International Journal of Applied Mathematical Sciences.
ISSN 0973-0176 Vol.2 No.2 (2005), pp. 149-158

http://www.gbspublisher.com/ijams.htm

Numerical Algorithm to Compute the Static Magnetic Vortices

M. Affouf
Department of Mathematics
Kean University
E-mail: maffouf@kean.edu

D. Petrescu
Department of Mathematical Sciences
Cheyney University of Pennsylvania
E-mail: dpetrescu@cheyney.edu

Abstract

The motion of gradient flows approach is used to study the multivortex solution of the Ginzburg-Landau equations of superconductivity. The static magnetic vortices of any multiplicity and any critical value are computed.

Key Words: Vortices, Ginzburg-Landau equations, Superconductivity, Numerical Algorithm.
AMS(MOS) Subject classifications. Primary 81J05, 65N30, 65M10.

1. Introduction

We construct a numerical algorithm to compute the static vortices solution of the 2-D radially symmetric Ginzburg-Landau equations. This algorithm is based on implementing the gradient flow and a transformation that reduces the order of the solution at the center of the vortex. This allows us to construct a finite difference scheme to compute the solution, with any desired degree of accuracy, while conserving the energy. First, we derive the equations of vortices and their boundary conditions. Second, we introduce the transformation (reduction transformation) and use the gradient flow approach to derive the equation of motion. Finally, we introduce the finite difference scheme and numerical results.

As a historical remark, we point out that the first attempt to compute the vortices were performed by Bogomolnyi and Vainshtein [3], by integrating the system (7) using a quasilinearization approach (Newton's method) and iterating until the difference between computed energies are acceptable.

Another approach was introduced by Jacobs and Rebbi [7]. They used variational approximations of the functions $S(r)$ and $R(r)$ in the form of polynomials of order 10 and then solved for their coefficients. They also computed the corresponding energy levels for various values of λ and n.

1.1. The Vortex Solution

It is known that the equilibrium state of certain types of superconductors is a multivortex configuration. These vortices (solitons) are important in elementary-particle physics, where they are defined as solutions of the Abelian Higgs model describing the interaction of a matter field with an Abelian gauge field. This model is similar to the Ginzburg-Landau macroscopic theory of superconductivity. In this theory, the matter field is described by a complex valued scalar function $\phi = \phi_1 + i\phi_2$ such that $|\phi|^2 = \phi\,\overline{\phi}$ is proportional to the density of the superconducting electrons. The (1-form) electromagnetic potential A is an Abelian gauge field [15]. The interaction between the matter field and the electromagnetic field is taken into account when we consider the covariant derivative (D) of the scalar field (weak coupling). That is, the covariant derivative $D\phi$ is given by

$$D_k\phi = (\partial_k - iA_k)\phi\ , \quad k = 0,\ 1,\ 2$$

where the index 0 represents the time component. Landau and Ginzburg proposed that the free energy density can be expanded in powers of $|\phi|$ and its derivatives. Taking into account the thermodynamic phase transitions and after scaling the free energy e has the form

$$e = \frac{1}{2}|D\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2 \tag{1}$$

where λ is the Ginzburg-Landau parameter proportional to the temperature. For $\lambda < 1$ equation (1) describes the free energy of type I superconductors which are characterized by vanishing of the magnetic field in the superconducting state. For $\lambda > 1$ equation (1) describes the free energy of type II supercomputers, for which the magnetic flux penetrates the superconducting state in tubes. We use the notation $|K|^2$ for $K_i \cdot K^i$, where $\cdot$ is a gauge invariant scalar product given by $A \cdot B = A\overline{B}$, repeated indices indicate summation. Adding the free energy density to the electromagnetic field we obtain the Ginzburg-Landau free energy

$$e = \frac{1}{2}|F|^2 + \frac{1}{2}|D\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2$$

where F is the electromagnetic tensor defined by

$$\begin{aligned} F = F_{ij} &= D_iA_j - D_jA_i = \partial_iA_j - \partial_jA_i \qquad (2) \\ &= \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0, & -B_3 \\ -E_2 & B_3 & 0 \end{pmatrix} \end{aligned}$$

Note that the potential function $V = \frac{\lambda}{8}(|\phi|^2 - 1)^2$ has minima at $|\phi| = 1$. States for which $|\phi| \approx 1$ correspond to superconductors, while states for which $|\phi| \approx 0$ correspond to normal metal. The quantized property of the magnetic flux penetrating the superconductors is given by the integral formula

$$\int_{R^2} F_{12}dxdy = \int_{R^2} B_3dxdy = 2\pi n$$

where n is the winding number (an integer) of the vortex (tube) around its axis of symmetry. Thus, the vortices are the two-dimensional cross section of the magnetic flux, and they are similarity solutions of a particular ansatz of the Ginzburg-Landau equations.

