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# Parametric Duality Models for Semiinfinite Discrete Minmax Fractional Programming Problems Containing Generalized $(\alpha, \eta, \rho)$-V-Invex Functions

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**Abstract**

In this paper, we present a fairly large number of parametric duality results under various generalized $(\alpha, \eta, \rho)$-V-invexity assumptions for a semiinfinite discrete minmax fractional programming problem.

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**Key Words:** Semiinfinite programming, discrete minmax fractional programming, generalized invex functions, infinitely many equality and inequality constraints, parametric duality models, duality theorems.

## 1. Introduction

In this paper, we construct several parametric duality models and establish numerous duality results under various generalized $(\alpha, \eta, \rho)$-V-invexity assumptions for the following semiinfinite discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x, t) \leq 0 \ \text{ for all } t \in T_j, \ \ j \in \underline{q},$$
$$H_k(x, s) = 0 \ \text{ for all } s \in S_k, \ \ k \in \underline{r},$$
$$x \in \mathbb{R}^n,$$

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where $p, q$, and $r$ are positive integers, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, for each $j \in \underline{q} \equiv \{1, 2, \ldots, q\}$ and $k \in \underline{r}$, $T_j$ and $S_k$ are compact subsets of complete metric spaces, for each $i \in \underline{p}$, $f_i$ and $g_i$ are real-valued functions defined on $\mathbb{R}^n$, for each $j \in \underline{q}$, $G_j(\cdot, t)$ is a real-valued function defined on $\mathbb{R}^n$ for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(\cdot, s)$ is a real-valued function defined on $\mathbb{R}^n$ for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, \cdot)$ and $H_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on $T_j$ and $S_k$ for all $x \in \mathbb{R}^n$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all $x$ satisfying the constraints of (P).

Nonlinear programming problems like (P) but with a finite number of constraints, that is, when the functions $G_j$ are independent of $t$, and the functions $H_k$ are independent of $s$, are known in the literature of mathematical programming as *generalized fractional programming problems.* Due to their generality (they contain the standard types of nonlinear programming problems, namely, problems with fractional, discrete max, and conventional objective functions, as special cases), mathematical tractability (they can be reformulated as parametric problems having nonfractional conventional objective functions), and modeling capabilities (they have been used as meaningful models in certain areas of economics, financial planning, multiobjective decision theory, and facility location problems), these problems have been the subject of intense investigations in the past two decades. For a fairly extensive list of references dealing with various aspects of generalized fractional programming, the reader is referred to [32].

A mathematical programming problem with a finite number of variables and infinitely many constraints is called a *semiinfinite programming problem.* Problems of this type have been utilized for the modeling and analysis of a staggering array of theoretical as well as concrete, real-world, practical problems. More specifically, semiinfinite programming concepts and techniques have found relevance and applications in approximation theory, statistics, game theory, engineering design (design of earthquake-resistant structures, design of control systems, digital filters, electronic circuits, etc.), boundary value problems, defect minimization for operator equations, geometry, random graphs, graphs related to Newton flows, wavelet analysis, reliability testing, environmental protection planning, decision making under uncertainty, semidefinite programming, geometric programming, disjunctive programming, optimal control problems, robotics, and continuum mechanics, among others. For a wealth of information pertaining to various aspects of semiinfinite programming, including areas of applications, optimality conditions, duality relations, and numerical algorithms, the reader is referred to [2, 5, 6, 9 - 13, 16 - 19, 28, 30]. From these and other related sources one can easily see that the two important trends, namely, the ubiquity of duality theories and generalized convexity concepts that have been playing significant roles in the evolution of optimization theory and methodology in general and in nonlinear programming in particular are conspicuously missing in the area of semiinfinite nonlinear programming. In fact, at present there are no publications dealing with nonlinear semiinfinite programming that make substantial use of any class of generalized convex functions in establishing sufficient optimality conditions or duality results. In particular, no sufficiency criteria or duality relations based on generalized convexity concepts have been investigated in the related literature for any kind of semiinfinite minmax programming problems.

In this paper we shall establish several parametric duality results for (P) under a variety of generalized $(\alpha, \eta, \rho)$-V-invexity assumptions. The nonparametric counterparts of these results are presented in [33], and their relevance to sufficient optimality conditions in semiinfinite discrete minmax fractional programming is discussed in [34].

The rest of this paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we consider two duality models with somewhat limited constraint structures, and prove weak, strong, and strict converse duality theorems under two sets of conditions. In Section 4, we formulate another two duality models with much more flexible constraint structures which allow for a greater variety of generalized $(\alpha, \eta, \rho)$-V-invexity hypotheses under which duality can be established. We continue our discussion of duality in Section 5 where we use a certain partitioning scheme and construct two generalized duality models and obtain several duality results under various generalized $(\alpha, \eta, \rho)$-V-invexity assumptions. Finally, in Section 6 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this paper.

Evidently, all the duality results established in this paper can easily be modified and restated for each one of the following seven classes of nonlinear programming problems, which are particular cases of (P):

$$(P1) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}}\ \frac{f_1(x)}{g_1(x)};$$

$$(P2) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}}\ \max_{1\leq i\leq p}\ f_i(x);$$

$$(P3) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}}\ f_1(x),$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in \mathbb{R}^n : G_j(x,t) \leq 0 \ \text{ for all } t \in T_j,\ j \in \underline{q},\\ H_k(x,s) = 0 \text{ for all } s \in S_k,\ \ k \in \underline{r}\};$$

$$(P4) \qquad \text{Minimize}\ \max_{1\leq i\leq p}\ \frac{f_i(x)}{g_i(x)}$$

subject to

$$\tilde{G}_j(x) \leq 0,\ \ j \in \underline{q}, \quad \tilde{H}_k(x) = 0,\ \ k \in \underline{r}, \quad x \in \mathbb{R}^n,$$

where $f_i$ and $g_i$, $i \in \underline{p}$, are as defined in the description of (P), $\tilde{G}_j$, $j \in \underline{q}$, and $\tilde{H}_k$, $k \in \underline{r}$, are real-valued functions defined on $\mathbb{R}^n$, and for each $i \in \underline{p}$, the denominators of the objective function of (P4) are positive for all feasible solutions;

$$(P5) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}}\ \frac{f_1(x)}{g_1(x)};$$

$$(P6) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \max_{1\le i\le p} f_i(x);$$

$$(P7) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ f_1(x),$$

where $\mathbb{G}$ (assumed to be nonempty) is the feasible set of (P4), that is,

$$\mathbb{G} = \{x \in \mathbb{R}^n : \tilde{G}_j(x) \le 0, \ j \in \underline{q}, \quad \tilde{H}_k(x) = 0, \quad k \in \underline{r}\}.$$

Since in most cases these results can easily be altered and rephrased for each one of the above seven problems, we shall not state them explicitly.

