Global Journal of Mathematics & Mathematical Sciences.
ISSN 0972-9836 Vol.1 No.1(2005), pp. 19-38

http://www.gbspublisher.com/gjmms.htm

Global Parametric Sufficient Optimality Conditions for Semiinfinite Discrete Minmax Fractional Programming Problems Containing Generalized (α, η, ρ)-V-Invex Functions

G. J. Zalmai[1] and **Qinghong Zhang**[2]
Department of Mathematics and Computer Science
Northern Michigan University
Marquette, Michigan 49855

Abstract

In this paper, we discuss a fairly large number of sets of global parametric sufficient optimality conditions under various generalized (α, η, ρ)-V-invexity assumptions for a semiinfinite discrete minmax fractional programming problem.

AMS Subject Classification 2000: primary: 90C30, 90C34; secondary: 90C29, 90C32.

Keywords: Semiinfinite programming, discrete minmax fractional programming, generalized invex functions, infinitely many equality and inequality constraints, sufficient optimality conditions.

1. Introduction

Our purpose in this paper is to formulate and prove several sets of global parametric sufficient optimality conditions under various generalized (α, η, ρ)-V-invexity assumptions for the following semiinfinite discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x,t) \leq 0 \ \text{ for all } t \in T_j, \ \ j \in \underline{q},$$
$$H_k(x,s) = 0 \ \text{ for all } s \in S_k, \ \ k \in \underline{r},$$

[1]E-mail address: gzalmai@nmu.edu
[2]Corresponding author. E-mail address: qzhang@nmu.edu

$$x \in \mathbb{R}^n,$$

where p, q, and r are positive integers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, for each $j \in \underline{q} \equiv \{1, 2, \ldots, q\}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on $\mathbb{R}^n$, for each $j \in \underline{q}$, $G_j(\cdot, t)$ is a real-valued function defined on $\mathbb{R}^n$ for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(\cdot, s)$ is a real-valued function defined on $\mathbb{R}^n$ for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, \cdot)$ and $H_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in \mathbb{R}^n$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

Nonlinear programming problems like (P) but with a finite number of constraints, that is, when the functions G_j are independent of t, and the functions H_k are independent of s, are known in the literature of mathematical programming as *generalized fractional programming problems.* Due to their generality (they contain the standard types of nonlinear programming problems, namely, problems with fractional, discrete max, and conventional objective functions, as special cases), mathematical tractability (they can be reformulated as parametric problems having nonfractional conventional objective functions), and modeling capabilities (they have been used as meaningful models in certain areas of economics, financial planning, multiobjective decision theory, and facility location problems), these problems have been the subject of intense investigations in the past two decades. For a fairly extensive list of references dealing with various aspects of generalized fractional programming, the reader is referred to [32].

A mathematical programming problem with a finite number of variables and infinitely many constraints is called a *semiinfinite programming problem.* Problems of this type have been utilized for the modeling and analysis of a staggering array of theoretical as well as concrete, real-world, practical problems. More specifically, semiinfinite programming concepts and techniques have found relevance and applications in approximation theory, statistics, game theory, engineering design (design of earthquake-resistant structures, design of control systems, digital filters, electronic circuits, etc.), boundary value problems, defect minimization for operator equations, geometry, random graphs, graphs related to Newton flows, wavelet analysis, reliability testing, environmental protection planning, decision making under uncertainty, semidefinite programming, geometric programming, disjunctive programming, optimal control problems, robotics, and continuum mechanics, among others. For a wealth of information pertaining to various aspects of semiinfinite programming, including areas of applications, optimality conditions, duality relations, and numerical algorithms, the reader is referred to [2, 5, 6, 9 - 13, 16 - 19, 28, 30]. From these and other related sources one can easily see that the two important trends, namely, the ubiquity of duality theories and generalized convexity concepts that have been playing significant roles in the evolution of optimization theory and methodology in general and in nonlinear programming in particular are conspicuously missing in the area of semiinfinite nonlinear programming. In fact, at present there are no publications dealing with nonlinear semiinfinite programming that make substantial use of any class of generalized convex functions in establishing sufficient optimality conditions or duality results. In particular, no sufficiency criteria or duality relations based on generalized convexity concepts have been investigated in the related literature for any kind of semiinfinite minmax programming problems.

Adopting a Dinkelbach-type [4] indirect parametric approach, in this paper we shall establish several global parametric sufficiency results for (P) under a variety of generalized (α, η, ρ)-V-invexity assumptions. The nonparametric counterparts of these results are presented in [33], and their relevance to duality in semiinfinite discrete minmax fractional programming is discussed in [34].

The rest of this paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we begin our discussion of sufficient optimality conditions where we formulate and prove a number of sufficiency criteria under a variety of generalized (α, η, ρ)-V-invexity assumptions that are placed on certain vector functions. Utilizing two partitioning schemes, in Section 4 we establish several sets of generalized sufficiency results each of which is in fact a family of such results whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 5 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this paper.

Evidently, all the sufficient optimality results established in this paper can easily be modified and restated for each one of the following seven classes of nonlinear programming problems, which are particular cases of (P):

$$(P1) \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \; \frac{f_1(x)}{g_1(x)};$$

$$(P2) \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \; \max_{1 \leq i \leq p} \; f_i(x);$$

$$(P3) \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \; f_1(x),$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in \mathbb{R}^n : G_j(x,t) \leq 0 \text{ for all } t \in T_j, \; j \in \underline{q}, \qquad H_k(x,s) = 0 \text{ for all } s \in S_k, \quad k \in \underline{r}\};$$

$$(P4) \qquad \text{Minimize} \; \max_{1 \leq i \leq p} \; \frac{f_i(x)}{g_i(x)}$$

subject to

$$\tilde{G}_j(x) \leq 0, \;\; j \in \underline{q}, \quad \tilde{H}_k(x) = 0, \;\; k \in \underline{r}, \quad x \in \mathbb{R}^n,$$

where f_i and g_i, $i \in \underline{p}$, are as defined in the description of (P), $\tilde{G}_j$, $j \in \underline{q}$, and $\tilde{H}_k$, $k \in \underline{r}$, are real-valued functions defined on $\mathbb{R}^n$, and for each $i \in \underline{p}$, the denominators of the objective function of (P4) are positive for all feasible solutions;

$$(P5) \qquad \underset{x \in \mathbb{G}}{\text{Minimize}} \; \frac{f_1(x)}{g_1(x)};$$

$$(P6) \qquad \underset{x \in \mathbb{G}}{\text{Minimize}} \; \max_{1 \leq i \leq p} \; f_i(x);$$

$$(P7) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}}\ f_1(x),$$

where $\mathbb{G}$ (assumed to be nonempty) is the feasible set of (P4), that is,

$$\mathbb{G} = \{x \in \mathbb{R}^n : \tilde{G}_j(x) \leq 0,\ j \in \underline{q}, \quad \tilde{H}_k(x) = 0, \quad k \in \underline{r}\}.$$

