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Hamiltonian Dynamics and Morse Topology of Humanoid Robots

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Abstract

Humanoid robots are anthropomorphic mechanisms with dynamics that resembles that of human musculo–skeletal dynamics. This paper proposes a generalised Hamiltonian model of humanoid dynamics and gives a Morse–type topological analysis of its momentum phase–space manifold.

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Key Words: Hamiltonian dynamics, Morse topology, humanoid robotics.

1. Introduction

Humanoid robots are anthropomorphic mechanisms with complex, muscle–driven dynamics with many degrees of freedom. Since the early papers of Vukobratovic (see [27] and the references therein), a large amount of research has been done on the kinematics, dynamics and control of biped, humanoid robots ([2, 6, 8, 9, 13, 19] and [24]–[26]). Some of the biped models have what has been referred to as the capability of passive dynamic walking [14] and others that of powered walking [17]. The 1980's were dominated by work on solutions to kinematic problems of redundancy and singularities [23, 29]. The 1990's were characterised by the extensive use of intelligent, neuro–fuzzy–genetic control of humanoid dynamics [3, 4, 7, 18, 20, 21], and computer–graphics animation [12].

Here we propose a generalised Hamiltonian model of humanoid dynamics and address the Morse topology analysis of its momentum phase–space manifold. In Section 2 we construct an N–dimensional configuration manifold Q^N using direct products of constrained rotational Lie groups $SO(n)$ with $n = 2, 3$. We derive a generalised Hamiltonian formulation of the momentum phase–space manifold or cotangent bundle T^*Q^N using the symplectic and Riemannian geometry of rotational Lie groups $SO(n)$ and their algebras $so(n)$. In Section 3 we introduce Hamiltonian's action principle and in Section 4 Morse functions. Finally, in Section 5, Morse homology and cohomology groups are derived.

2. The Configuration Manifold

The kinematics of an n–segment humanoid chain are usually defined by a map between external (usually end–effector) coordinates x^r $(r = 1, \dots, n)$ and internal (joint) coordinates $q^i\,(i = 1, \dots, N)$ [10, 11]. The forward kinematics are defined as a nonlinear map $x^r = x^r(q^i)$ with corresponding linear vector functions $dx^r = \partial x^r/\partial q^i\; dq^i$ of differentials, and $\dot{x}^r = (\partial x^r/\partial q^i)\; \dot{q}^i$ of velocities. Here and subsequently the summation convention is understood over repeated indices. When the rank of the configuration–dependent Jacobian matrix $J \equiv \partial x^r/\partial q^i$ is less than n kinematic singularities occur; the onset of this condition could be detected by a manipulability measure. The inverse kinematics are defined conversely by a nonlinear map $q^i = q^i(x^r)$ with corresponding linear vector functions $dq^i = \partial q^i/\partial x^r\; dx^r$ of differentials and $\dot{q}^i = (\partial q^i/\partial x^r)\; \dot{x}^r$ of velocities. Again in the case $n < N$ of redundancy, the inverse kinematic problem admits infinitely many solutions. Often the pseudo–inverse configuration–control $\dot{q}^i = J^*\; \dot{x}^r$ is used instead, where $J^* = J^T(J\,J^T)^{-1}$ denotes the Moore–Penrose pseudo–inverse of the Jacobian matrix J.

Humanoid joints, that is, internal coordinates q^i $(i = 1, \dots, N)$, constitute a smooth configuration manifold Q^N, described as follows. Uniaxial, 'hinge' joints represent constrained, rotational Lie groups $SO(2)^i_{cnstr}$, parameterised by constrained angles $q^i_{cnstr} \equiv q^i \in [q^i_{\min}, q^i_{\max}]$. Three–axial, 'ball–and–socket' joints represent constrained rotational Lie groups $SO(3)^i_{cnstr}$, parameterised by constrained Euler angles $q^i = q^{\phi_i}_{cnstr}$. In the sequel the subscript 'cnstr' will be omitted for simplicity, and always assumed in relation to internal coordinates q^i.