## 2. Preliminaries

In this section we recall, for convenience of reference, the definitions of certain classes of generalized convex functions which will be needed in the sequel. We begin by defining an invex function, which has been instrumental in creating a vast array of interesting and important classes of generalized convex functions.

**DEFINITION 0.1.** *Let $f$ be a real-valued differentiable function defined on $\mathbb{R}^n$. Then $f$ is said to be $\eta$-invex at $y$ if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,*

$$f(x) - f(y) \ge \langle \nabla f(y), \eta(x, y)\rangle,$$

*where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)$ is the gradient of $f$ at $y$, and $\langle a, b\rangle$ denotes the inner product of the vectors $a$ and $b$; $f$ is said to be $\eta$-invex on $\mathbb{R}^n$ if the above inequality holds for all $x, y \in \mathbb{R}^n$.*

From this definition it is clear that every real-valued differentiable convex function is invex with $\eta(x, y) = x-y$. This generalization of the concept of convexity was originally proposed by Hanson [14] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \ \text{ subject to } \ g_i(x) \le 0, \ i \in \underline{m}, \ \ x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the same function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [3] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define $\eta$-pseudoinvex and $\eta$-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need a simple extension of invexity, namely, $\rho$-invexity which was originally defined in [20].

Let $h$ be a differentiable real-valued function defined on $\mathbb{R}^n$.

**DEFINITION 0.2.** *The function $h$ is said to be (strictly) $(\eta, \rho)$-invex at $x^*$ if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,*

$$h(x) - h(x^*)(>) \ge \langle \nabla h(x^*), \eta(x, x^*)\rangle + \rho\|x - x^*\|^2.$$

**DEFINITION 0.3.** *The function $h$ is said to be (strictly)* $(\eta, \rho)$-pseudoinvex *at $x^* \in \mathbb{R}^n$ if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$ $(x \neq x^*)$,*

$$\langle \nabla h(x^*), \eta(x, x^*) \rangle \geq -\rho \|x - x^*\|^2 \;\Rightarrow\; h(x)(>) \geq h(x^*).$$

**DEFINITION 0.4.** *The function $h$ is said to be (prestrictly)* $(\eta, \rho)$-quasiinvex *at $x^* \in \mathbb{R}^n$ if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,*

$$h(x)(<) \leq h(x^*) \;\Rightarrow\; \langle \nabla h(x^*), \eta(x, x^*) \rangle \leq -\rho \|x - x^*\|^2.$$

From the above definitions it is clear that if $h$ is $(\eta, \rho)$-invex at $x^*$, then it is both $(\eta, \rho)$-pseudoinvex and $(\eta, \rho)$-quasiinvex at $x^*$, if $h$ is $(\eta, \rho)$-quasiinvex at $x^*$, then it is prestrictly $(\eta, \rho)$-quasiinvex at $x^*$, and if $h$ is strictly $(\eta, \rho)$-pseudoinvex at $x^*$, then it is $(\eta, \rho)$-quasiinvex at $x^*$.

Let the function $F = (F_1, F_2, \ldots, F_N) : \mathbb{R}^n \to \mathbb{R}^N$ be differentiable at $x^*$. The following generalizations of the notions of invexity, pseudoinvexity, and quasiinvexity for vector-valued functions were originally proposed in [21].

**DEFINITION 0.5.** *The function $F$ is said to be* (strictly) $(\alpha, \eta, \bar{\rho})$-V-invex at $x^*$ *if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\} \equiv (0, \infty)$, and $\bar{\rho}_i \in \mathbb{R}$, $i \in \underline{N}$, such that for each $x \in \mathbb{R}^n$ and $i \in \underline{N}$,*

$$F_i(x) - F_i(x^*)\,(>) \geq \langle \alpha_i(x, x^*) \nabla F_i(x^*), \eta(x, x^*) \rangle + \bar{\rho}_i \|x - x^*\|^2.$$

**DEFINITION 0.6.** *The function $F$ is said to be* (strictly) $(\beta, \eta, \tilde{\rho})$-V-pseudoinvex at $x^*$ *if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{N}$, and $\tilde{\rho} \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$ $(x \neq x^*)$,*

$$\Big\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x, x^*) \Big\rangle \geq -\tilde{\rho} \|x - x^*\|^2 \;\;\Rightarrow$$
$$\sum_{i=1}^{N} \beta_i(x, x^*) F_i(x)(>) \geq \sum_{i=1}^{N} \beta_i(x, x^*) F_i(x^*).$$

**DEFINITION 0.7.** *The function $F$ is said to be* (prestrictly) $(\gamma, \eta, \hat{\rho})$-V-quasiinvex at $x^*$ *if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{N}$, and $\hat{\rho} \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,*

$$\sum_{i=1}^{N} \gamma_i(x, x^*) F_i(x)(<) \leq \sum_{i=1}^{N} \gamma_i(x, x^*) F_i(x^*) \;\;\Rightarrow$$
$$\Big\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x, x^*) \Big\rangle \leq -\hat{\rho} \|x - x^*\|^2.$$

In contrast to the case of $(\eta,\rho)$-invex, $(\eta,\rho)$-pseudoinvex, and $(\eta,\rho)$-quasiinvex functions, the relationships among the three classes of functions specified in Definitions 0.5 - 0.7 are not immediately obvious. However, the underlying relationships can be determined by appropriate choices of the functions $\alpha_i, \beta_i$, and $\gamma_i,\ i \in \underline{N}$, and the real numbers $\bar{\rho}_i,\ i \in \underline{N},\ \tilde{\rho}$, and $\hat{\rho}$. Indeed, it is easily seen that an $(\alpha,\eta,\bar{\rho})$-V-invex function is both $(\beta,\eta,\tilde{\rho})$-V-pseudoinvex and $(\gamma,\eta,\hat{\rho})$-V-quasiinvex if we choose $\gamma_i = \beta_i = 1/\alpha_i,\ i \in \underline{N}$, and $\hat{\rho} = \tilde{\rho} = \sum_{i=1}^{N} \bar{\rho}_i/\alpha_i$.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\beta,\eta,\tilde{\rho})$-V-pseudoinvexity can be defined in the following equivalent way:

The function $F$ is said to be *$(\beta,\eta,\tilde{\rho})$-V-pseudoinvex at $x^*$* if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\backslash\{0\},\ i \in \underline{N}$, and $\tilde{\rho} \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{N} \beta_i(x,x^*)F_i(x) < \sum_{i=1}^{N} \beta_i(x,x^*)F_i(x^*) \Rightarrow \Big\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x,x^*)\Big\rangle < -\tilde{\rho}\|x - x^*\|^2.$$

Apparently, the primary motivation for introducing the notion of V-invexity was to relax the rather stringent requirement that in an invex programming problem the invexity property be satisfied for both the objective function and the constraints for the same kernel function $\eta$. It was demonstrated in [21] that this improvement enables one to investigate the optimality and duality aspects of a number of mathematical programming problems, including pseudolinear multiobjective problems and certain types of multiobjective fractional programming problems, in a unified framework.