Since in most cases these results can easily be altered and rephrased for each one of the above seven problems, we shall not state them explicitly.

2. Preliminaries

In this section we recall, for convenience of reference, the definitions of certain classes of generalized convex functions which will be needed in the sequel. We begin by defining an invex function, which has been instrumental in creating a vast array of interesting and important classes of generalized convex functions.

DEFINITION 0.1. *Let f be a real-valued differentiable function defined on $\mathbb{R}^n$. Then f is said to be η-invex at y if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,*

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y)\rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)$ is the gradient of f at y, and $\langle a, b\rangle$ denotes the inner product of the vectors a and b; f is said to be η-invex on $\mathbb{R}^n$ if the above inequality holds for all $x, y \in \mathbb{R}^n$.

From this definition it is clear that every real-valued differentiable convex function is invex with $\eta(x, y) = x-y$. This generalization of the concept of convexity was originally proposed by Hanson [14] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0,\ i \in \underline{m}, \quad x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the same function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [3] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need a simple extension of invexity, namely, ρ-invexity which was originally defined in [20].

Let h be a differentiable real-valued function defined on $\mathbb{R}^n$.

DEFINITION 0.2. *The function h is said to be (strictly) (η, ρ)-invex at x^* if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$ $(x \neq x^*)$,*

$$h(x) - h(x^*)(>) \geq \langle \nabla h(x^*), \eta(x, x^*)\rangle + \rho\|x - x^*\|^2.$$

DEFINITION 0.3. *The function h is said to be (strictly)* (η,ρ)-pseudoinvex *at $x^* \in \mathbb{R}^n$ if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$ $(x \neq x^*)$,*

$$\langle \nabla h(x^*), \eta(x,x^*)\rangle \geq -\rho\|x-x^*\|^2 \;\Rightarrow\; h(x)(>) \geq h(x^*).$$

DEFINITION 0.4. *The function h is said to be (prestrictly)* (η,ρ)-quasiinvex *at $x^* \in \mathbb{R}^n$ if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,*

$$h(x)(<) \leq h(x^*) \;\Rightarrow\; \langle \nabla h(x^*), \eta(x,x^*)\rangle \leq -\rho\|x-x^*\|^2.$$

From the above definitions it is clear that if h is (η,ρ)-invex at x^*, then it is both (η,ρ)-pseudoinvex and (η,ρ)-quasiinvex at x^*, if h is (η,ρ)-quasiinvex at x^*, then it is prestrictly (η,ρ)-quasiinvex at x^*, and if h is strictly (η,ρ)-pseudoinvex at x^*, then it is (η,ρ)-quasiinvex at x^*.

Let the function $F = (F_1, F_2, \ldots, F_N) : \mathbb{R}^n \to \mathbb{R}^N$ be differentiable at x^*. The following generalizations of the notions of invexity, pseudoinvexity, and quasiinvexity for vector-valued functions were originally proposed in [21].

DEFINITION 0.5. *The function F is said to be* (strictly) $(\alpha,\eta,\bar\rho)$-V-invex at x^* *if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\backslash\{0\} \equiv (0,\infty)$, and $\bar\rho_i \in \mathbb{R}$, $i \in \underline{N}$, such that for each $x \in \mathbb{R}^n$ and $i \in \underline{N}$,*

$$F_i(x) - F_i(x^*)\,(>) \geq \langle \alpha_i(x,x^*)\nabla F_i(x^*), \eta(x,x^*)\rangle + \bar\rho_i\|x-x^*\|^2.$$

DEFINITION 0.6. *The function F is said to be* (strictly) $(\beta,\eta,\tilde\rho)$-V-pseudoinvex at x^* *if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\backslash\{0\}$, $i \in \underline{N}$, and $\tilde\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$ $(x \neq x^*)$,*

$$\Big\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x,x^*)\Big\rangle \geq -\tilde\rho\|x-x^*\|^2 \quad\Rightarrow$$
$$\sum_{i=1}^{N} \beta_i(x,x^*)F_i(x)(>) \geq \sum_{i=1}^{N} \beta_i(x,x^*)F_i(x^*).$$

DEFINITION 0.7. *The function F is said to be* (prestrictly) $(\gamma,\eta,\hat\rho)$-V-quasiinvex at x^* *if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\backslash\{0\}$, $i \in \underline{N}$, and $\hat\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,*

$$\sum_{i=1}^{N} \gamma_i(x,x^*)F_i(x)(<) \leq \sum_{i=1}^{N} \gamma_i(x,x^*)F_i(x^*) \quad\Rightarrow$$
$$\Big\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x,x^*)\Big\rangle \leq -\hat\rho\|x-x^*\|^2.$$

In contrast to the case of (η,ρ)-invex, (η,ρ)-pseudoinvex, and (η,ρ)-quasiinvex functions, the relationships among the three classes of functions specified in Definitions 0.5 - 0.7 are not immediately obvious. However, the underlying relationships can be determined by appropriate choices of the functions α_i, β_i, and $\gamma_i,\ i \in \underline{N}$, and the real numbers $\bar{\rho}_i,\ i \in \underline{N},\ \tilde{\rho}$, and $\hat{\rho}$. Indeed, it is easily seen that an $(\alpha,\eta,\bar{\rho})$-V-invex function is both $(\beta,\eta,\tilde{\rho})$-V-pseudoinvex and $(\gamma,\eta,\hat{\rho})$-V-quasiinvex if we choose $\gamma_i = \beta_i = 1/\alpha_i,\ i \in \underline{N}$, and $\hat{\rho} = \tilde{\rho} = \sum_{i=1}^{N} \bar{\rho}_i/\alpha_i$.

