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On positive solutions to the Cauchy problem for semilinear third order equations with nonreal "characteristic" roots

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Abstract

The Cauchy problem for a third order semilinear nonautonomous ODE on the positive half-line is considered. It is assumed that the linear part of the equation is variable and has a pair of complex "characteristic" roots. Conditions for the existence of positive solutions are derived. Moreover, lower solution estimates are established.

AMS Subject classification: 34C10, 34C11.

Key Words: ordinary differential equation, third order semilinear equation, Cauchy problem, positive solutions.

1. Introduction and statement of the main result

The problem of existence of positive of solutions to ordinary differential equations (ODEs) continues to attract the attention of many specialists despite its long history, cf. [1, 2, 3, 7, 8, 10] and references therein. It is still one of the most burning problems of theory of ODEs, because of the absence of its complete solution.

Let $p_k(t)$ $(t \geq 0;\ k = 1, ..., 3)$ be real continuous scalar-valued functions defined and bounded on $[0, \infty)$ and $p_0 \equiv 1$. Let $F : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ be a continuous function.

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In the present paper we investigate positive solutions to the semilinear equation

$$\sum_{k=0}^{3} p_k(t)\frac{d^{3-k}x(t)}{dt^{3-k}} = F(t,x) \ \ (t>0,\ x=x(t)) \tag{1.1}$$

with the initial conditions

$$x^{(j)}(0) = x_j \in \mathbf{R} \ \ (j=0,...,2). \tag{1.2}$$

A solution of problem (1.1), (1.2) is a function $x(.)$ defined on $[0,\infty)$, having continuous derivatives up to the 3-th order. In addition, $x(.)$ satisfies (1.2) and (1.1) for all $t>0$. The existence of solutions is assumed.

As it is well-known, the existence of positive solutions for such equations is proved mainly in the case when p_k are constant, cf. [7, 8, 4]. In [6] the positivity conditions were derived for a class of semilinear nonautonomous equations in the divergent form. In [9], in particular, the following remarkable result is established: solutions of the equation

$$\sum_{k=0}^{3} p_k(t)\frac{d^{3-k}v(t)}{dt^{3-k}} = 0 \ \ (t>0). \tag{1.3}$$

do not oscillate, if the roots of the polynomial

$$P(t,z) = \sum_{k=0}^{3} p_{n-k}(t)z^k \ \ (z\in\mathbf{C}, t\geq 0)$$

are real and not intersecting. In the present paper, *we do not assume that the roots of* $P(t,z)$ *are real.* Besides, we particularly generalize the corresponding result from [6].

Assume that $P(t,z)$ has the roots $\rho_k(t)$ $(k=1,2,3)$ with the property

$$Re\ \rho_k(t) \geq -\mu \ \ (t\geq 0;\ k=1,2,3) \tag{1.4}$$

with some $\mu>0$. Denote

$$q_1(t) = 3\mu - p_1(t),\ q_2(t) = 3\mu^2 - 2\mu p_1(t) + p_2(t)$$

and

$$q_3(t) = \mu^3 - p_1(t)\mu^2 + p_2(t)\mu - p_3(t).$$

Put

$$d_0 = 1, d_1 = \inf_{t\geq 0} q_1(t), d_2 = \sup_{t\geq 0} q_2(t), \text{ and } d_3 = \inf_{t\geq 0} q_3(t).$$

Now we are in a position to formulate the main result of the paper.

Theorem 0.1 *Let the polynomial*

$$\tilde{Q}(z) = \sum_{k=0}^{3}(-1)^k d_k z^{3-k} \ \ (d_0=1,\ z\in\mathbf{C})$$

have a pair of complex conjugate roots: $\tilde{\gamma} \pm i\tilde{\omega}$ $(\tilde{\gamma}, \tilde{\omega} > 0)$, *and a positive root* $\tilde{z}_0$ *satisfying the condition*

$$\tilde{z}_0 > \tilde{\gamma} + \tilde{\omega}. \tag{1.5}$$

Then equation (1.1) has a nonnegative solution $x(t)$, *satisfying the inequality*

$$x(t) \geq const\ e^{(-m+\tilde{z}_0)t}\ (t \geq 0),$$

provided

$$F(t, y) \geq 0\ (t, y \geq 0). \tag{1.6}$$

This theorem is proved in the next two sections.