The Ginzburg-Landau equations can be derived from the following two-dimensional, Abelian Higgs Lagrangian

$$\mathscr{L} = \frac{1}{2}(\eta^{ij}\eta^{mn}F_{im} \cdot F_{jn} - \eta^{kl}D_k\phi \cdot D_l\phi + \frac{\lambda}{4}(\phi\overline{\phi} - 1)^2) \tag{3}$$

where η^{ij} is the Minkowski metric, given in cartesian coordinates by $diag(-,+,+)$. The Euler-Lagrange equations for this Lagrangian are

$$\begin{aligned} \partial_i F_{jk} + \partial_j F_{kj} + \partial_k F_{ij} &= 0 \\ \partial_i F^{ij} &= Im(\phi\overline{D^j\phi}) \\ D^k D_k\phi &= \frac{\lambda}{2}r(|\phi|^2 - 1)\phi \end{aligned} \tag{4}$$

The static vortices [1] are the time independent solutions of (4) which are rotationally symmetric functions of the following form

$$A_t = 0, \quad A_r = 0, \quad A_\theta = S(r), \quad \phi = e^{in\theta}R(r) \tag{5}$$

Substituting these functions in (3), we obtain the Lagrangian

$$L(R, R_r, S, S_r) = \pi\int_0^\infty \left[\left(\frac{S_r}{r}\right)^2 + (R_r)^2 + \frac{1}{r^2}(S-n)^2R^2 + \frac{\lambda}{4}(1-R^2)^2)\right] rdr \tag{6}$$

From (6) we derive the following coupled ordinary differential equations

$$\begin{aligned} S_{rr} &- \frac{1}{r}S_r - (S-n)R^2 = 0 \\ R_{rr} &+ \frac{1}{r}R_r - \frac{1}{r^2}(S-n)^2R - \frac{\lambda}{2}(R^2-1)R = 0. \end{aligned} \tag{7}$$

The functions $S(r)$ and $R(r)$ must satisfy the boundary conditions:

$$\begin{aligned} &R(0) = 0 \qquad S(0) = 0 \\ &R(r) \to 1 \qquad S(r) \to n \quad \text{as } r \to \infty \end{aligned} \tag{8}$$

The conditions (8) are consequence of the symmetry and the continuity of $S(r)$ and $R(r)$ at the center of the vortex. The asymptotic behavior of these functions can be described by expanding their Taylor's series around $r = 0$ and at $r \longrightarrow \infty$, using the assumption that the magnetic field is zero in the superconducting state. The asymptotic limits of the functions $S(r)$ and $R(r)$ with multiplicity n and critical value λ are described in Tinkham [15] as follows:

$$\begin{array}{rcl} S(r) = & O(r^2) & \text{for } r \longrightarrow 0 \\ R(r) = & O(r^n) & \text{for } r \longrightarrow 0 \\ S(r) = & n + O(e^{-r}) & \text{for } r \longrightarrow \infty \\ R(r) = & 1 + O(e^{-\lambda r}) & \text{for } r \longrightarrow \infty \end{array}$$

Additional properties of $S(r)$ and $R(r)$ have been shown by Berger and Chen [1].

Remark 1 *For decreased values of λ, the corresponding energy values require an increasing domain.*

Remark 2 *As n increases the function $R(r)$ approaches zero in the order r^n as $r \longrightarrow 0$.*

In this paper we introduce an efficient numerical method for solving the system of differential equations (7). Our approach is based first on the transformation $S(r) = r\sigma(r)$ and on minimizing the energy functional using finite difference scheme.

2. Numerical Algorithm

We work with the gradient flow frame work, since it is known [1] that solutions of system (7) are minima of the energy (6). Also, in order to construct efficient numerical scheme we apply an intermediate substitution. This approach is described as follows

2.1. Reduction Transformation

The function $S(r)$ approaches zero in the order of r^2, so we reduce the order of $S(r)$ by one using the transformation

$$S(r) = r\sigma(r) \tag{9}$$

with the corresponding boundary conditions

$$\begin{array}{rcl} \sigma(0) & = & \sigma(\infty) = 0 \\ \int_0^\infty \partial_r(\sigma^2)dr & = & \sigma^2|_0^\infty = 0 \end{array} \tag{10}$$