In the proofs of the sufficiency theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\beta,\eta,\tilde{\rho})$-V-pseudoinvexity can be defined in the following equivalent way:

The function F is said to be $(\beta,\eta,\tilde{\rho})$*-V-pseudoinvex at* x^* if there exist functions $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\backslash\{0\},\ i \in \underline{N}$, and $\tilde{\rho} \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{N} \beta_i(x,x^*)F_i(x) < \sum_{i=1}^{N} \beta_i(x,x^*)F_i(x^*) \ \Rightarrow\ \Big\langle \sum_{i=1}^{N} \nabla F_i(x^*), \eta(x,x^*) \Big\rangle < -\tilde{\rho}\|x-x^*\|^2.$$

Apparently, the primary motivation for introducing the notion of V-invexity was to relax the rather stringent requirement that in an invex programming problem the invexity property be satisfied for both the objective function and the constraints for the same kernel function η. It was demonstrated in [21] that this improvement enables one to investigate the optimality and duality aspects of a number of mathematical programming problems, including pseudolinear multiobjective problems and certain types of multiobjective fractional programming problems, in a unified framework.

The concept of ρ-invexity has been extended in many directions, and various types of generalized ρ-invex functions have been utilized for establishing a wide range of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult [1, 3, 7, 8, 15, 20, 23, 24, 26, 29], and for recent surveys of these and related functions, the reader is referred to [22, 27].

We conclude this section by recalling a necessary optimality result for (P) which was established in [35] by employing a Dinkelbach-type indirect approach and utilizing the necessary optimality conditions given in [6] for a semiinfinite programming problem. The contents and style of this result will serve as our guide in this paper for formulating various sufficient optimality conditions for (P).

THEOREM 0.1. *[35] Let* $x^* \in \mathbb{F}$, *let the functions* f_i *and* $g_i,\ i \in \underline{p}$, *be continuously differentiable at* x^*, *for each* $j \in \underline{q}$, *let* $G_j(z,t)$ *be differentiable, its partial derivatives be continuous jointly in* z *and* t, *and for each* $k \in \underline{r}$, *let the function* $H_k(\cdot,s)$ *be linear for all* $s \in S_k$. *If* x^* *is an optimal solution of* (P) *and if either one of the two constraint qualifications specified in [6] holds at* x^*, *then there exist* $\lambda^* \in \mathbb{R},\ u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^{p} u_i = 1\}$ *and integers* ν_0 *and* ν, *with* $0 \leq \nu_0 \leq \nu \leq n$, *such that there exist* ν_0 *indices* j_m, *with* $1 \leq j_m \leq q$, *together with* ν_0 *points* $t^m \in \hat{T}_{j_m}(x^*) \equiv \{t \in T_{j_m} : G_{j_m}(x^*,t) = 0\},\ m \in \underline{\nu_0},\ \nu-\nu_0$ *indices* k_m, *with* $1 \leq k_m \leq r$, *together with $\nu-\nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, with the property that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m) = 0,$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \quad i \in \underline{p}.$$

3. Sufficient Optimality Conditions

In this section, we present several sets of sufficiency results in which various generalized (α, η, ρ)-V-invexity assumptions are imposed on certain vector functions whose components are the individual as well as some combinations of the problem functions.

In the proofs of our sufficiency theorems, we shall make use of the following auxiliary result which provides an alternative expression for the objective function of (P); its proof is straightforward and hence omitted.

LEMMA 0.1. *For each $x \in \mathbb{R}^n$,*

$$\varphi(x) \equiv \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^p u_i f_i(x)}{\sum_{i=1}^p u_i g_i(x)}.$$

THEOREM 0.2. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \ge 0$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \le \nu_0 \le \nu \le n$, such that there exist ν_0 indices j_m, with $1 \le j_m \le q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \le k_m \le r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, with the property that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m) = 0, \tag{0.1}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \quad i \in \underline{p}. \tag{0.2}$$

Assume, furthermore, that either one of the following two sets of conditions holds:

(a) *(i)* $(f_1, \ldots, f_p)$ *is* $(\theta, \eta, \bar{\rho})$*-V-invex at* x^**;*

(ii) $(-g_1, \ldots, -g_p)$ *is* $(\xi, \eta, \tilde{\rho})$*-V-invex at* x^**;*

(iii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is* $(\pi, \eta, \hat{\rho})$*-V-invex at* x^**;*

(iv) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is* $(\delta, \eta, \breve{\rho})$*-V-invex at* x^**;*

(v) $\theta_i = \xi_j = \pi_k = \delta_l = \sigma$ *for all* i, $j \in \underline{p}, k \in \underline{\nu_0}$, *and* $l \in \underline{\nu}\backslash\underline{\nu_0}$*;*

(vi) $\sum_{i=1}^{p} u_i^*(\bar{\rho}_i + \lambda^* \tilde{\rho}_i) + \sum_{m=1}^{\nu_0} \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m \geq 0$;

(b) *(i)* $\big(L_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, L_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is* $(\theta, \eta, 0)$*-V-pseudoinvex at* x^**, where*

$$L_i(z, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) = u_i^* \Big[f_i(z) - \lambda^* g_i(z) + \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(z, s^m) \Big], \quad i \in \underline{p}.$$