2. Preliminaries

Let $a_k(t)$ $(t \geq 0;\ k = 1, 2, 3)$ be real continuous scalar-valued functions bounded on $[0, \infty)$ and $a_0 \equiv 1$. Consider the equation

$$\sum_{k=0}^{3} (-1)^k a_k(t) \frac{d^{3-k} u(t)}{dt^{3-k}} = 0 \ \ (t > 0). \tag{2.1}$$

Put

$$c_1 = \inf_{t \geq 0} a_1(t), c_2 = \sup_{t \geq 0} a_2(t), c_3 = \inf_{t \geq 0} a_3(t). \tag{2.2}$$

Let the polynomial

$$Q(z) = \sum_{k=0}^{3} (-1)^k c_k z^{3-k} \ \ (c_0 = 1,\ z \in \mathbf{C})$$

have a pair of complex conjugate roots: $\gamma \pm i\omega$ $(\gamma, \omega > 0)$, and a positive root z_0. In addition, let

$$z_0 > \gamma + \omega. \tag{2.3}$$

Denote

$$K(t) := c_0[e^{z_0 t} - e^{\gamma t}(cos\ (\omega t) + b_0\ sin\ (\omega t))],$$

where

$$b_0 := \frac{z_0 - \gamma}{\omega} \text{ and } c_0 := \frac{1}{(z_0 - \gamma)^2 + \omega^2}.$$

Lemma 0.2 *Let* $Q_(\lambda)$ *have a pair of complex conjugate roots* $\gamma \pm i\omega$ $(\gamma, \omega > 0)$, *and a positive root* z_0. *In addition, let condition (2.3) hold. Then*

$$K^{(j)}(t) \geq 0 \ \ (j = 0, 1, 2;\ t \geq 0).$$

Proof: Put $b_1 = z_0 - \gamma$. So $b_1 = \omega b_0$. Since

$$K(t) = c_0 e^{\gamma t} f(t) \text{ where } f(t) = e^{b_1 t} - cos(\omega\ t) - b_0\ sin(\omega t),$$

it is enough to check that f and its derivatives are positive. By virtue of the Taylor series

$$e^{b_1 t} = 1 + b_1 t + g(t) \ \ (g(t) > 0).$$

Since $|cos\ s| \leq 1, |sin\ s| \leq s\ (s \geq 0)$, we have

$$f(t) = 1 + b_1 t + g(t) - cos\ \omega t - b_0\ sin\ \omega t. \tag{2.4}$$

Thus, one can assert that $f \geq 0$ for all $t \geq 0$. Furthermore, due to (2.3), $b_0 \geq 1$ and $b_1 \geq \omega$. Thus

$$\dot{f}(t) = b_1 e^{b_1 t} + \omega sin(\omega t) - b_0 \omega\ cos(\omega t) =$$

$$b_1(1 + b_1 t + g(t)) + \omega sin(\omega t) - \omega b_0\ cos(\omega t) \geq 0.$$

Moreover,

$$\ddot{f}(t) = b_1^2 e^{b_1 t} + \omega^2 cos(\omega t) +$$

$$\omega^2 b_0\ sin(\omega t) = b_1^2(1 + b_1 t + g(t)) + \ \omega^2 cos(t\omega) + b_0 \omega^2 sin(t\omega) \geq 0.$$

As claimed. □

Lemma 0.3 *Under the condition (2.3), a solution u of (2.1) with the initial conditions*

$$u(0) = \dot{u}(0) = 0;\ \ddot{u}(0) = 1 \tag{2.5}$$

satisfies the inequalities $u^{(j)}(t) \geq K^{(j)}(t)\ \ (j = 0, 1, 2;\ t > 0)$.