The concept of $\rho$-invexity has been extended in many directions, and various types of generalized $\rho$-invex functions have been utilized for establishing a wide range of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult [1, 3, 7, 8, 15, 20, 23, 24, 26, 29], and for recent surveys of these and related functions, the reader is referred to [22, 27].

We conclude this section by recalling a necessary optimality result for (P) which was established in [35] by employing a Dinkelbach-type [4] indirect approach and utilizing the necessary optimality conditions given in [6] for a semiinfinite programming problem. The contents and style of this result will serve as our guide in this paper for constructing several parametric duality models for (P) and proving appropriate duality theorems.

**THEOREM 0.1.** *[35] Let $x^* \in \mathbb{F}$, let the functions $f_i$ and $g_i$, $i \in \underline{p}$, be continuously differentiable at $x^*$, for each $j \in \underline{q}$, let $G_j(z,t)$ be differentiable, its partial derivatives be continuous jointly in $z$ and $t$, and for each $k \in \underline{r}$, let the function $H_k(\cdot,s)$ be linear for all $s \in S_k$. If $x^*$ is an optimal solution of $(P)$ and if either one of the two constraint qualifications specified in [6] holds at $x^*$, then there exist $\lambda^* \in \mathbb{R}$, $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^{p} u_i = 1\}$ and integers $\nu_0$ and $\nu$, with $0 \leq \nu_0 \leq \nu \leq n$, such that there exist $\nu_0$ indices $j_m$, with $1 \leq j_m \leq q$, together with $\nu_0$ points $t^m \in$*

$\hat{T}_{j_m}(x^*) \equiv \{t \in T_{j_m} : G_{j_m}(x^*,t)=0\}$, $m \in \underline{\nu_0}$, $\nu-\nu_0$ *indices* $k_m$, *with* $1 \leq k_m \leq r$, *together with* $\nu-\nu_0$ *points* $s^m \in S_{k_m}$ *for* $m \in \underline{\nu}\backslash\underline{\nu_0}$, *and* $\nu$ *real numbers* $v_m^*$ *with* $v_m^* > 0$ *for* $m \in \underline{\nu_0}$, *with the property that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*,s^m) = 0,$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \quad i \in \underline{p}.$$

For brevity, we shall henceforth call $x$ a *normal* feasible solution if $x$ satisfies all the constraints of (P) and either one of the two constraint qualifications given in [6] holds at $x$.

We notice that in the above theorem, it is assumed that for each $k \in \underline{r}$, the function $H_k(\cdot,s)$ is linear for all $s \in S_k$. Of course, this restriction is imposed by Theorem 0.1, which is used in the proofs of strong and strict converse duality theorems. However, for establishing weak duality results for (P), it is not necessary to require that $H_k(\cdot,s)$ be linear. Therefore, in the statements of weak duality theorems, we shall only assume that for each $k \in \underline{r}$, $H_k(\cdot,s)$ is differentiable.

## 3. Duality Model I

In this section, we consider a dual problem with a relatively simple constraint structure and prove weak, strong, and strict converse duality theorems under $(\alpha,\eta,\rho)$-V-invexity conditions. More general duality models for (P) will be discussed in subsequent sections.

Let

$$\mathbb{H} = \Big\{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s}) : y \in \mathbb{R}^n;\ u \in U;\ \lambda \in \mathbb{R}_+;\ 0 \leq \nu_0 \leq \nu \leq n;$$
$$v \in \mathbb{R}^\nu, v_i > 0, 1 \leq i \leq \nu_0;\ J_{\nu_0} = (j_1,j_2,\ldots,j_{\nu_0}), 1 \leq j_i \leq q;\ K_{\nu\backslash\nu_0} = (k_{\nu_0+1},\ldots,k_\nu),$$
$$1 \leq k_i \leq r;\ \bar{t} = (t^1,t^2,\ldots,t^{\nu_0}), t^i \in T_{j_i};\ \bar{s} = (s^{\nu_0+1},\ldots,s^\nu), s^i \in S_{k_i}\Big\}.$$

Consider the following two problems:

(DI) $$\sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y,s^m) = 0, \quad (0.1)$$

$$u_i[f_i(y) - \lambda g_i(y)] + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y,s^m) \geq 0, \quad i \in \underline{p}; \quad (0.2)$$

$$(\tilde{D}\text{I}) \qquad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to (0.2) and

$$\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda\nabla g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m), \eta(x,y)\Big\rangle \geq 0 \text{ for all } x \in \mathbb{F}, \quad (0.3)$$

where $\eta$ is a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$.

The structures of the two problems designated above as (DI) and ($\tilde{D}$I), which can be proved under appropriate $(\alpha, \eta, \rho)$-V-invexity hypotheses to be dual problems for (P), are based directly on the form and contents of the necessary optimality conditions of Theorem 0.1. This is, of course, the standard method for constructing Wolfe-type dual problems. Comparing (D) and ($\tilde{D}$I), we see that ($\tilde{D}$I) is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for ($\tilde{D}$I), but the converse is not necessarily true. Furthermore, we observe that (0.1) is a system of $n$ equations, whereas (0.3) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to ($\tilde{D}$I) because of the dependence of (0.3) on the feasible set of (P). Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for (P) - (DI) and (P) - ($\tilde{D}$I) are almost identical and, therefore, we shall consider only the pair (P) - (DI).

In the sequel, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P); its proof is straightforward and hence omitted.

**LEMMA 0.1.** *For each $x \in \mathbb{R}^n$,*

$$\varphi(x) \equiv \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)} = \max_{u\in U} \frac{\sum_{i=1}^p u_i f_i(x)}{\sum_{i=1}^p u_i g_i(x)}.$$

The next two theorems show that (DI) is a dual problem for (P).