Then x^* *is an optimal solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).
(a): Keeping in mind that $u^* \geq 0$ and $\lambda^* \geq 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \\
&= \sum_{i=1}^{p} u_i^* \{ f_i(x) - f_i(x^*) - \lambda^*[g_i(x) - g_i(x^*)] \} \ \text{(by (0.2))} \\
&\geq \sum_{i=1}^{p} u_i^* [\sigma(x, x^*) \langle \nabla f_i(x^*) - \lambda^* \nabla g_i(x^*), \eta(x, x^*) \rangle \\
&\qquad + (\bar{\rho}_i + \lambda^* \tilde{\rho}_i) \|x - x^*\|^2] \ \text{(by (i), (ii), and (v))} \\
&= -\sigma(x, x^*) \Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(x, x^*) \Big\rangle \\
&\geqq \sum_{m=1}^{\nu_0} v_m [G_{j_m}(x^*, t^m) - G_{j_m}(x, t^m)] + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(x^*, s^m) \\
&\quad + \Big[\sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda^* \tilde{\rho}_i) + \sum_{m=1}^{\nu_0} \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m \Big] \|x - x^*\|^2 \\
&\qquad \text{(by (iii), (iv), (v) and primal feasibility of } x) \\
&\geqq 0,
\end{aligned} \tag{0.3}$$

where the last inequality follows from (vi), the primal feasibility of x, and the fact that $t^m \in \hat{T}_{j_m}(x^*)$. Now using (0.3) and Lemma 0.1, we see that

$$\varphi(x^*) = \lambda^* \leq \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} = \varphi(x).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an optimal solution of (P).

(b): By our $(\theta,\eta,0)$-V-pseudoinvexity assumption, (0.1) implies that

$$\sum_{i=1}^{p}\theta_i(x,x^*)\Big\{u_i^*[f_i(x)-\lambda^* g_i(x)]+\sum_{m=1}^{\nu_0}v_m^* G_{j_m}(x,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m^* H_{k_m}(x,s^m)\Big\}$$
$$\geq \sum_{i=1}^{p}\theta_i(x,x^*)\Big\{u_i^*[f_i(x^*)-\lambda^* g_i(x^*)]+\sum_{m=1}^{\nu_0}v_m^* G_{j_m}(x^*,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m^* H_{k_m}(x^*,s^m)\Big\}.$$

Because $x^*\in\mathbb{F},\ t^m\in\hat{T}_{j_m}(x^*),m\in\underline{\nu_0}$, and (0.2) holds, the right-hand side of this inequality is equal to zero, and so we have that

$$\sum_{i=1}^{p}\theta_i(x,x^*)\Big\{u_i^*[f_i(x)-\lambda^* g_i(x)]+\sum_{m=1}^{\nu_0}v_m^* G_{j_m}(x,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m^* H_{k_m}(x,s^m)\Big\}\geq 0.$$

But $x\in\mathbb{F}$ and $v_m^*>0$ for each $m\in\underline{\nu_0}$, and hence the above inequality reduces to

$$\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)[f_i(x)-\lambda^* g_i(x)]\geq 0. \tag{0.4}$$

Using this inequality and Lemma 0.1, we see that

$$\begin{aligned}\varphi(x)&=\max_{1\leq i\leq p}\frac{f_i(x)}{g_i(x)}=\max_{1\leq i\leq p}\frac{\theta_i(x,x^*)f_i(x)}{\theta_i(x,x^*)g_i(x)}\ \ (\text{since }\theta_i(x,x^*)>0,\ i\in\underline{p})\\&=\max_{u\in U}\frac{\sum_{i=1}^{p}u_i\theta_i(x,x^*)f_i(x)}{\sum_{i=1}^{p}u_i\theta_i(x,x^*)g_i(x)}\ \ (\text{by Lemma 0.1})\\&\geq\frac{\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)f_i(x)}{\sum_{i=1}^{p}u_i^*\theta_i(x,x^*)g_i(x)}\\&\geq\lambda^*\ \ (\text{by (0.4)})\\&=\varphi(x^*).\end{aligned}$$

Since $x\in\mathbb{F}$ was arbitrary, we conclude from the above inequality that x^* is an optimal solution of (P). □

In Theorem 0.2, separate (α,η,ρ)-V-invexity assumptions were imposed on the vector functions $(f_1,\ldots,f_p)$ and $(-g_1,\ldots,-g_p)$. In the remainder of this section, we shall formulate some sufficient optimality conditions in which various generalized (α,η,ρ)-V-invexity requirements will be placed on the vector function $(\mathcal{E}_1(\cdot,\lambda,u),\ldots,\mathcal{E}_p(\cdot,\lambda,u))$, where for each $i\in\underline{p}$, the component function $\mathcal{E}_i(\cdot,\lambda,u)$ is defined, for fixed λ and u, on $\mathbb{R}^n$ by

$$\mathcal{E}_i(z,\lambda,u)=u_i[f_i(z)-\lambda g_i(z)].$$

THEOREM 0.3. *Let $x^*\in\mathbb{F}$, let $\lambda^*=\varphi(x^*)$, let the functions $f_i,\ g_i,\ i\in\underline{p},\ G_j(\cdot,t)$, and $H_k(\cdot,s)$ be differentiable at x^* for all $t\in T_j$ and $s\in S_k,\ j\in\underline{q},\ k\in\underline{r}$, and assume that there exist $u^*\in U$ and integers ν_0 and ν, with $0\leq\nu_0\leq\nu\leq n$, such that there exist ν_0 indices j_m, with $1\leq j_m\leq q$, together with ν_0 points $t^m\in$*

$\hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.1) and (0.2) hold. Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:

(a) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is (θ, η, ρ)-V-pseudoinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \hat{\rho})$-V-quasiinvex at x^*;*

(iv) $\rho + \tilde{\rho} + \hat{\rho} \geq 0$;

(b) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly (θ, η, ρ)-V-quasiinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \hat{\rho})$-V-quasiinvex at x^*;*

(iv) $\rho + \tilde{\rho} + \hat{\rho} > 0$;

(c) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly (θ, η, ρ)-V-quasiinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is strictly $(\pi, \eta, \tilde{\rho})$-V-pseudoinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is $(\delta, \eta, \hat{\rho})$-V-quasiinvex at x^*;*

(iv) $\rho + \tilde{\rho} + \hat{\rho} \geq 0$;

(d) *(i)* $\big(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*)\big)$ *is prestrictly (θ, η, ρ)-V-quasiinvex at x^*;*

(ii) $\big(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0})\big)$ *is $(\pi, \eta, \hat{\rho})$-V-quasiinvex at x^*;*

(iii) $\big(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu)\big)$ *is strictly $(\delta, \eta, \hat{\rho})$-V-pseudoinvex at x^*;*

(iv) $\rho + \tilde{\rho} + \hat{\rho} \geq 0$.