Proof: Simple calculations show that $K(t)$ is a solution of the equation

$$Q(D)x(t) = 0 \ \ (D = d/dt), \tag{2.6}$$

with the initial condition $K(0) = \dot{K}(0) = 0,\ \ddot{K}(0) = 1$. So K is the Green function to (2.6). Furthermore, we have

$$m_k(t) := (-1)^k (c_k - a_k(t)) \geq 0 \ \ (k = 1, 2, 3).$$

Rewrite equation (2.1) in the form

$$\sum_{k=0}^{3} (-1)^k c_k \frac{d^{3-k} u}{dt^{3-k}} = \sum_{k=1}^{3} m_k(t) \frac{d^{3-k} u}{dt^{3-k}}. \tag{2.7}$$

Put

$$v(t) := \sum_{k=0}^{3} c_k \frac{d^{3-k} u(t)}{dt^{3-k}}. \tag{2.8}$$

Since, K is the Green functions to (2.6), thanks to the Variation of Constants Formula and condition (2.5), we have

$$u(t) = K(t) + \int_0^t K(t-s) v(s) ds. \tag{2.9}$$

But $\dot{K}(0) = K(0) = 0$. Consequently,

$$\frac{d^j}{dt^j} \int_0^t K(t-s)v(s)ds = \frac{d}{dt} \int_0^t K^{(j-1)}(t-s)v(s)ds =$$

$$\int_0^t K^{(j)}(t-s)v(s)ds \quad (j = 1, 2).$$

Hence thanks to (2.7) and (2.8),

$$v(t) = \sum_{k=1}^{3} m_k(t)[K^{(3-k)}(t) + \int_0^t K^{(3-k)}(t-s)v(s)ds] =$$

$$K_0(t,t) + \int_0^t K_0(t, t-s)v(s)ds, \tag{2.10}$$

where

$$K_0(t,\tau) = \sum_{k=1}^{3} m_k(t) K_0^{(3-k)}(\tau) \quad (t, \tau \geq 0).$$

According to the previous lemma, $K_0(t,\tau) \geq 0 \ (t, \tau \geq 0)$. Put $h(t) = K_0(t,t)$. Let V be the Volterra operator with the kernel $K_0(t, t-s)$. Then thanks to (2.10) and the Neumann series,

$$v(t) = h(t) + \sum_{k=1}^{\infty} (V^k h)(t) \geq h(t) \geq 0.$$

Hence (2.9) and the previous lemma yield

$$u^{(j)}(t) = K^{(j)}(t) + \int_0^t K^{(j)}(t-s)v(s)ds \geq$$

$$K^{(j)}(t) \geq 0 \quad (j = 0, 1, 2), \tag{2.11}$$

as claimed. □

Recall that the function $W(t,\tau)$ defined for $t \geq \tau \geq 0$ is *the Green function to equation (2.1)* if it satisfies (2.1) for $t > \tau$ and the conditions

$$W(t,t) = 0; \ \lim_{t \downarrow \tau} \frac{\partial W(t,\tau)}{\partial t} = 0; \ \lim_{t \downarrow \tau} \frac{\partial^2 W(t,\tau)}{\partial t^2} = 1.$$

Lemma 0.4 *Let polynomial $Q(z)$ have a pair of complex conjugate roots: $\gamma \pm i\omega$ $(\gamma, \omega > 0)$, and a positive root z_0 satisfying condition (2.3). Then the Green function $W(t,\tau)$ to equation (2.1) and its first, and second derivatives are nonnegative. Moreover,*

$$\frac{\partial^k W(t,\tau)}{\partial t^k} \geq \frac{\partial^k K(t-\tau)}{\partial t^k} \quad (k = 0, 1, 2; \ t > \tau \geq 0). \tag{2.12}$$

Proof: For a $\tau > 0$, take the initial conditions

$$u(\tau) = \dot{u}(\tau) = 0, j = 0, 1;\ \ddot{u}(\tau) = 1.$$

Then the corresponding solution $u(t)$ to (2.1) is equal to $W(t, \tau)$. Repeat the arguments of the proof of Lemma 2.2. Then according to (2.11) we have the required result. As claimed. □

3. Proof of Theorem 1.1

Under (1.5) put

$$\tilde{b}_0 := \frac{\tilde{z}_0 - \tilde{\gamma}}{\tilde{\omega}} \text{ and } \tilde{c}_0 := \frac{1}{(\tilde{z}_0 - \tilde{\gamma})^2 + \tilde{\omega}^2}.$$