Substitution (9) is essential for the stability of the proposed discretization scheme to solve the system of equations (7). Using the substitution (9) and the identity

$$(\partial_r S)^2 = \sigma^2 + 2r\sigma\sigma_r + r^2\sigma_r^2 \tag{11}$$

in the Lagrangian (6) we obtain

$$\begin{aligned} L(R,R_r,\sigma,\sigma_r) &= \pi\int_0^\infty [r(\sigma_r^2+R_r^2)+\frac{\sigma^2}{r}+2\sigma\sigma_r \\ &+ \frac{1}{r}(r\sigma-n)^2R^2+\frac{\lambda}{4}(R^2-1)^2r]dr \end{aligned} \tag{12}$$

next we use the gradient flow approach

$$(R_t,\sigma_t)=\delta L_{(R,\sigma)}(R,R_r,\sigma,\sigma_r)$$

to derive the variational equations

$$\begin{aligned} \sigma_t &= r\sigma_{rr}+\sigma_r-\frac{\sigma}{r}-(r\sigma-n)R^2 \\ R_t &= rR_{rr}+R_r-\frac{1}{r}(r\sigma-n)^2R-r\frac{\lambda}{2}(R^3-R) \end{aligned} \tag{13}$$

2.2. Energy Discretization

We discretize the line of integration uniformly with step size h on a finite interval $[0,H]$, with H chosen such that $\max(e^{-H},e^{-\lambda H})$ is small enough to insure that the solutions $R(r)$ and $\sigma(r)$ attain their asymptotic limit at H. Let σ_i and R_i denote the value of these functions at $r_i=ih$. We start with this particular discretization of the Lagrangian (12), because the conservation of energy is important in the dynamics of vortex interaction

$$\begin{aligned} L(R,\sigma) &= \pi\sum_i\left[\left(\frac{r_{i+1}+r_i}{2}\right)\left(\left(\frac{\sigma_{i+1}-\sigma_i}{h}\right)^2\left(\frac{R_{i+1}-R_i}{h}\right)^2\right)\right. \\ &+ \left. 2\sigma_i\frac{\sigma_{i+1}-\sigma_i}{h}+\frac{\sigma_i^2}{r_i}+\frac{1}{r_i}(r_i\sigma_i-n)^2R_i^2+\frac{\lambda}{4}r_i(R_i^2-1)^2\right] \end{aligned} \tag{14}$$

Proposition 1 *The finite difference variations of $L(R_i,\sigma_i)$ with respect to R_i and σ_i are consistent with the coupled system of differential equations (13) with order of accuracy $0(h^2)$.*

Proof: We take the first finite variation of L with respect to σ_i that is

$$\begin{aligned} \delta L_{\sigma_i} &= \pi\sum_i\left[\left(\frac{r_{i+1}+r_i}{2}\right)\frac{1}{h^2}(2(\sigma_{i+1}-\sigma_i)(\delta\sigma_{i+1}-\delta\sigma_i)+\frac{2}{r_i}\sigma_i\delta\sigma_i\right. \\ &+ \left.\frac{2}{r_i}(r_i\sigma_i-n)r_i\delta\sigma_i\cdot R_i^2\right]h \end{aligned}$$

Denote the last two terms by $G_i\delta\sigma_i$ and rearrange the terms as follows

$$\begin{aligned}\delta L_{\sigma_i} &= \pi\sum_i\left[\frac{r_{i+1}+r_i}{h^2}(\sigma_{i+1}-\sigma_i)\delta\sigma_{i+1}-\frac{(r_{i+1}+r_i)}{h^2}\cdot(\sigma_{i+1}-\sigma_i)\delta\sigma_i+G_i\delta\sigma_i\right]h\\ &= \pi\sum_i\left[\frac{r_i+r_{i-1}}{h^2}(\sigma_i-\sigma_{i-1})\delta\sigma_i-\frac{(r_{i+1}+r_i)}{h^2}(\sigma_{i+1}-\sigma_i)\delta\sigma_i+G_i\delta\sigma_i\right]h\\ &= \pi h\sum_i\left(\frac{(2r_i-h)}{h^2}(\sigma_i-\sigma_{i-1})-\frac{(2r_i+h)}{h^2}(\sigma_{i+1}-\sigma_i)+G_i\right)\delta\sigma_i\end{aligned}$$

Finally we obtain

$$\delta L_{\sigma_i}=2\pi h\sum_i\left[-\frac{r_i(\sigma_{i+1}-2\sigma_i+\sigma_{i-1})}{h^2}-\frac{\sigma_{i+1}-\sigma_{i-1}}{2h}+\frac{\sigma_i}{r_i}+(r_i\sigma_i-n)R_i^2\right]\delta\sigma_i$$