**THEOREM 0.2.** *Let $x$ and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that either one of the following two sets of hypotheses is satisfied:*

*(a)* *(i)* $(f_1, \ldots, f_p)$ *is* $(\theta, \eta, \bar{\rho})$*-V-invex at* $y$*;*

*(ii)* $(-g_1, \ldots, -g_p)$ *is* $(\xi, \eta, \tilde{\rho})$*-V-invex at* $y$*;*

*(iii)* $\big(v_1 G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0} G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is* $(\pi, \eta, \hat{\rho})$*-V-invex at* $y$*;*

*(iv)* $\big(v_{\nu_0+1} H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu H_{k_\nu}(\cdot, s^\nu)\big)$ *is* $(\delta, \eta, \breve{\rho})$*-V-invex at* $y$*;*

*(v)* $\theta_i = \xi_j = \pi_k = \delta_l = \sigma$ *for all* $i,\ j \in \underline{p}, k \in \underline{\nu_0}$*, and* $l \in \underline{\nu}\backslash\underline{\nu_0}$*;*

(vi) $\sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda\tilde{\rho}_i) + \sum_{m=1}^{\nu_0} \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m \geq 0$;

(b) (i) $\big(L_1(\cdot, u, v, \lambda, \bar{t}, \bar{s}), \ldots, L_p(\cdot, u, v, \lambda, \bar{t}, \bar{s})\big)$ *is* $(\theta, \eta, 0)$*-V-pseudoinvex at* $y$*, where*

$$L_i(z, u, v, \lambda, \bar{t}, \bar{s}) = u_i\Big[f_i(z) - \lambda g_i(z) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(z, s^m)\Big], \quad i \in \underline{p}.$$

*Then* $\varphi(x) \geq \lambda$.

*Proof.* (a): Keeping in mind that $u \geq 0$ and $\lambda \geq 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} u_i\{[f_i(x) - \lambda g_i(x)] - [f_i(y) - \lambda g_i(y)]\} \\
&= \sum_{i=1}^{p} u_i\{f_i(x) - f_i(y) - \lambda[g_i(x) - g_i(y)]\} \\
&\geq \sum_{i=1}^{p} u_i[\sigma(x, y)\langle \nabla f_i(y) - \lambda \nabla g_i(y), \eta(x, y)\rangle \\
&\quad + (\bar{\rho}_i + \lambda\tilde{\rho}_i)\|x - y\|^2] \ \text{ (by (i), (ii), and (v))} \\
&= -\sigma(x, y)\Big\langle \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m), \eta(x, y)\Big\rangle \\
&\quad + \sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda\tilde{\rho}_i)\|x - y\|^2 \ \text{ (by (0.1))} \\
&\geq \sum_{m=1}^{\nu_0} v_m[G_{j_m}(y, t^m) - G_{j_m}(x, t^m)] + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m) \\
&\quad + \Big[\sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda\tilde{\rho}_i) + \sum_{m=1}^{\nu_0} \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m\Big]\|x - y\|^2 \\
&\quad \text{(by (iii), (iv), (v) and primal feasibility of } x) \\
&\geq \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m) \ \text{ ( by (vi) and primal feasibility of } x).
\end{aligned}$$

Therefore, in view of (0.2), we have

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0. \tag{0.4}$$

Now using (0.4) and Lemma 0.1, we see that

$$\varphi(x) = \max_{d\in U} \frac{\sum_{i=1}^{p} d_i f_i(x)}{\sum_{i=1}^{p} d_i g_i(x)} \geq \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} \geq \lambda.$$

(b): By our $(\theta, \eta, 0)$-V-pseudoinvexity assumption, (0.1) implies that

$$\sum_{i=1}^{p} \theta_i(x,y)\Big\{u_i[f_i(x) - \lambda g_i(x)] + \sum_{m=1}^{\nu_0} v_m G_{j_m}(x,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(x,s^m)\Big\}$$
$$\geq \sum_{i=1}^{p} \theta_i(x,y)\Big\{u_i[f_i(y) - \lambda g_i(y)] + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y,s^m)\Big\}.$$

Because of (0.2), the right-hand side of this inequality is greater than or equal to zero, and so we have that

$$\sum_{i=1}^{p} \theta_i(x,y)\Big\{u_i[f_i(x) - \lambda g_i(x)] + \sum_{m=1}^{\nu_0} v_m G_{j_m}(x,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(x,s^m)\Big\} \geq 0.$$

But $x \in \mathbb{F}$ and $v_m > 0$ for each $m \in \underline{\nu_0}$, and hence the above inequality reduces to

$$\sum_{i=1}^{p} u_i \theta_i(x,y)[f_i(x) - \lambda g_i(x)] \geq 0. \tag{0.5}$$

Using this inequality and Lemma 0.1, we see that

$$\begin{aligned}\varphi(x) &= \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)} = \max_{1\leq i\leq p} \frac{\theta_i(x,y) f_i(x)}{\theta_i(x,y) g_i(x)} \ \ (\text{since } \theta_i(x,y) > 0,\ i \in \underline{p}) \\ &= \max_{d\in U} \frac{\sum_{i=1}^{p} d_i \theta_i(x,y) f_i(x)}{\sum_{i=1}^{p} d_i \theta_i(x,y) g_i(x)} \ \ (\text{by Lemma 0.1}) \\ &\geq \frac{\sum_{i=1}^{p} u_i \theta_i(x,y) f_i(x)}{\sum_{i=1}^{p} u_i \theta_i(x,y) g_i(x)} \\ &\geq \lambda \ \ (\text{by (0.5)}).\end{aligned}$$

□

**THEOREM 0.3.** *(Strong Duality) Let $x^*$ be a normal optimal solution of (P), let the functions $f_i$ and $g_i, i \in \underline{p}$, be continuously differentiable at $x^*$, for each $j \in \underline{q}$, let the function $G_j(z,t)$ be differentiable for all $t \in T_j$, its partial derivatives be continuous jointly in $z$ and $t$, and for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be linear for all $s \in S_k$. Assume that for each feasible solution $(y, u, v, \lambda, \nu, \nu_0,\ J_{\nu_0}, K_{\nu\backslash\nu_0},\ \bar{t}, \bar{s})$ of (DI), either one of the two sets of conditions specified in Theorem 0.2 is satisfied. Then there exist $u^*, v^*, \lambda^*,\ \nu^*, \nu_0^*, J_{\nu_0^*},\ K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*$ such that $(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*,$ $J_{\nu_0^*},\ K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$.*

*Proof.* Since $x^*$ is a normal optimal solution of (P), by Theorem 0.1, there exist $u^*, v^*, \lambda^* (= \varphi(x^*))$, $\nu^*$, $\nu_0^*, J_{\nu_0^*}$, $K_{\nu^*\backslash\nu_0^*}, \bar{t}^*$, $\bar{s}^*$ such that $(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is a feasible solution of (DI). Since $\varphi(x^*) = \lambda^*$, the optimality of $(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ for (DI) follows from Theorem 0.2. □

We also have the following converse duality result for (P) - (DI).