All $SO(n)$–joints are Hausdorff C^∞–manifolds with atlases (U_α, u_α). In other words, they are paracompact and metrisable smooth manifolds, admitting a Riemannian metric.

Let A and B be two smooth manifolds described by smooth atlases (U_α, u_α) and (V_β, v_β), respectively. Then for $(\alpha, \beta) \in A \times B$ the family

$$(U_\alpha \times V_\beta, u_\alpha \times v_\beta : U_\alpha \times V_\beta \longrightarrow \mathbb{R}^m \times \mathbb{R}^n)$$

is a smooth atlas for the direct product $A \times B$. If A and B are two Lie groups $SO(n)$, then their direct product $G = A \times B$ is also their direct product as smooth manifolds and their direct product as algebraic groups, with the product law

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2), \quad a_{1,2} \in A,\; b_{1,2} \in B.$$

On generalising the direct product to N rotational joint groups, we can draw an anthropomorphic product–tree (see Figure 1) using a line segment '–' to represent direct products of the humanoid's $SO(n)$–joints. This is our basic model for the humanoid configuration manifold Q^N.

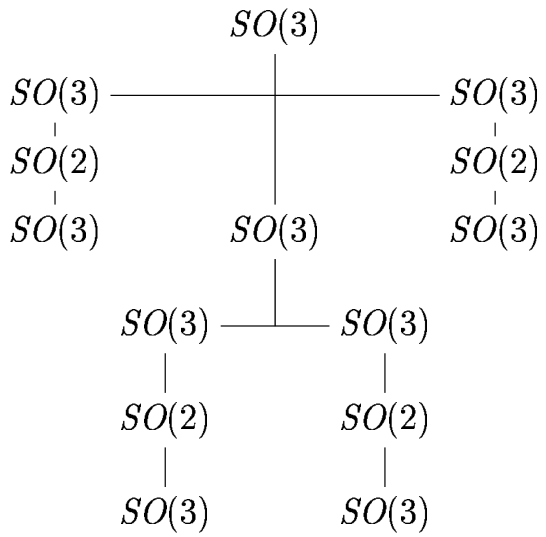

Figure 1: Q^N modelled as an anthropomorphic product–tree of constrained $SO(n)$ groups

Let T_qQ^N be a tangent space to Q^N at the point q. The tangent bundle TQ^N represents a union $\cup_{q\in Q^N} T_qQ^N$ together with the standard topology on TQ^N and a natural smooth manifold structure, the dimension of which is twice that of Q^N. A vector field X on Q^N represents a section $X : Q^N \to TQ^N$ of the tangent bundle TQ^N.

Analogously let $T_q^*Q^N$ be a cotangent space to Q^N at q, the dual to its tangent space T_qQ^N. The cotangent bundle T^*Q^N represents a union $\cup_{q\in Q^N} T_q^*Q^N$, together with the standard topology on T^*Q^N and a natural smooth manifold structure, the dimension of which is twice that of Q^N. A one–form θ on Q^N represents a section $\theta : Q^N \to T^*Q^N$ of the cotangent bundle T^*Q^N.

We refer to the tangent bundle TQ^N of the configuration manifold Q^N as the velocity phase–space manifold, and to its cotangent bundle T^*Q^N as the momentum phase–space manifold.

If we apply the functor `Lie` on the category $\mathcal{S}^\bullet[SO(n)^i]$ (for $n = 2, 3$ and $i = 1, \ldots, N$) of rotational Lie groups $SO(n)^i$ (and their homomorphisms) we obtain the category $\mathcal{S}_\bullet[so(n)_i]$ of corresponding tangent Lie algebras $so(n)_i$ (and their homomorphisms). If we further apply the isomorphic functor `Dual` to the category $\mathcal{S}_\bullet[so(n)_i]$ we obtain the dual category $\mathcal{S}^*_\bullet[so(n)^*_i]$ of cotangent or canonical Lie algebras $so(n)^*_i$ (and their homomorphisms). To go directly from $\mathcal{S}^\bullet[SO(n)^i]$ to $\mathcal{S}^*_\bullet[so(n)^*_i]$ we use the canonical functor `Can`. Therefore, we have a commutative diagram as depicted in Figure 2.

