup_{(t,s)\in J_a\times J_b} |p(t,s)| + K'ab < 1,$$

then the problem (7)–(8) has a unique fuzzy solution on $[-r, a] \times [-r, b]$.

Proof. Let the operator $N_1' : C([-r, a] \times [-r, b], E^n) \to C([-r, a] \times [-r, b], E^n)$ defined by:

$$N_1'(u)(x,y) := \begin{cases} \varphi(x,y), & \text{if } (x,y) \in [-r,0] \times [-r,0] \\ \varphi(0,0) + \displaystyle\int_0^x p(s,y) u_{(s,y)} ds & \\ + \displaystyle\int_0^x \int_0^y f(t,s,u_{(t,s)}) dsdt, & \text{if } (x,y) \in J_a \times J_b. \end{cases}$$

We shall show that N_1' is a contraction operator. Indeed, consider $u, \overline{u} \in C(J_a \times J_b, E^n)$ and $\alpha \in (0,1]$, then

$$N_1'(u)(x,y) = \varphi(0,0) + \int_0^x p(s,y) u_{(s,y)} ds + \int_0^x \int_0^y f(t,s,u_{(t,s)}) dsdt,$$

and

$$N_1'(\overline{u})(x,y) = \varphi(0,0) + \int_0^x p(s,y) \overline{u}_{(s,y)} ds + \int_0^x \int_0^y f(t,s,\overline{u}(t,s)) dsdt.$$

Then

$$\begin{aligned} H_d([N_1'(u)(t,s)]^\alpha & , \quad [N_1'(\overline{u})(t,s)]^\alpha) = H_d\left([\varphi(0,0)]^\alpha + \left[\int_0^x p(s,y) u_{(s,y)} ds) \right]^\alpha \right. \\ & + \quad \left[\int_0^x \int_0^y f(t,s,u_{(t,s)}) dsdt \right]^\alpha, \\ & \quad\; [\varphi(0,0)]^\alpha + \left[\int_0^x p(s,y) \overline{u}_{(s,y)} ds \right]^\alpha \\ & + \quad \left. \left[\int_0^x \int_0^y f(t,s,\overline{u}_{(t,s)}) dsdt \right]^\alpha \right) \\ & \leq \quad H_d\left(\left[\int_0^x p(s,y) u_{(s,y)} ds \right]^\alpha \right. \\ & + \quad \left[\int_0^x \int_0^y f(t,s,u_{(t,s)}) dsdt \right]^\alpha, \\ & \quad\; \left[\int_0^x p(s,y) \overline{u}_{(s,y)} ds \right]^\alpha \\ & + \quad \left. \left[\int_0^x \int_0^y f(t,s,\overline{u}_{(t,s)}) dsdt \right]^\alpha \right) \\ & \leq \quad \sup_{(t,s) \in J_a \times J_b} |p(t,s)| \int_0^a H_d([u(t+w, s+\overline{w})]^\alpha, \\ & \quad\; [\overline{u}(t+w, s+\overline{w})]^\alpha) dt \\ & + \quad \int_0^x \int_0^y H_d([f(t,s,u_{(t,s)})]^\alpha, [f(t,s,\overline{u}_{(t,s)}]^\alpha) dsdt \end{aligned}$$

$$
\begin{aligned}
\leq\ & \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a H_d\left([u(t+w,s+\overline{w})]^\alpha,\right.\\
& \left.[\overline{u}(t+w,s+\overline{w})]^\alpha\right) dt\\
+\ & \int_0^x \int_0^y K' H_d\left([u(t+w,s+\overline{w})]^\alpha,\right.\\
& \left.[\overline{u}(t+w,s+\overline{w})]^\alpha\right) dsdt\\
\leq\ & \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a \sup_{\alpha\in[0,1]} H_d([u(t+w,s+\overline{w})]^\alpha,\\
& [\overline{u}(t+w,s+\overline{w})]^\alpha)dt\\
+\ & \int_0^a \int_0^b K' \sup_{\alpha\in(0,1]} H_d([u(t+w,s+\overline{w})]^\alpha,\\
& [\overline{u}(t+w,s+\overline{w})]^\alpha)dsdt\\
\leq\ & \sup_{(t,s)\in J_a\times J_b} |p(t,s)| \int_0^a \sup_{\alpha\in[0,1]} d_\infty([u(t+w,s+\overline{w})]^\alpha,\\
& [\overline{u}(t+w,s+\overline{w})]^\alpha)dt\\
+\ & K' \int_0^a \int_0^b d_\infty(u(t+w,s+\overline{w}),\overline{u}(t+w,s+\overline{w}))dsdt\\
\leq\ & \left(a \sup_{(t,s)\in J_a\times J_b} |p(t,s)| + K'ab\right) H_1(u,\overline{u}).
\end{aligned}
$$