**THEOREM 0.4.** *(Strict Converse Duality) Let $x^*$ be a normal optimal solution of (P), let the functions $f_i$ and $g_i, i \in \underline{p}$, be continuously differentiable at $x^*$, for each $j \in \underline{q}$, let the function $G_j(z,t)$ be differentiable for all $t \in T_j$, its partial derivatives be continuous jointly in $z$ and $t$, for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be linear for all $s \in S_k$, and let $(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu_0}, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\backslash\tilde{\nu}_0}, \tilde{\bar{t}}, \tilde{\bar{s}})$ be an optimal solution of (DI). Assume that either one of the following two sets of hypotheses is satisfied:*

- *(a) The assumptions of part (a) of Theorem 0.2 are satisfied for the feasible solution $(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\backslash\tilde{\nu}_0}, \tilde{\bar{t}}, \tilde{\bar{s}})$ of (DI), $(f_1, \dots, f_p)$ is strictly $(\theta, \eta, \bar{\rho})$-V-invex at $\tilde{x}$, or $(-g_1, \dots, -g_p)$ is strictly $(\xi, \eta, \tilde{\rho})$-V-invex at $\tilde{x}$, or $\big(\tilde{v}_1 G_{j_1}(\cdot, \tilde{t}^1), \dots, \tilde{v}_{\tilde{\nu}_0} G_{j_{\tilde{\nu}_0}}(\cdot, \tilde{t}^{\tilde{\nu}_0})\big)$ is strictly $(\pi, \eta, \hat{\rho})$-V-invex at $\tilde{x}$, or $\big(\tilde{v}_{\tilde{\nu}_0+1} H_{k_{\tilde{\nu}_0+1}}(\cdot, \tilde{s}^{\tilde{\nu}_0+1}), \dots, \tilde{v}_{\tilde{\nu}} H_{k_{\tilde{\nu}}}(\cdot, \tilde{s}^{\tilde{\nu}})\big)$ is strictly $(\delta, \eta, \breve{\rho})$-V-invex at $\tilde{x}$, or $\sum_{i=1}^{p} \tilde{u}_i(\bar{\rho}_i + \tilde{\lambda}\tilde{\rho}_i) + \sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m \hat{\rho}_m + \sum_{m=\tilde{\nu}_0+1}^{\tilde{\nu}} \breve{\rho}_m > 0$.*
- *(b) The function $L(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\bar{t}}, \tilde{\bar{s}})$ is strictly $(\theta, \eta, 0)$-V-pseudoinvex at $\tilde{x}$.*

*Then $\tilde{x} = x^*$, that is, $\tilde{x}$ is an optimal solution of (P), and $\varphi(x^*) = \tilde{\lambda}$.*

*Proof.* (a): Suppose to the contrary that $\tilde{x} \neq x^*$. By Theorem 0.1, there exist $u^*, v^*, \lambda^*, \nu^*, \nu_0^*$, $J_{\nu_0^*}$, $K_{\nu^*\backslash\nu_0^*}, \bar{t}^*$, $\bar{s}^*$ such that $(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is a feasible solution of (DI) and $\varphi(x^*) = \lambda^*$. Now proceeding as in the proof of Theorem 0.2 (with $x$ replaced by $x^*$ and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ by $(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\backslash\tilde{\nu}_0}, \tilde{\bar{t}}, \tilde{\bar{s}})$) and using any of the conditions set forth above, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i[f_i(x^*) - \tilde{\lambda} g_i(x^*)] > 0.$$

Using Lemma 0.1 and this inequality, as in the proof of Theorem 0.2, we obtain $\varphi(x^*) > \tilde{\lambda}$, which contradicts the fact that $\varphi(x^*) = \lambda^* \leq \tilde{\lambda}$. Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

(b): The proof is similar to that of part (a). □

## 4. Duality Model II

In this section, we consider certain variants of (DI) and ($\tilde{D}$I) that allow for a greater variety of generalized $(\alpha, \eta, \rho)$-V-invexity conditions under which duality can be established. These duality models have the following forms:

(DII) $\sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$

subject to (0.1) and

$$u_i[f_i(y) - \lambda g_i(y)] \geq 0, \quad i \in \underline{p}, \tag{0.6}$$

$$G_{j_m}(y, t^m) \geq 0, \quad m \in \underline{\nu_0}, \tag{0.7}$$

$$v_m H_{k_m}(y, s^m) \geq 0, \quad m \in \underline{\nu}\backslash\underline{\nu_0}; \tag{0.8}$$

($\tilde{D}$II) $\sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$

subject to (0.3) and (0.6) - (0.8).

The remarks and observations made earlier about the relationships between (DI) and ($\tilde{D}$I) are, of course, also valid for (DII) and ($\tilde{D}$II). Since the constraint inequalities of (DII) are formed by splitting the inequality (0.2) into three inequalities (0.6), (0.7), and (0.8), it is clear that Theorems 0.2 - 0.4 are valid for the pair (P) - (DII). Below, we shall establish some duality results in which various generalized $(\alpha, \eta, \rho)$-V-invexity requirements will be placed on the vector function $(\mathcal{E}_1(\cdot, \lambda, u), \ldots, \mathcal{E}_p(\cdot, \lambda, u))$, where for each $i \in \underline{p}$, the component function $\mathcal{E}_i(\cdot, \lambda, u)$ is defined, for fixed $\lambda$ and $u$, on $\mathbb{R}^n$ by

$$\mathcal{E}_i(z, \lambda, u) = u_i[f_i(z) - \lambda g_i(z)].$$

**THEOREM 0.5.** *Let $x$ and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

*(a)* *(i)* $\big(\mathcal{E}_1(\cdot, \lambda, u), \ldots, \mathcal{E}_p(\cdot, \lambda, u)\big)$ *is $(\theta, \eta, \rho)$-V-pseudoinvex at $y$;*

*(ii)* $\big(v_1 G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0} G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at $y$;*

*(iii)* $\big(v_{\nu_0+1} H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \hat{\rho})$-V-quasiinvex at $y$;*

*(iv)* $\rho + \tilde{\rho} + \hat{\rho} \geq 0$*;*

*(b)* *(i)* $\big(\mathcal{E}_1(\cdot, \lambda, u), \ldots, \mathcal{E}_p(\cdot, \lambda, u)\big)$ *is prestrictly $(\theta, \eta, \rho)$-V-quasiinvex at $y$;*

*(ii)* $\big(v_1 G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0} G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at $y$;*