A similar computation with respect to R_i yields

$$\begin{aligned}\delta L_{R_i} &= 2\pi h\sum_i\left[-r_i\frac{R_{i+1}-2R_i+R_{i-1}}{h^2}-\frac{R_{i+1}-R_{i-1}}{2h}\right.\\ &+ \left.\frac{1}{r_i}(r_i\sigma_i-n)^2R_i+\frac{\lambda}{2}r_i(R_i^3-R_i)\right]\delta R_i\end{aligned}$$

We obtain the finite difference equations by imposing $\delta L_{\sigma_i}=0$, for all $\delta\sigma_i$, and $\delta L_{R_i}=0$, for all δR_i. •

2.3. Implicit Difference Schemes

We point out that the energy functional $L(R,R_r,\sigma,\sigma_r)$ is positive definite and it has unique minimum [1]. This minimum is the stationary limit of the nonlinear parabolic system of equations

$$\begin{aligned}\frac{d}{dt}\sigma &= -\delta L_\sigma\\ \frac{d}{dt}R &= -\delta L_R\end{aligned}\tag{15}$$

Having already introduced a second order approximation to the right hand side of (15), the functions $R(r)$ and $\sigma(r)$ are the evolution limits of the following implicit scheme

$$\begin{aligned}\frac{\sigma_m^{n+1}-\sigma_m^n}{\Delta t} &= \frac{r_m}{h^2}(\sigma_{m+1}^{n+1}-2\sigma_m^{n+1}+\sigma_{m-1}^{n+1})+G_\sigma(\sigma^n,R^n)\\ \frac{R_m^{n+1}-R_m^n}{\Delta t} &= \frac{r_m}{h^2}(R_{m+1}^{n+1}-2R_m^{n+1}+R_{m-1}^{n+1})+G_R(\sigma^n,R^n)\end{aligned}\tag{16}$$

where $\sigma_m^n=\sigma(n\Delta t,mh)$, $R_m^n=R(n\Delta t,mh)$, and G_σ and G_R are nonlinear remaining terms from L_σ and L_R evaluated at time step $n\Delta t$.

Proposition 2 *The implicit scheme (16) is unconditionally stable and of order* $O(\Delta t)+O(h^2)$*.*

Proof: We linearize the equations (16) and apply the Fourier transform by replacing σ_m^n and R_m^n by $\xi^n e^{im\beta h}$ to obtain

$$\xi = \frac{1}{1+4r_m\alpha sin^2(\beta h/2)}$$

For all values of $\alpha = \frac{\Delta t}{h^2}$, clearly the magnitude $|\xi| \leq 1$ which implies the unconditional linearized stability. We observe that the lower order terms in (6) are divided by the variable r which imposes certain restrictions on time steps Δt. To resolve this restriction we construct a second implicit scheme of higher order $O(\Delta t^2)+O(h^2)$:

$$\frac{3\sigma_m^{n+1}-2\sigma_m^n}{2\Delta t} - \frac{2\sigma_m^n-\sigma_m^{n-1}}{2\Delta t} = r_m\frac{\sigma_{m+1}^{n+1}-2\sigma_m^{n+1}+\sigma_{m-1}^{n+1}}{h^2}+G_\sigma(\sigma^n,R^n) \quad (17)$$

$$\frac{3R_m^{n+1}-2R_m^n}{2\Delta t} - \frac{2R_m^n-R_m^{n-1}}{2\Delta t} = r_m\frac{R_{m+1}^{n+1}-2R_m^{n+1}+R_{m-1}^{n+1}}{h^2}+G_R(\sigma^n,R^n) \quad (18)$$

By using similar substitution in the linearized equations of (17) and (18) we obtain

$$(3+4\alpha r_m sin^2(\beta h/2))\xi^2-4\xi+1=0$$

for both equations. Solving for ξ we get

$$\xi = \frac{2\mp\sqrt{1-4\alpha r_m sin^2(\beta h/2)}}{2+1+4\alpha r_m sin^2(\beta h/2)} = \frac{2\mp\sqrt{1-w}}{2+1+w}$$

If $1-w$ is nonnegative, then we have

$$|\xi| \leq \frac{2\mp\sqrt{1-w}}{2+1+w} \leq 1$$

if $1-w$ is negative, then

$$\xi = \frac{2\mp i\sqrt{w-1}}{2+w+1}$$

for which

$$|\xi|^2 = (\frac{2}{2+w+1})^2+\frac{w-1}{(2+w+1)^2} = \frac{4+w-1}{(2+w+1)^2} \leq 1$$

Thus for any value of α or β both roots are bounded by 1. This proves the unconditional stability of (17) •