Hence for each $(t,s)\in J_a\times J_b$

$$
H_1(N_1'(u),N_1'(\overline{u})) \leq \left(a \sup_{(t,s)\in J_a\times J_b} |p(t,s)| + K'ab\right) H_1(u,\overline{u}).
$$

So, N_1' is a contraction and thus, by Banach fixed point theorem, N_1' has a unique fixed point which is fuzzy solution to (7)–(8).

Now we extend the problem (5)-(6) by considering non-local hyperbolic problems. More precisely, we consider the following non-local problem

$$
\frac{\partial^2 u(x,y)}{\partial x\partial y} = f(x,y,u_{(x,y)}),\quad t\in J_a\times J_b := [0,a]\times[0,b] \tag{15}
$$

$$
u(x,y) + \sum_{i=1}^{p} g_i(x)u(x,b_i+y) = \varphi(x,y),\quad (x,y)\in J_a\times[-r,0],\ i=1,\ldots,p, \tag{16}
$$

$$
u(x,y) + \sum_{j=1}^{m} h_j(y)u(a_j+x,y) = \psi(x,y),\quad y\in[-r,0]\times J_b,\ j=1,\ldots,m. \tag{17}
$$

where φ is as in problem (5)-(6), $\eta,\ g_i\in C([-r,0]\times[-r,0],E^n),\ i=1,\ldots,m.$

Definition 4.5 *By a fuzzy solution of (15)–(17) we mean a function* $u(.,.)\in C([-r,a]\times[-r,b],E^n)$ *such that* $\dfrac{\partial^2 u(x,y)}{\partial x\partial y} = f(x,y,u_{(x,y)})$ *on* $J_a\times J_b$ *and the conditions (16), (17).*

Theorem 4.6 *Assume (H2) is satisfied. If*

$$\sum_{i=1}^{p}\sup_{x\in J_a}|g_i(x)|+\sum_{j=1}^{m}\sup_{y\in J_b}|h_j(y)|+K'ab<1,$$

then the problem (15)-(17) has a unique fuzzy solution on $[-r,a]\times[-r,b]$.

Proof. Consider the operator $N_2':C([-r,a]\times[-r,b],E^n)\to C([-r,a]\times[-r,b],E^n)$ defined by:

$$N_2'(u)(x,y):=\begin{cases}\varphi(x,y)-\displaystyle\sum_{i=1}^{p}g_i(x)u(x,b_i+y)), & (x,y)\in J_a\times[-r,0]\\ \varphi(x,y)-\displaystyle\sum_{j=1}^{m}h_j(y)u(a_j+x,y)=\varphi(x,y), & (x,y)\in[-r,0]\times J_b,\\ \varphi(x,0)+\varphi(0,y)-\varphi(0,0) & \\ -\displaystyle\sum_{i=1}^{p}g_i(x)u(x,y+b_i)-\sum_{j=1}^{m}h_j(y)u(a_j+x,y) & \\ +\displaystyle\int_0^x\int_0^y f(t,s,u(t,s))dsdt, & (x,y)\in[0,a]\times[0,b].\end{cases}$$

The reasoning used in the proof of Theorem 4.4 shows that N_2' is a contraction operator, and hence, it has a unique fixed point which is a fuzzy solution of (15)-(17).

Remark 4.7 *The arguments used in Theorem 4.4 can be applied to obtain a uniqueness result for the following fuzzy hyperbolic problem with nonlocal condition*

$$\frac{\partial^2u(x,y)}{\partial x\partial y}=\frac{\partial(p(x,y)u(x,y))}{\partial y}+f(x,y,u_{(x,y)}),\quad t\in J_a\times J_b:=[0,a]\times[0,b]\quad(18)$$

$$u(x,y)+\sum_{i=1}^{p}g_i(x)u(x,b_i+y)=\varphi(x,y),\quad (x,y)\in J_a\times[-r,0],\ i=1,\dots,p,\quad(19)$$

$$u(x,y)+\sum_{j=1}^{m}h_j(y)u(a_j+x,y)=\varphi(x,y),\quad y\in[-r,0]\times J_b,\ j=1,\dots,m.\quad(20)$$

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