*(iii)* $\big(v_{\nu_0+1} H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \hat{\rho})$-V-quasiinvex at $y$;*

*(iv)* $\rho + \tilde{\rho} + \hat{\rho} > 0$*;*

*(c)* *(i)* $\big(\mathcal{E}_1(\cdot, \lambda, u), \ldots, \mathcal{E}_p(\cdot, \lambda, u)\big)$ *is prestrictly $(\theta, \eta, \rho)$-V-quasiinvex at $y$;*

*(ii)* $\big(v_1 G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0} G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is strictly $(\pi, \eta, \tilde{\rho})$-V-pseudoinvex at $y$;*

*(iii)* $\big(v_{\nu_0+1} H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \hat{\rho})$-V-quasiinvex at $y$;*

*(iv)* $\rho + \tilde{\rho} + \hat{\rho} \geq 0$*;*

*(d)* *(i)* $\big(\mathcal{E}_1(\cdot,\lambda,u),\ldots,\mathcal{E}_p(\cdot,\lambda,u)\big)$ *is prestrictly* $(\theta,\eta,\rho)$*-V-quasiinvex at* $y$*;*

*(ii)* $\big(v_1G_{j_1}(\cdot,t^1),\ldots,v_{\nu_0}G_{j_{\nu_0}}(\cdot,t^{\nu_0})\big)$ *is* $(\pi,\eta,\tilde{\rho})$*-V-quasiinvex at* $y$*;*

*(iii)* $\big(v_{\nu_0+1}H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\ldots,v_\nu H_{k_\nu}(\cdot,s^\nu)\big)$ *is strictly* $(\delta,\eta,\hat{\rho})$*-V-pseudoinvex at* $y$*;*

*(iv)* $\rho+\tilde{\rho}+\hat{\rho}\geq 0$.

*Then* $\varphi(x)\geq\lambda$.

*Proof.* (a): Because of (0.7) and primal feasibility of $x$, we have $G_{j_m}(x,t^m)\leq 0\leq G_{j_m}(y,t^m)$, and hence

$$\sum_{m=1}^{\nu_0} v_m\pi_m(x,y)G_{j_m}(x,t^m)\leq\sum_{m=1}^{\nu_0} v_m\pi_m(x,x)G_{j_m}(y,t^m),$$

which in view of (ii) implies that

$$\Big\langle\sum_{m=1}^{\nu_0} v_m\nabla G_{j_m}(y,t^m),\eta(x,y)\Big\rangle\leq-\tilde{\rho}\|x-y\|^2. \tag{0.9}$$

Similarly, we can show that our assumptions in (iii) combined with the primal feasibility of $x$ and (0.8) lead to the following inequality:

$$\Big\langle\sum_{m=\nu_0+1}^{\nu} v_m\nabla H_{k_m}(y,s^m),\eta(x,y)\Big\rangle\leq-\hat{\rho}\|x-y\|^2. \tag{0.10}$$

Now because of (0.9), (0.10), and (iv), (0.1) reduces to

$$\Big\langle\sum_{i=1}^{p} u_i[\nabla f_i(y)-\lambda\nabla g_i(y)],\eta(x,y)\Big\rangle\geq-\rho\|x-y\|^2,$$

which in view of (i) implies that

$$\sum_{i=1}^{p} u_i\theta_i(x,y)[f_i(x)-\lambda g_i(x)]\geq\sum_{i=1}^{p} u_i\theta_i(x,y)[f_i(y)-\lambda g_i(y)]\geq 0,$$

where the last inequality follows from (0.6). In the proof of part (b) of Theorem 0.2 it was shown that this inequality leads to the desired conclusion that $\varphi(x)\geq\lambda$.

(b) - (d): The proofs are similar to that of part (a). □

**THEOREM 0.6.** *(Strong Duality) Let* $x^*$ *be a normal optimal solution of (P), let the functions* $f_i$ *and* $g_i, i\in\underline{p}$*, be continuously differentiable at* $x^*$*, for each* $j\in\underline{q}$*, let the function* $G_j(z,t)$ *be differentiable for all* $t\in T_j$*, its partial derivatives be continuous jointly in* $z$ *and* $t$*, and for each* $k\in\underline{r}$*, let the function* $H_k(\cdot,s)$ *be linear for all* $s\in S_k$*. Assume that for each feasible solution* $(y,u,v,\lambda,\nu,\nu_0,\,J_{\nu_0},K_{\nu\backslash\nu_0},\,\bar{t},\bar{s})$ *of (DII), any one of the four sets of conditions specified in Theorem 0.5 is satisfied. Then there exist* $u^*,v^*,\lambda^*,\ \nu^*,\nu_0^*,J_{\nu_0^*},\ K_{\nu^*\backslash\nu_0^*},\bar{t}^*,\bar{s}^*$ *such that* $(x^*,u^*,v^*,\lambda^*,\nu^*,\nu_0^*,$ $J_{\nu_0^*},\ K_{\nu^*\backslash\nu_0^*},\bar{t}^*,\bar{s}^*)$ *is an optimal solution of (DII) and* $\varphi(x^*)=\lambda^*$.

*Proof.* The proof is similar to that of Theorem 0.3. □

We also have the following converse duality result for (P) - (DII).

**THEOREM 0.7.** *(Strict Converse Duality) Let $x^*$ be a normal optimal solution of (P), let the functions $f_i$ and $g_i, i \in \underline{p}$, be continuously differentiable at $x^*$, for each $j \in \underline{q}$, let the function $G_j(z,t)$ be differentiable for all $t \in T_j$, its partial derivatives be continuous jointly in $z$ and $t$, for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be linear for all $s \in S_k$, and let $\tilde{w} \equiv (\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu_0},\ J_{\tilde{\nu}_0},\ K_{\tilde{\nu}\backslash\tilde{\nu}_0},\ \bar{\tilde{t}}, \bar{\tilde{s}})$ be an optimal solution of (DII). Assume that any one of the following four sets of hypotheses is satisfied:*

*(a) The assumptions specified in part (a) of Theorem 0.5 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\mathcal{E}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{E}_p(\cdot, \tilde{\lambda}, \tilde{u})\big)$ is strictly $(\theta, \eta, \rho)$-V-pseudoinvex at $\tilde{x}$.*

*(b) The assumptions specified in part (b) of Theorem 0.5 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\mathcal{E}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{E}_p(\cdot, \tilde{\lambda}, \tilde{u})\big)$ is $(\theta, \eta, \rho)$-V-quasiinvex at $\tilde{x}$.*