Then x^* is an optimal solution of (P).

Proof. (a): Let x be an arbitrary feasible solution of (P). Since $x \in \mathbb{F}$ and $t^m \in \hat{T}_{j_m}(x^*), m \in \underline{\nu_0}$, we have $G_{j_m}(x, t^m) \leq 0 = G_{j_m}(x^*, t^m)$, and hence

$$\sum_{m=1}^{\nu_0} v_m^* \pi_m(x, x^*) G_{j_m}(x, t^m) \leq \sum_{m=1}^{\nu_0} v_m^* \pi_m(x, x^*) G_{j_m}(x^*, t^m),$$

which in view of (ii) implies that

$$\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m), \eta(x, x^*) \Big\rangle \leq -\tilde{\rho} \|x - x^*\|^2. \tag{0.5}$$

Similarly, we can show that our assumptions in (iii) combined with the feasibility of x and x^* lead to the following inequality

$$\Big\langle \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(x, x^*) \Big\rangle \leq -\hat{\rho}\|x - x^*\|^2. \tag{0.6}$$

Now because of (0.5), (0.6), and (iv), (0.1) reduces to

$$\Big\langle \sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], \eta(x, x^*) \Big\rangle \geq -\rho\|x - x^*\|^2,$$

which in view of (i) implies that

$$\sum_{i=1}^{p} u_i^* \theta_i(x, x^*)[f_i(x) - \lambda^* g_i(x)] \geq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*)[f_i(x^*) - \lambda^* g_i(x^*)] = 0,$$

where the equality follows from (0.2). In the proof of part (b) of Theorem 0.2 it was shown that this inequality leads to the desired conclusion that x^* is an optimal solution of (P).

(b) - (d): The proofs are similar to that of part (a). □

In the remainder of this section, we briefly discuss certain modifications of Theorems 0.2 and 0.3 obtained by replacing (0.1) with an inequality.

THEOREM 0.4. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.2) and the following inequality hold:*

$$\Big\langle \sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(x, x^*) \Big\rangle \geq 0 \quad \text{for all } x \in \mathbb{F}, \tag{0.7}$$

where $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function. Furthermore, assume that either one of the two sets of conditions specified in Theorem 0.2 is satisfied. Then x^* is an optimal solution of (P).

Although the proofs of Theorems 0.2 and 0.4 are essentially the same, their contents are somewhat different. This can easily be seen by comparing (0.1) with (0.7). We observe that any quadruple $(x^*, u^*, v^*, \lambda^*)$ that satisfies (0.1) and (0.2) also satisfies (0.2) and (0.7), but the converse is not necessarily true. Moreover, (0.1) is a system of n equations, whereas (0.7) is a single inequality. Evidently, from a computational point of view, (0.1) is preferable to (0.7) because of the dependence of the latter on the feasible set of (P).

THEOREM 0.5. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.2) and (0.7) hold. Furthermore, assume that any one of the four sets of conditions specified in Theorem 0.3 is satisfied. Then x^* is an optimal solution of (P).*

4. Generalized Sufficiency Criteria

In this section, we discuss several families of sufficient optimality results under various generalized (α, η, ρ)-V-invexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions. This is accomplished by employing a certain type of partitioning scheme which was originally proposed in [25] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let ν_0 and ν be integers, with $1 \leq \nu_0 \leq \nu \leq n$, and let $\{J_0, J_1, \ldots, J_M\}$ and $\{K_0, K_1, \ldots, K_M\}$ be partitions of the sets $\underline{\nu_0}$ and $\underline{\nu} \backslash \underline{\nu_0}$, respectively; thus, $J_i \subset \underline{\nu_0}$ for each $i \in \underline{M} \cup \{0\}$, $J_i \cap J_j = \emptyset$ for each $i, j \in \underline{M} \cup \{0\}$ with $i \neq j$, and $\cup_{i=0}^{M} J_i = \underline{\nu_0}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_M\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{\nu_0}$ and $\underline{\nu} \backslash \underline{\nu_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$ for $i > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i(\cdot, \lambda, u, v, \bar{t}, \bar{s})$, $i \in \underline{p}$, and $\Lambda_\tau(\cdot, v, \bar{t}, \bar{s})$, $\tau \in \underline{M}$, defined, for fixed $u, v, \lambda, \bar{t} \equiv (t^1, t^2, \ldots, t^{\nu_0})$, and $\bar{s} \equiv (s^{\nu_0+1}, s^{\nu_0+2}, \ldots, s^{\nu})$, on $\mathbb{R}^n$ as follows:

$$\Phi_i(z, u, v, \lambda, \bar{t}, \bar{s}) = u_i \Big[f_i(z) - \lambda g_i(z) + \sum_{m \in J_0} v_m G_{j_m}(z, t^m) + \sum_{m \in K_0} v_m H_{k_m}(z, s^m) \Big], \quad i \in \underline{p},$$

$$\Lambda_\tau(z, v, \bar{t}, \bar{s}) = \sum_{m \in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(z, s^m), \quad \tau \in \underline{M}.$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized sufficiency results for (P) as follows.

THEOREM 0.6. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.1) and (0.2) hold. Assume, furthermore, that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $\big(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is* $(\theta, \eta, \bar{\rho})$*-V-pseudoinvex at* x^**;*

(ii) $\big(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \ldots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is* $(\pi, \eta, \tilde{\rho})$*-V-quasiinvex at* x^**;*

(iii) $\bar{\rho} + \tilde{\rho} \geq 0$*;*

(b) (i) $\big(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is prestrictly* $(\theta, \eta, \bar{\rho})$*-V-quasiinvex at* x^**;*

(ii) $\big(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \ldots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is* $(\pi, \eta, \tilde{\rho})$*-V-quasiinvex at* x^**;*

(iii) $\bar{\rho} + \tilde{\rho} > 0$*;*

(c) (i) $\big(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is prestrictly* $(\theta, \eta, \bar{\rho})$*-V-quasiinvex at* x^**;*

(ii) $\big(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \ldots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is strictly* $(\pi, \eta, \tilde{\rho})$*-V-pseudoinvex at* x^**;*

(iii) $\bar{\rho} + \tilde{\rho} \geq 0$*;*

Then x^* *is an optimal solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).