On applying the functor `Lie` to the configuration manifold Q^N (Figure 1), we get the product–tree of the same anthropomorphic structure but with tangent Lie algebras $so(n)_i$ as vertices instead of the groups $SO(n)^i$. Again, applying the functor `Can` on Q^N gives the product–tree of the same anthropomorphic structure, but with cotangent Lie algebras $so(n)^*_i$ as vertices. Both the tangent algebras $so(n)_i$ and the cotangent algebras $so(n)^*_i$ contain infinitesimal group generators – angular velocities

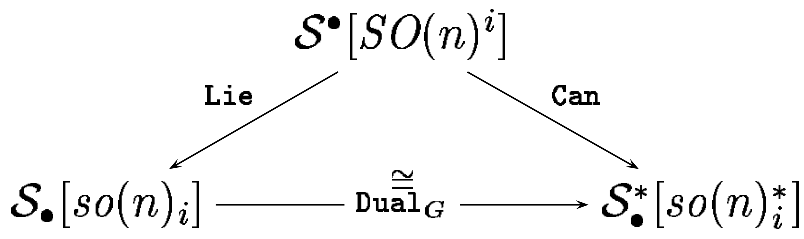

Figure 2: commutative triangle

$\dot{q}^i = \dot{q}^{\phi_i}$ in the first case and canonical angular momenta $p_i = p_{\phi_i}$ in the second. As Lie group generators, the angular velocities and angular momenta satisfy the commutation relations $[\dot{q}^{\phi_i}, \dot{q}^{\psi_i}] = \epsilon_\theta^{\phi\psi}\,\dot{q}^{\theta_i}$ and $[p_{\phi_i}, p_{\psi_i}] = \epsilon^\theta_{\phi\psi}\,p_{\theta_i}$, respectively, where the structure constants $\epsilon_\theta^{\phi\psi}$ and $\epsilon^\theta_{\phi\psi}$ are totally antisymmetric third–order tensors.

In this way the functor $\mathtt{Dual}_G : \mathtt{Lie} \cong \mathtt{Can}$ establishes a unique geometrical duality between the kinematics of the angular velocities $\dot{q}^i$ involved in the Lagrangian formalism on the tangent bundle of Q^N) and the kinematics of the angular momenta p_i involved in the Hamiltonian formalism on the cotangent bundle of Q^N. In other words, we have two functors, Lie and Can, from the category of Lie groups (of which $\mathcal{S}^\bullet[SO(n)^i]$ is a subcategory) into the category of their Lie algebras (of which $\mathcal{S}_\bullet[so(n)_i]$ and $\mathcal{S}^*_\bullet[so(n)^*_i]$ are subcategories), and a unique natural equivalence between them defined by the functor $\mathtt{Dual}_G$. As the angular momenta p_i are in a bijective correspondence with the angular velocities $\dot{q}^i$, every component of the functor $\mathtt{Dual}_G$ is invertible.

3. The Hamiltonian Action

The Riemannian metric $g =<,>$ on the configuration manifold Q^N is a positive–definite quadratic form $g : TQ^N \to \mathbb{R}$, given in local coordinates $q^i \in U$ (U open in Q^N) as

$$g_{ij} \mapsto g_{ij}(q,m)\, dq^i dq^j.$$

Here

$$g_{ij}(q,m) = \sum_{\mu=1}^{n} m_\mu \delta_{rs} \frac{\partial x^r}{\partial q^i} \frac{\partial x^s}{\partial q^j}$$

is the covariant material metric tensor defining a relation between internal and external coordinates and including n segmental masses m_μ. The quantities x^r are external coordinates ($r, s = 1, \ldots, 6n$) and $i, j = 1, \ldots, N \equiv 6n - h$, where h denotes the number of holonomic constraints.