*(c) The assumptions specified in part (c) of Theorem 0.5 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\mathcal{E}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{E}_p(\cdot, \tilde{\lambda}, \tilde{u})\big)$ is $(\theta, \eta, \rho)$-V-quasiinvex at $\tilde{x}$.*

*(d) The assumptions specified in part (d) of Theorem 0.5 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\mathcal{E}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{E}_p(\cdot, \tilde{\lambda}, \tilde{u})\big)$ is $(\theta, \eta, \rho)$-V-quasiinvex at $\tilde{x}$.*

*Then $\tilde{x} = x^*$, that is, $\tilde{x}$ is an optimal solution of (P), and $\varphi(x^*) = \tilde{\lambda}$.*

*Proof.* The proof is similar to that of Theorem 0.4. □

## 5. Duality Model III

In this section, we discuss several families of duality results under various generalized $(\alpha, \eta, \rho)$-V-invexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions. This is accomplished by employing a certain type of partitioning scheme which was originally proposed in [25] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\nu_0$ and $\nu$ be integers, with $1 \leq \nu_0 \leq \nu \leq n$, and let $\{J_0, J_1, \ldots, J_M\}$ and $\{K_0, K_1, \ldots, K_M\}$ be partitions of the sets $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively; thus, $J_i \subset \underline{\nu_0}$ for each $i \in \underline{M} \cup \{0\}$, $J_i \cap J_j = \emptyset$ for each $i, j \in \underline{M} \cup \{0\}$ with $i \neq j$, and $\cup_{i=0}^{M} J_i = \underline{\nu_0}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_M\}$. Moreover, if $m_1$ and $m_2$ are the numbers of the partitioning sets of $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$ for $i > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i(\cdot, u, v, \lambda, \bar{t}, \bar{s}),\ i \in \underline{p}$, and $\Lambda_\tau(\cdot, v, \bar{t}, \bar{s})$ defined, for fixed $u, v, \lambda, \bar{t} \equiv (t^1, t^2, \ldots, t^{\nu_0})$, and $\bar{s} \equiv (s^{\nu_0+1},\ s^{\nu_0+2},\ \ldots, s^\nu)$, on $\mathbb{R}^n$ as follows:

$$\Phi_i(z, u, v, \lambda, \bar{t}, \bar{s}) = u_i\Big[f_i(z) - \lambda g_i(z) + \sum_{m\in J_0} v_m G_{j_m}(z, t^m) + \sum_{m\in K_0} v_m H_{k_m}(z, s^m)\Big],\quad i \in \underline{p},$$

$$\Lambda_\tau(z, v, \bar{t}, \bar{s}) = \sum_{m\in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(z, s^m),\quad \tau \in \underline{M}.$$

Making use of the sets and functions defined above, we can state our general duality models as follows:

(DIII) $\displaystyle\sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$

subject to (0.1) and

$$f_i(y) - \lambda g_i(y) + \sum_{m\in J_0} v_m G_{j_m}(y, t^m) + \sum_{m\in K_0} v_m H_{k_m}(y, s^m) \geq 0,\quad i \in \underline{p}, \tag{0.11}$$

$$\sum_{m\in J_\tau} v_m G_{j_m}(y, t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(y, s^m) \geq 0,\quad \tau \in \underline{M}; \tag{0.12}$$

($\tilde{D}$III) $\displaystyle\sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$

subject to (0.3), (0.11), and (0.12).

The remarks made earlier about the relationships between (DI) and ($\tilde{D}$I) are, of course, also valid for (DIII) and ($\tilde{D}$III).

The next two theorems show that (DIII) is a dual problem for (P).

**THEOREM 0.8.** *Let $x$ and $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

*(a)* *(i)* $\big(\Phi_1(\cdot, u, v, \lambda, \bar{t}, \bar{s}), \ldots, \Phi_p(\cdot, u, v, \lambda, \bar{t}, \bar{s})\big)$ *is $(\theta, \eta, \bar{\rho})$-V-pseudoinvex at $y$;*

*(ii)* $\big(\Lambda_1(\cdot, v, \bar{t}, \bar{s}), \ldots, \Lambda_M(\cdot, v, \bar{t}, \bar{s})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at $y$;*

*(iii)* $\bar{\rho} + \tilde{\rho} \geq 0$*;*

*(b)* *(i)* $\big(\Phi_1(\cdot, u, v, \lambda, \bar{t}, \bar{s}), \ldots, \Phi_p(\cdot, u, v, \lambda, \bar{t}, \bar{s})\big)$ *is prestrictly $(\theta, \eta, \bar{\rho})$-V-quasiinvex at $y$;*

*(ii)* $\big(\Lambda_1(\cdot, v, \bar{t}, \bar{s}), \ldots, \Lambda_M(\cdot, v, \bar{t}, \bar{s})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at $y$;*

*(iii)* $\bar{\rho} + \tilde{\rho} > 0$*;*

*(c)* *(i)* $\big(\Phi_1(\cdot, u, v, \lambda, \bar{t}, \bar{s}), \ldots, \Phi_p(\cdot, u, v, \lambda, \bar{t}, \bar{s})\big)$ *is prestrictly $(\theta, \eta, \bar{\rho})$-V-quasiinvex at $y$;*

*(ii)* $\big(\Lambda_1(\cdot, v, \bar{t}, \bar{s}), \ldots, \Lambda_M(\cdot, v, \bar{t}, \bar{s})\big)$ *is strictly* $(\pi, \eta, \tilde{\rho})$*-V-pseudoinvex at* $y$*;*

*(iii)* $\bar{\rho} + \tilde{\rho} \geq 0$*;*

*Then* $\varphi(x) \geq \lambda$.

*Proof.* (a): It is clear that (0.1) can be expressed as follows:

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{m \in J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{k_m}(y, s^m)$$
$$+ \sum_{\tau=1}^{M} \Big[ \sum_{m \in J_\tau} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{k_m}(y, s^m) \Big] = 0. \quad (0.13)$$

Since $x \in \mathbb{F}$, $v_m > 0$, $m \in \nu_0$, and (0.12) holds, it follows that for each $\tau \in \underline{M}$,

$$\begin{aligned} \Lambda_\tau(x, v, \bar{t}, \bar{s}) &= \sum_{m \in J_\tau} v_m G_{j_m}(x, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(x, s^m) \\ &\leq 0 \leq \sum_{m \in J_\tau} v_m G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(y, s^m) \\ &= \Lambda_\tau(y, v, \bar{t}, \bar{s}), \end{aligned}$$

and hence

$$\sum_{\tau=1}^{M} \pi_\tau(x, y) \Lambda_\tau(x, v, \bar{t}, \bar{s}) \leq \sum_{\tau=1}^{M} \pi_\tau(x, y) \Lambda_\tau(y, v, \bar{t}, \bar{s}),$$

which because of (ii) implies that

$$\Big\langle \sum_{\tau=1}^{M} \Big[ \sum_{m \in J_\tau} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{k_m}(y, s^m) \Big], \eta(x, y) \Big\rangle \leq -\tilde{\rho} \|x - y\|^2. \quad (0.14)$$