(a): It is clear that (0.1) can be expressed as follows:

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m \in J_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m \in K_0} v_m^* \nabla H_{k_m}(x^*, s^m)$$
$$+ \sum_{\tau=1}^{M} \Big[\sum_{m \in J_\tau} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* \nabla H_{k_m}(x^*, s^m) \Big] = 0. \quad (0.8)$$

Since $x, x^* \in \mathbb{F}$, $v_m^* > 0$, and $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \nu_0$, it follows that for each $\tau \in \underline{M}$,

$$\begin{aligned}\Lambda_\tau(x, v^*, \bar{t}, \bar{s}) &= \sum_{m \in J_\tau} v_m^* G_{j_m}(x, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(x, s^m) \\ &\leq 0 \\ &= \sum_{m \in J_\tau} v_m^* G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(x^*, s^m) \\ &= \Lambda_\tau(x^*, v^*, \bar{t}, \bar{s}),\end{aligned}$$

and hence

$$\sum_{\tau=1}^{M} \pi_\tau(x, x^*) \Lambda_\tau(x, v^*, \bar{t}, \bar{s}) \leq \sum_{\tau=1}^{M} \pi_\tau(x, x^*) \Lambda_\tau(x^*, v^*, \bar{t}, \bar{s}),$$

which because of (ii) implies that

$$\Big\langle \sum_{\tau=1}^{M} \Big[\sum_{m\in J_\tau} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m\in K_\tau} v_m^* \nabla H_{k_m}(x^*, s^m)\Big], \eta(x, x^*)\Big\rangle \leq -\tilde{\rho}\|x - x^*\|^2. \tag{0.9}$$

Combining (0.8) and (0.9), and using (iii) we get

$$\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m\in J_0} v_m^* \nabla G_{j_m}(x^*, t^m) \\ + \sum_{m\in K_0} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(x, x^*)\Big\rangle \geq \tilde{\rho}\|x - x^*\|^2 \geq -\bar{\rho}\|x - x^*\|^2, \tag{0.10}$$

which by virtue of (i) implies that

$$\sum_{i=1}^{p} \theta_i(x, x^*)\Phi_i(x, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) \geq \sum_{i=1}^{p} \theta_i(x, x^*)\Phi_i(x^*, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) = 0, \tag{0.11}$$

where the equality follows from the feasibility of x^*, the fact that $t^m \in \hat{T}_{j_m}(x^*)$, and (0.2). Because $x \in \mathbb{F}$ and $v_m^* > 0,\ m \in \underline{\nu_0}$, this inequality reduces to

$$0 \leq \sum_{i=1}^{p} u_i^* \theta_i(x, x^*)[f_i(x) - \lambda^* g_i(x)].$$

Now using this inequality and Lemma 0.1, as in the proof of Theorem 0.2, we obtain $\varphi(x^*) \leq \varphi(x)$. Since x was arbitrary, we conclude that x^* is an optimal solution of (P).

(b): Proceeding in exactly the same manner as in the proof of part (a), we obtain (0.10) in which the second inequality is strict. By (i), this implies that (0.11) holds and, therefore, the rest of the proof is identical to that of part (a).

(c) and (d): The proofs are similar to those of parts (a) and (b). □

We next state the altered version of Theorem 0.6 obtained by replacing (0.1) with (0.7).

THEOREM 0.7. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k,\ j \in \underline{q},\ k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.2) and (0.7) hold. Assume, furthermore, that any one of the three sets of hypotheses specified in Theorem 0.6 is satisfied. Then x^* is an optimal solution of (P).*

Each one of the six sets of conditions given in Theorems 0.6 and 0.7 can be viewed as a family of sufficient optimality conditions whose members can easily be identified by appropriate choices of the partitioning sets J_μ and K_μ, $\mu \in \underline{M} \cup \{0\}$.

In the remainder of this section we present two other collections of sufficiency results which are somewhat different from those stated in Theorems 0.6 and 0.7. It appears that sufficiency results of this type have not been considered previously for any kind of minmax optimization problems. These results are formulated by utilizing a partition of $\underline{p}$ in addition to those of $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, and by placing appropriate generalized (α, η, ρ)-V-invexity requirements on certain vector functions involving $\mathcal{E}_i(\cdot, \lambda, u)$, G_j, and H_k.

Let $\{I_0, I_1, \ldots, I_d\}$, $\{J_0, J_1, \ldots, J_e\}$, and $\{K_0, K_1, \ldots, K_e\}$ be partitions of $\underline{p}$, $\underline{\nu_0}$, and $\underline{\nu}\backslash\underline{\nu_0}$, respectively, such that $D = \{0, 1, 2, \ldots, d\} \subset E = \{0, 1, \ldots, e\}$, and let the function $\Pi_\tau(\cdot, u, v, \lambda, \bar{t}, \bar{s}) : \mathbb{R}^n \to \mathbb{R}$ be defined, for fixed $u, v, \lambda, \bar{t}$, and $\bar{s}$, by

$$\Pi_\tau(z, u, v, \lambda, \bar{t}, \bar{s}) = \sum_{i \in I_\tau} u_i[f_i(z) - \lambda g_i(z)] + \sum_{m \in J_\tau} v_m G_{j_m}(z, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(z, s^m), \quad \tau \in D.$$

THEOREM 0.8. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$, $u^* > 0$, and integers ν_0 and ν, with $0 \le \nu_0 \le \nu \le n$, such that there exist ν_0 indices j_m, with $1 \le j_m \le q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \le k_m \le r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.1) and (0.2) hold. Assume, furthermore, that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $\big(\Pi_0(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, \Pi_d(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is $(\theta, \eta, \bar{\rho})$-V-pseudoinvex at x^*;*

(ii) $\big(\Lambda_{d+1}(\cdot, v^*, \bar{t}, \bar{s}), \ldots, \Lambda_e(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\bar{\rho} + \tilde{\rho} \ge 0$;