The Lagrangian of the system is a quadratic form $L : TQ^N \to \mathbb{R}$ dependent on the velocity v and such that $L = \frac{1}{2} < v, v >$. In local coordinates q^i, $v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N), it is given by

$$L = \frac{1}{2} g_{ij}(q,m)\, v^i v^j.$$

The Hamiltonian function $H : T^*Q^N \to \mathbb{R}$ is given in local canonical coordinates $q^i, p_i \in U_p$ on the momentum phase–space manifold T^*Q^N by

$$H = \frac{1}{2} g^{ij}(q,m)\, p_i p_j + V(q).$$

Here

$$g^{ij}(q,m) = \sum_{\chi=1}^{n} m_\chi \delta_{rs} \frac{\partial q^i}{\partial x^r} \frac{\partial q^j}{\partial x^s}$$

denotes the contravariant material metric tensor relating internal and external coordinates and includes n segmental masses m_χ.

For any smooth function L on TQ^N, the fibre derivative or Legendre transformation is a diffeomorphism $FL : TQ^N \to T^*Q^N$, $F(w) \cdot v = < w, v >$, from the momentum phase–space manifold to the velocity phase–space manifold associated with the metric $g = <,>$. In local coordinates q^i, $v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N), FL is given by $(q^i,\, v^i) \mapsto (q^i,\, p_i)$.

The following exist on the momentum phase–space manifold T^*Q^N [1]:
(i) A unique canonical one–form θ with the property that, for any one–form β on the configuration manifold Q^N, we have $\beta^*\theta = \beta$. In local canonical coordinates $q^i, p_i \in U_p$ (U_p open in T^*Q^N) it is given by $\theta = p_i dq^i$.
(ii) A unique nondegenerate Hamiltonian symplectic two–form ω_H, which is closed ($d\omega = 0$) and exact ($\omega = d\theta = dp_i \wedge dq^i$). Each body segment has, in the general $SO(3)$ case, a sub–phase–space manifold $T^*SO(3)$ with

$$\omega^{(sub)} = dp_\phi \wedge d\phi + dp_\psi \wedge d\psi + dp_\theta \wedge d\theta.$$

The space

$$\Omega = \{c \equiv \left(q^i, p_i\right) : [0,1] \to T^*Q^N \mid c(0) \in o_{Q^N}\}$$

of the humanoid's joint paths $c(t) \equiv (q^i(t), p_i(t))$ in T^*Q^N, starting at the zero Q^N–section, may be considered as a fibration over Q^N, given by

$$\pi_\Omega : \Omega \to Q^N, \quad \pi_\Omega(c) = \pi_{T^*Q^N}(c(1)).$$

The Hamiltonian action functional $\mathcal{A}_H : \Omega \to \mathbb{R}$ is now defined by

$$\mathcal{A}_H = \int_c \theta - \int_0^1 H(c(t))dt.$$

If ξ is a vector field along $c \in \Omega$, the first variation of $\mathcal{A}_H$ in the ξ–direction is

$$d\mathcal{A}_H(c)\xi = \int_0^1 \left[\omega\left(\frac{dc}{dt}, \xi\right) - dH(c(t), t)\xi\right] dt + \theta(\xi(1)).$$

The fibre derivative FL of $\mathcal{A}_H$ vanishes on the set

$$\Sigma_{\mathcal{A}_H} = \left\{ c \in \Omega \ : \ \left.\frac{dc}{dt}\right|_\omega = dH(c) \right\}$$

of generalised Hamiltonian orbits in T^*Q^N. In local canonical coordinates $q^i, p_i \in U_p$, U_p open in T^*Q^N, this is given for $i = 1, \ldots, N$ by dissipative, driven Hamilton equations

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} + \frac{\partial R}{\partial p_i}, \\ \dot{p}_i &= F_i - \frac{\partial H}{\partial q^i} + \frac{\partial R}{\partial q^i}, \\ q^i(0) &= q_0^i, \quad p_i(0) = p_i^0 \end{aligned}$$

(see [10, 11]). Here $R = R(q,p)$ denotes the Rayleigh biquadratic dissipation function and $F_i = F_i(t,q,p)$ are covariant driving torques of equivalent muscular actuators, resembling muscular excitation and contraction dynamics in rotational form.