Combining (0.13) and (0.14), and using (iii) we get

$$\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{m \in J_0} v_m \nabla G_{j_m}(y, t^m)$$
$$+ \sum_{m \in K_0} v_m \nabla H_{k_m}(y, s^m), \eta(x, y) \Big\rangle \geq \tilde{\rho} \|x - y\|^2 \geq -\bar{\rho} \|x - y\|^2, \quad (0.15)$$

which by virtue of (i) implies that

$$\sum_{i=1}^{p} \theta_i(x, y) \Phi_i(x, u, v, \lambda, \bar{t}, \bar{s}) \geq \sum_{i=1}^{p} \theta_i(x, y) \Phi_i(y, u, v, \lambda, \bar{t}, \bar{s}). \quad (0.16)$$

Because $x \in \mathbb{F}$, $v_m > 0$, $m \in \underline{\nu_0}$, and (0.11) holds, this inequality reduces to

$$0 \leq \sum_{i=1}^{p} u_i \theta_i(x, y)[f_i(x) - \lambda g_i(x)].$$

Now using this inequality and Lemma 0.1, as in the proof of Theorem 0.2, we obtain $\varphi(x) \geq \lambda$.

(b): Proceeding in exactly the same manner as in the proof of part (a), we obtain (0.15) in which the second inequality is strict. By (i), this implies that (0.16) holds and, therefore, the rest of the proof is identical to that of part (a).

(c): The proof is similar to those of parts (a) and (b). □

**THEOREM 0.9.** *(Strong Duality) Let $x^*$ be a normal optimal solution of (P), let the functions $f_i$ and $g_i, i \in \underline{p}$, be continuously differentiable at $x^*$, for each $j \in \underline{q}$, let the function $G_j(z,t)$ be differentiable for all $t \in T_j$, its partial derivatives be continuous jointly in $z$ and $t$, and for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be linear for all $s \in S_k$. Assume that for each feasible solution $(y, u, v, \lambda, \nu, \nu_0,\ J_{\nu_0}, K_{\nu\backslash\nu_0},\ \bar{t}, \bar{s})$ of (DIII), any one of the three sets of conditions specified in Theorem 0.8 is satisfied. Then there exist $u^*, v^*, \lambda^*,\ \nu^*, \nu_0^*, J_{\nu_0^*},\ K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*$ such that $(x^*, u^*, v^*, \lambda^*, \nu^*, \nu_0^*, J_{\nu_0^*},\ K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is an optimal solution of (DIII) and $\varphi(x^*) = \lambda^*$.*

*Proof.* The proof is similar to that of Theorem 0.3. □

We also have the following converse duality result for (P) - (DIII).

**THEOREM 0.10.** *(Strict Converse Duality) Let $x^*$ be a normal optimal solution of (P), let the functions $f_i$ and $g_i, i \in \underline{p}$, be continuously differentiable at $x^*$, for each $j \in \underline{q}$, let the function $G_j(z,t)$ be differentiable for all $t \in T_j$, its partial derivatives be continuous jointly in $z$ and $t$, for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be linear for all $s \in S_k$, and let $\tilde{w} \equiv (\tilde{x}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0,\ J_{\tilde{\nu}_0},\ K_{\tilde{\nu}\backslash\tilde{\nu}_0},\ \bar{\bar{t}}, \bar{\bar{s}})$ be an optimal solution of (DIII). Assume that any one of the following three sets of hypotheses is satisfied:*

*(a) The assumptions specified in part (a) of Theorem 0.8 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\Phi_1(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\bar{t}}, \bar{\bar{s}}), \ldots, \Phi_p(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\bar{t}}, \bar{\bar{s}})\big)$ is strictly $(\theta, \eta, \bar{\rho})$-V-pseudoinvex at $\tilde{x}$.*

*(b) The assumptions specified in part (b) of Theorem 0.8 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\Phi_1(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\bar{t}}, \bar{\bar{s}}), \ldots, \Phi_p(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\bar{t}}, \bar{\bar{s}})\big)$ is $(\theta, \eta, \bar{\rho})$-V-quasiinvex $\tilde{x}$.*

*(c) The assumptions specified in part (c) of Theorem 0.8 are satisfied for the feasible solution $\tilde{w}$. Moreover, assume that $\big(\Phi_1(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\bar{t}}, \bar{\bar{s}}), \ldots, \Phi_p(\cdot, \tilde{u}, \tilde{v}, \tilde{\lambda}, \bar{\bar{t}}, \bar{\bar{s}})\big)$ is $(\theta, \eta, \bar{\rho})$-V-quasiinvex $\tilde{x}$.*

*Then $\tilde{x} = x^*$, that is, $\tilde{x}$ is an optimal solution of (P), and $\varphi(x^*) = \tilde{\lambda}$.*

*Proof.* The proof is similar to that of Theorem 0.4 □

Each one of the three sets of results given in Theorem 0.8 can be viewed as a family of duality results whose members can easily be identified by appropriate choices of the partitioning sets $J_\mu$ and $K_\mu,\ \mu \in \underline{M} \cup \{0\}$.

## 6. Concluding Remarks

Taking a Dinkelbach-type [4] parametric approach, in this study we have established a fairly large number of parametric duality results under various generalized $(\alpha, \eta, \rho)$-V-invexity hypotheses for a semiinfinite discrete minmax fractional programming problem. It appears that all these results are new in the area of semiinfinite programming. Since all the results obtained here can be modified and restated in a straightforward manner for each one of the seven problems designated as (P1) - (P7) in Section 1, they collectively subsume a fairly large number of existing results in the areas of conventional nonlinear programming and semiinfinite nonlinear programming. Furthermore, the style and techniques employed in this paper can be utilized to establish similar results for some other classes of related optimization problems. For example, it seems reasonable to expect that a similar approach can be applied to investigate the optimality and duality aspects of the following two closely related classes of semiinfinite continuous minmax fractional and multiobjective fractional programming problems:

$$\underset{x \in \mathbb{F}}{\text{Minimize}} \ \max_{y \in Y} \frac{f(x, y)}{g(x, y)},$$

$$\underset{x \in \mathbb{F}}{\text{Minimize}} \ \left( \frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)} \right).$$

We shall investigate these two classes of semiinfinite programming problems in subsequent papers.

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