(b) *(i)* $\big(\Pi_0(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, \Pi_d(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is prestrictly $(\theta, \eta, \bar{\rho})$-V-quasiinvex at x^*;*

(ii) $\big(\Lambda_{d+1}(\cdot, v^*, \bar{t}, \bar{s}), \ldots, \Lambda_e(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is strictly $(\pi, \eta, \tilde{\rho})$-V-pseudoinvex at x^*;*

(iii) $\bar{\rho} + \tilde{\rho} \ge 0$;

(c) *(i)* $\big(\Pi_0(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \ldots, \Pi_d(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s})\big)$ *is prestrictly $(\theta, \eta, \bar{\rho})$-V-quasiinvex at x^*;*

(ii) $\big(\Lambda_{d+1}(\cdot, v^*, \bar{t}, \bar{s}), \ldots, \Lambda_e(\cdot, v^*, \bar{t}, \bar{s})\big)$ *is $(\pi, \eta, \tilde{\rho})$-V-quasiinvex at x^*;*

(iii) $\bar{\rho} + \tilde{\rho} > 0$.

Then x^* is an optimal solution of (P).

Proof. (a): Suppose to the contrary that x^* is not an optimal solution of (P). Then there is $\bar{x} \in \mathbb{F}$ such that $\varphi(\bar{x}) < \varphi(x^*)$, and so it follows that

$$f_i(\bar{x}) - \lambda^* g_i(\bar{x}) < 0, \quad i \in \underline{p}.$$

Since $u^* > 0$, we see that for each $\tau \in D$,

$$\sum_{i \in I_\tau} u_i^*[f_i(\bar{x}) - \lambda^* g_i(\bar{x})] < 0. \tag{0.12}$$

Now using this inequality, we see that

$$\begin{aligned}
&\Pi_\tau(\bar{x}, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) \\
&= \sum_{i \in I_\tau} u_i^*[f_i(\bar{x}) - \lambda^* g_i(\bar{x})] + \sum_{m \in J_\tau} v_m^* G_{j_m}(\bar{x}, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(\bar{x}, s^m) \\
&\leq \sum_{i \in I_\tau} u_i^*[f_i(\bar{x}) - \lambda^* g_i(\bar{x})] \ \ \text{(by the feasibility of } \bar{x} \text{ and positivity of } v_m^*,\ m \in \underline{\nu_0}) \\
&< 0 \quad \text{(by (0.12))} \\
&= \sum_{i \in I_\tau} u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] + \sum_{m \in J_\tau} v_m^* G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(x^*, s^m) \\
&\qquad \text{(since (0.2) holds, } x^* \in \mathbb{F}, \text{ and } t^m \in \hat{T}_{j_m}(x^*),\ m \in \underline{\nu_0}) \\
&= \Pi_\tau(x^*, u^*, v^*, \lambda^*, \bar{t}, \bar{s}),
\end{aligned}$$

and hence

$$\sum_{\tau \in D} \theta_\tau(x, x^*)\Pi_\tau(\bar{x}, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) < \sum_{\tau \in D} \theta_\tau(x, x^*)\Pi_\tau(x^*, u^*, v^*, \lambda^*, \bar{t}, \bar{s}),$$

which in view of (i) implies that

$$\begin{aligned}
\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda \nabla g_i(x^*)] + \sum_{\tau \in D} \Big[\sum_{m \in J_\tau} v_m^* \nabla G_{j_m}(x^*, t^m) \\
+ \sum_{m \in K_\tau} v_m^* \nabla H_{k_m}(x^*, s^m) \Big], \eta(\bar{x}, x^*) \Big\rangle < -\bar{\rho}\|\bar{x} - x^*\|^2.
\end{aligned} \tag{0.13}$$

As shown in the proof of Theorem 0.6, for each $\tau \in E \backslash D$, $\Lambda_\tau(\bar{x}, v^*, \bar{t}, \bar{s}) \leq \Lambda_\tau(x^*, v^*, \bar{t}, \bar{s})$, and hence

$$\sum_{\tau \in E \backslash D} \pi_\tau(x, x^*)\Lambda_\tau(\bar{x}, v^*, \bar{t}, \bar{s}) \leq \sum_{\tau \in E \backslash D} \pi_\tau(x, x^*)\Lambda_\tau(x^*, v^*, \bar{t}, \bar{s}),$$

which in view of (ii) implies that

$$\begin{aligned}
\Big\langle \sum_{\tau \in E \backslash D} \Big[\sum_{m \in J_\tau} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* \nabla H_{k_m}(x^*, s^m) \Big], \eta(\bar{x}, x^*) \Big\rangle \\
\leq -\tilde{\rho}\|\bar{x} - x^*\|^2.
\end{aligned} \tag{0.14}$$

Now combining (0.13) and (0.14) and using (iii), we see that

$$\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) +$$
$$\sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m), \eta(\bar{x}, x^*) \Big\rangle < -(\bar{\rho} + \tilde{\rho}) \|\bar{x} - x^*\|^2 \leq 0,$$

which contradicts (0.1). Therefore, x^* is an optimal solution of (P).

(b) and (c): The proofs are similar to that of part (a). □

The following modified version of Theorem 0.8 obtained by replacing (0.1) with (0.7) can be proved in an identical manner.

THEOREM 0.9. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions $f_i, g_i,\ i \in \underline{p},\ G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^* for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U, u^* > 0$, and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that (0.2) and (0.7) hold. Assume, furthermore, that any one of the three sets of hypotheses specified in Theorem 0.8 is satisfied. Then x^* is an optimal solution of (P).*

As we mentioned previously, one can readily identify numerous special cases of the six families of sufficiency results stated in Theorems 0.8 and 0.9.