If φ_t^H is the Hamiltonian isotopy generated by H, then

$$\varphi_1^H(o_{Q^N}) = \{d\mathcal{A}_H(c) \mid c \in \Sigma_{\mathcal{A}_H}\}.$$

4. Morse Functions

Let $f : Q^N \to \mathbb{R}$ represent a C^∞) function on the configuration manifold Q^N. Recall that $z = (q,p) \in Q^N$ is called the *critical point* of f if $df(z) \equiv df[(q,p)] = 0$. In local coordinates $(x^1, ..., x^n) = (q^1, ..., q^n, p_1, ..., p_n)$ in a neighborhood of z, this means $\frac{\partial f}{\partial x^i}(z) = 0$ for $i = 1, ..., n$. The Hessian of f at a critical point z defines a symmetric bilinear form $\nabla df(z) = d^2 f(z)$ on $T_z Q^N$, given in local coordinates $(x^1, ..., x^n)$ by the matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$. If z is not a critical point of f, the Hessian of f at z depends on the choice of metric. The index and nullity of this matrix are called the index and nullity of the critical point z of f.

We assume that all critical points $z_1, ..., z_n$ of f are nondegenerate in the sense that the Hessians $d^2 f(z_i)$ $(i = 1, ..., m)$ have maximal rank. Let z be such a critical point of f of Morse index s (= number of negative eigenvalues of $d^2 f(z_i)$, counted with multiplicity) [15]. The eigenvectors corresponding to these negative eigenvalues then span a subspace $V_z \subset T_z Q^N$ of dimension s. We choose a basis $e_1, ..., e_s$ of V_z, for example, an orthonormal basis with respect to the Riemannian metric $g = <, >$ on Q^N, with dual basis $dx^1, ..., dx^s$. This basis then defines an orientation of V_z, which we may also represent by the s–form $dx^1 \wedge ... \wedge dx^s$. Let z' be another critical point of f, of Morse index $s-1$. We consider paths $c(t)$ of f of steepest descent from z to z', that is, integral curves of the vector field $-\nabla f(c)$. Thus $c(t) \equiv (q^i(t), p_i(t))$ in T^*Q^N solves for the gradient flow of f, that is,

$$\dot{c}(t) = -\nabla f\,(c(t))$$

with

$$\lim_{t \to -\infty} c(t) = z, \qquad \lim_{t \to \infty} c(t) = z'.$$

Let M^s be the set of critical points of f of Morse index s and let H_f^s be the vector space over $\mathbb{R}$ spanned by that set. We can define a boundary operator

$$\delta : H_f^{s-1} \to H_f^s$$

by putting

$$\delta(z') = \sum_{p\in M^q} n(z',z)\, z$$

for $z' \in M^{s-1}$ and extending δ by linearity.

This operator satisfies $\delta^2 = 0$ and therefore defines a cohomology theory. Using Conley's continuation principle, Floer [5] showed that the resulting cohomology theories resulting from different choices of f are canonically isomorphic.

Witten [28] considered the operators

$$d_t = e^{-tf} d e^{tf},$$

their formal adjoints

$$d_t^* = e^{tf} d e^{-tf}$$

and the corresponding Laplacian

$$\Delta_t = d_t d_t^* + d_t^* d_t.$$

For $t = 0$, Δ_0 is the standard Hodge–de Rham Laplacian, while for $t \to \infty$ we have the expansion

$$\Delta_t = dd^* + d^*d + t^2 \left\|df\right\|^2 + t \sum_{k,j} \frac{\partial^2 h}{\partial x^k \partial x^j} \left[i\left(\frac{\partial}{\partial x^k}\right), dx^j \right].$$

Here $\left(\frac{\partial}{\partial x^k}\right)_{k=1,\ldots,n}$ is an orthonormal frame at the point under consideration. The expression Δ_t becomes large for $t \to \infty$, except at the critical points of f. Therefore the eigenvalues of Δ_t will concentrate near the critical points of f for $t \to \infty$ and we obtain an interpolation between the de Rham cohomology and the Morse cohomology.