In both schemes, the discretized approximations lead to tridiagonal system of equations with variable entries depending on the distance r_m from the center of the vortex. These equations are solved by standard techniques. We terminate the iteration process whenever the difference between consecutive energy values becomes smaller than a preassigned number. For example, when $h = 0.01$, we terminate the iterations when the difference is less than 10^{-8}, that is

$$|L(\sigma^{n+1},R^{n+1})-L(\sigma^n,R^n)| < 10^{-8}$$

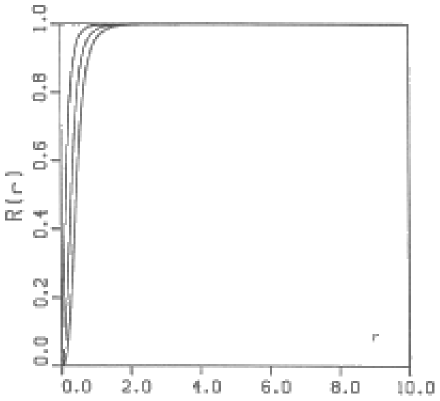

Figure 1: *R(r) for* $n = 1,2,3$ *and for* $\lambda = 100$

3. Numerical Results

We present two examples obtained by the method just described.

1. For various values of $\lambda = .5, 1, 100$ and vortex number $n = 1$, we compute the energy value on the interval $[0,20]$ using (14) for which we have the following values $E_i(.5,1) = .8667198, E_i(1,1) = 1.000043$, $E_i(100,1) = 2.5366966$. These values agree with the computed values obtained in [7].
2. In Figures 1 and 2 we plot the graphs of the functions $S(r)$ and $R(r)$ for fixed value of $\lambda = 100$ but for different vortex numbers $n = 1,\ 2,\ 3$.

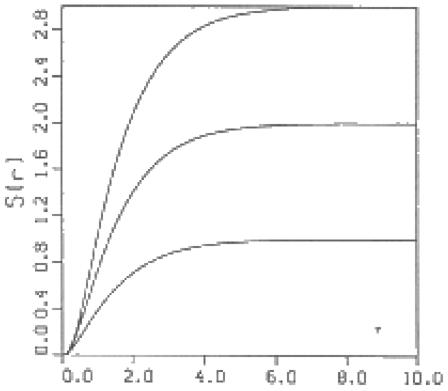

Figure 2: *S(r) for* $n = 1,2,3$ *and for* $\lambda = 100$

References

[1] M. Berger, Y. Chen, *Symmetric vortices for the Ginzburg-Landau equations of super conductivity*, J. Functional Analysis 82 (1989), 259–295

[2] E. Bogomolnyi, *The stability of classical solutions*, Sov. J. Nucl. Phys. 24 (1976), 449–454

[3] E. Bogomolnyi and A. Vainshtein, *Stability of strings in gauge Abelian theory,* Sov. J. Nucl. Phys. 23 (1976), 588–591

[4] E. Bogomolnyi and M. Marionov, *Calculation of the monopole mass in gauge theory,* Sov. J. Nucl. Phys. 23 (1976), 676–680

[5] Q. Du, M. Gunzburger, J. Peterson, *Analysis and approximation of the Ginzburg-Landau model of superconductivity*, SIAM Rev. 34(1)(1992) 54–81

[6] S. Gustafson, I.M. Sigal,*The stability of magnetic vortices* Comm. Math. Phys. 212 (2000) 257-275

[7] L. Jacobs, C. Rebbi, *Interaction energy of super conductivity vortices*, Phys. Rev. B19 (1979) 4486–4494

[8] A. Jaffe, C. Taubes, *Vortices and Monopoles*, Birthauser (1980)

[9] S. Y. Lin, Y. Yang, *Computation of superconductivity in thin films*, J. Comput. Phys. 89(2) (1990), 257–275

[10] K. Moriarty, E. Myers, C. Rebbi, *Dynamical interaction of superconducting flux vortices*, J. Comp. phys. 81 (1989), 481–488

[11] F. Odeh, *Existence and bifurcation theorem for the Ginzburg-Landau equations*, J. Math. Phys. 8 (1967), 2351–2356

[12] J. Shatah, M. Struwe, *Well-posedness in the energy space for semilinear wave equations with critical growth*, Internat. Math. Res. Notices no(7) (1994)

[13] J. Shatah, M. Struwe, *Geometric wave equations*, Courant Lect. Notes no(2) (1998)

[14] ’t Hooft, *Magnetic monopoles in unified gauge theories*, Nucl. Phys.B79, (1974), 276–283

[15] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill (1975)