5. Concluding Remarks

Taking a Dinkelbach-type [4] parametric approach, in this study we have established a fairly large number of sets of global sufficient optimality conditions under various generalized (α, η, ρ)-V-invexity hypotheses for a semiinfinite discrete minmax fractional programming problem. It appears that all these results are new in the area of semiinfinite programming. Since all the results obtained here can be modified and restated in a straightforward manner for each one of the seven problems designated as (P1) - (P7) in Section 1, they collectively subsume a fairly large number of existing results in the areas of conventional nonlinear programming and semiinfinite nonlinear programming. Furthermore, the style and techniques employed in this paper can be utilized to establish similar results for some other classes of related optimization problems. For example, it seems reasonable to expect that a similar approach can be applied to investigate the optimality and duality aspects of the following two closely related classes of semiinfinite continuous minmax fractional and multiobjective fractional programming problems:

$$\underset{x \in \mathbb{F}}{\text{Minimize}} \ \max_{y \in Y} \frac{f(x, y)}{g(x, y)},$$

$$\underset{x \in \mathbb{F}}{\text{Minimize}} \ \Big(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)} \Big).$$

We shall investigate these two classes of semiinfinite programming problems in subsequent papers.

References

[1] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc. Ser. B. 28 (1986), 1 - 9.

[2] B. Brosowski, Parametric Semiinfinite Optimization, Peter Lang, Frankfurt a. M., 1982.

[3] B. D. Craven, Invex functions and constrained local minima, Bull. Austral. Math. Soc. 24 (1981), 357 - 366.

[4] W. Dinkelbach, On nonlinear fractional programming, Management Sci. 13 (1967), 492 - 498.

[5] A. V. Fiacco and K. O. Kortanek (Eds.), Semi-infinite Programming and Applications, Lecture Notes in Economics and Mathematical Systems, Vol. 215, Springer, Berlin, 1983.

[6] K. R. Gehner, Necessary and sufficient conditions for the Fritz John problem with linear equality constraints, SIAM J. Control 12 (1974), 140 - 149.

[7] G. Giorgi and A. Guerraggio, Various types of nonsmooth invex functions, J. Inform. Optim. Sci. 17 (1996), 137 - 150.

[8] G. Giorgi and S. Mititelu, Convexités généralisées et propriétés, Rev. Roumaine Math. Pures Appl. 38 (1993), 125 - 172.

[9] K. Glashoff and S. A. Gustafson, Linear Optimization and Approximation, Springer, Berlin, 1983.

[10] M. A. Goberna and M.A. López, Linear Semi-Infinite Optimization, Wiley, New York, 1998.

[11] M. A. Goberna and M.A. López (Eds.), Semi-infinite Programming - Recent Advances, Kluwer, Dordrecht, 2001.

[12] S. A. Gustafson and K. O. Kortanek, Semi-infinite programming and applications, in Mathematical Programming: The State of the Art, edited by A. Bachem *et al.*, Springer, Berlin, 1983, 132 - 157.

[13] P. R. Gribik, Selected applications of semi-infinite programming, in Constructive Approaches to Mathematical Models, Edited by C. V. Coffman and G. J. Fix, Academic Press, New York, 1979, 171 - 187.

[14] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981), 545 - 550.

[15] M. A. Hanson and B. Mond, Further generalizations of convexity in mathematical programming, J. Inform. Optim. Sci. 3 (1982), 25 - 32.

[16] R. Henn and P. Kischka, Über einige Anwendungen der semi-infiniten Optimierung, Zeitschrift Oper. Res. 20 (1976), 39 - 58.

[17] R. Hettich (Ed.), Semi-infinite Programming, Lecture Notes in Control and Information Sciences, Vol. 7, Springer, Berlin, 1976.

[18] R. Hettich and K. O. Kortanek, Semi-infinite programming: theory, methods, and applications, SIAM Review 35 (1993), 380 - 429.

[19] R. Hettich and P. Zencke, Numerische Methoden der Approximation und semi-infinite Optimierung, Teubner, Stuttgart, 1982.

[20] V. Jeyakumar, Strong and weak invexity in mathematical programming, Methods Oper. Res. 55 (1985), 109 - 125.

[21] V. Jeyakumar and B. Mond, On generalized convex mathematical programming, J. Austral. Math. Soc. Ser. B 34 (1992), 43 - 53.

[22] P. Kanniappan and P. Pandian, On generalized convex functions in optimization theory - A survey, Opsearch 33 (1996), 174 - 185.

[23] D. H. Martin, The essence of invexity, J. Optim. Theory Appl. 47 (1985), 65 - 76.

[24] S. Mititelu and I. M. Stancu-Minasian, Invexity at a point: Generalizations and classification, Bull. Austral. Math. Soc. 48 (1993), 117 - 126.

[25] B. Mond and T. Weir, Generalized concavity and duality, in Generalized Concavity in Optimization and Economics, edited by S. Schaible and W. T. Ziemba, Academic Press, New York, 1981, pp. 263 - 279.

[26] R. Pini, Invexity and generalized convexity, *Optimization* **22** (1991), 513 - 525.

[27] R. Pini and C. Singh, A survey of recent [1985 - 1995] advances in generalized convexity with applications to duality theory and optimality conditions, Optimization 39 (1997), 311 - 360.

[28] R. Reemtsen and J. -J. Rückmann (Eds.), Semi-Infinite Programming, Kluwer, Boston, 1998.

[29] T. W. Reiland, Nonsmooth invexity, Bull. Austral. Math. Soc. 42 (1990), 437 - 446.

[30] O. Stein, Bilevel Strategies in Semi-infinite Programming, Kluwer, Boston, 2003.

[31] G. -W. Weber, Generalized semi-infinite optimization: theory and applications in optimal control and discrete optimization, J. Stat, Management Syst. 5 (2002), 359 - 388.

[32] G. J. Zalmai, Optimality principles and duality models for a class of continuous-time generalized fractional programming problems with operator constraints, J. Stat. Management Syst. 1 (1998), 61 - 100.

[33] G. J. Zalmai and Qinghong Zhang, Global nonparametric sufficient optimality conditions for semiinfinite discrete minmax fractional programming problems containing generalized (α, η, ρ)-V-invex functions, submitted for publication.

[34] G. J. Zalmai and Qinghong Zhang, Parametric duality models for semiinfinite discrete minmax fractional programming problems containing generalized (α, η, ρ)-V-invex functions, submitted for publication.

[35] G. J. Zalmai and Qinghong Zhang, Global parametric sufficient optimality conditions for semiinfinite discrete minmax fractional programming problems involving generalized (η, ρ)-invex functions, submitted for publication.