Lemma 0.5 *Under (1.4), let polynomial $\tilde{Q}(z)$ have a pair of complex conjugate roots: $\tilde{\gamma} \pm i\tilde{\omega}$ ($\tilde{\gamma}, \tilde{\omega} > 0$), and a positive root $\tilde{z}_0$ satisfying condition (1.5). Then the Green function $\tilde{W}(t, \tau)$ to equation (1.3) is nonnegative. Moreover, for all $t > \tau \geq 0$, we have*

$$\tilde{W}(t, \tau) \geq \tilde{c}_0[e^{(\tilde{z}_0-\mu)(t-\tau)} - e^{(\tilde{\gamma}-\mu)(t-\tau)}(cos\ (\tilde{\omega}\ (t-\tau)) + \tilde{b}_0\ sin(\tilde{\omega}\ (t-\tau)))] \geq 0$$

Proof: Put $x(t) = e^{-\mu t}u(t)$ in (1.3). Then

$$0 = e^{\mu t}\sum_{k=0}^{3} p_k(t)\frac{d^{3-k}e^{-\mu t}u}{dt^{3-k}} = \sum_{k=0}^{3} p_k(t)(\frac{d}{dt} - \mu)^{3-k}u.$$

That is, equation (1.3) is reduced the equation

$$P(t, \frac{d}{dt} - \mu)u \equiv \sum_{k=0}^{3} p_k(t)(\frac{d}{dt} - \mu)^{3-k}u = 0. \tag{3.1}$$

But

$$P(t, z-\mu) = \sum_{k=0}^{3} p_k(t)(z-\mu)^{3-k} = \sum_{k=0}^{3} p_k(t)\sum_{j=0}^{3-k} C_{3-k}^{j}(-\mu)^j z^{3-k-j} =$$

$$\sum_{k=0}^{3} p_k(t)\sum_{m=k}^{3} C_{3-k}^{m-k}(-\mu)^{m-k}z^{3-m} = \sum_{m=0}^{3} z^{3-m}\sum_{k=0}^{m} p_k(t)C_{3-k}^{m-k}(-\mu)^{k-m}.$$

where $C_n^k = n!/k!(n-k)!$. That is,

$$P(t, z-\mu) = \sum_{m=0}^{3}(-1)^m q_m(t)z^{3-m},$$

where

$$q_m(t) = \sum_{k=0}^{m} p_k(t)C_{3-k}^{m-k}(-1)^k\mu^{m-k}\ \ (m = 1, 2, 3),\ q_0 \equiv 1.$$

Take into account that

$$P(t, z-\mu) = \prod_{k=1}^{3} (z - \rho_k(t) - \mu),$$

where according to (1.4), $Re\ \rho_k(t) + \mu \geq 0$ and apply Lemma 2.3 to equation (3.1). Due to the substitution $x(t) = e^{-\mu t}u(t)$, Lemma 2.3 proves the required result. □

Lemma 0.6 *Under the hypothesis of Theorem 1.1, let a solution $v(t)$ of the linear nonautonomous problem (1.2), (1.3) be positive. Then a solution $x(t)$ of problem (1.1), (1.2) is also positive. Moreover, $x(t) \geq v(t)$, $t \geq 0$.*

Proof: Thanks to the Variation of Constants Formula, equation (1.1) can be rewritten as

$$x(t) = v(t) + \int_0^t \tilde{W}(t, s) F(s, x(s)) ds.$$

Since $\tilde{W}(t, s)$ is positive due to the previous lemma, there is a sufficiently small $t_0 \geq 0$, such that $x(t) \geq 0$, $t \leq t_0$. Hence, $x(t) \geq v(t)$, $t \leq t_0$. Extending this inequality to all $t \geq 0$, we prove the lemma. □

Proof of Theorem 1.1: An arbitrary solution of (2.1) takes the form

$$u(t) = \sum_{k=1}^{3} l_k \frac{\partial^k \tilde{W}(t, 0)}{\partial t^k} \quad (l_k = const).$$

Take $u(t) = \tilde{W}(t, 0)$. Now Lemma 3.2 implies the required result. □

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