5. Morse Homology and Cohomology

For any Morse function f on the configuration manifold Q^N, we denote by $\mathrm{Crit}_z(f)$ the set of critical points of index z and define $C_z(f)$ as the free Abelian group generated by $\mathrm{Crit}_z(f)$ [15]. Consider the gradient flow generated by the equation

$$\dot{c} = -\nabla f, \tag{1}$$

where ∇f is defined with respect to the given Riemannian metric $g = \langle , \rangle$ on Q^N by

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}.$$

Denote by $\mathcal{M}_{f,g}(M)$ the set of all $c : \mathbb{R} \to M$ satisfying (1) such that

$$\int_{-\infty}^{+\infty} \left| \frac{dc}{dt} \right|^2 dt < \infty.$$

Define

$$\partial : C_z(f) \to C_{z-1}(f),$$
$$\partial x = \sum_{y \in \mathrm{Crit}_{z-1}(f)} n(x,y)y,$$

where $n(x,y)$ is the number of points in the zero–dimensional manifold $\widehat{\mathcal{M}}_{f,g}(x,y)$ counted with signs with respect to the orientation σ. The proof of $\partial \circ \partial = 0$ is based on glueing and cobordism arguments [22]. The Morse homology groups are defined by

$$H_z(f) = \mathrm{Ker}(\partial)/\mathrm{Im}(\partial).$$

For generic choices of Morse functions f_1 and f_2, the groups $H_z(f_1)$ and $H_z(f_2)$ are both isomorphic to the singular homology group of Q^N [16].

The Morse function f is decreasing along the trajectories solving the autonomous gradient equation (1). Therefore the boundary operator ∂ preserves the downward filtration given by level sets of f. In other words, if

$$\mathrm{Crit}_p^\lambda(f) := \mathrm{Crit}_p(f) \cap f^{-1}((-\infty, \lambda])$$

and

$$C_p^\lambda(f) := \text{ free abelian group generated by } \mathrm{Crit}_p^\lambda(f),$$

then the boundary operator ∂ restricts to

$$\partial^\lambda : C_p^\lambda(f) \to C_{p-1}^\lambda(f).$$

Clearly $\partial^\lambda \circ \partial^\lambda = 0$, so we can define the relative Morse homology groups

$$H_p^\lambda(f) := \mathrm{Ker}(\partial^\lambda)/\mathrm{Im}(\partial^\lambda).$$

An obvious inclusion

$$j^\lambda : \mathrm{Crit}_p^\lambda(f) \to \mathrm{Crit}_p(f)$$

generates the homomorphism

$$j_\sharp^\lambda : C_p^\lambda(f) \to C_p(f)$$

which commutes with ∂, that is, $\partial^\lambda \circ j_\sharp^\lambda = j_\sharp^\lambda \circ \partial$. Hence we have an inclusion homomorphism

$$j_*^\lambda : H_p^\lambda(f) \to H_p(f).$$

In a similar way, using the standard algebraic construction [22], we can define Morse cohomology. Setting

$$C_\lambda^z(f) = \mathrm{Hom}(C_z^\lambda(f), \mathbb{Z})$$

and

$$\delta^\lambda : C_\lambda^z(f) \to C_\lambda^{z+1}(f), \quad \langle \delta^\lambda a, x \rangle = \langle a, \partial^\lambda x \rangle,$$

we can define the relative Morse cohomology groups as

$$H_\lambda^z(f) = \mathrm{Ker}(\delta^\lambda)/\mathrm{Im}(\delta^\lambda).$$

Since $\mathrm{Crit}_z(f)$ is finite, for large λ we have $H_z^\lambda(f) = H_p(f)$ and $H_\lambda^z(f) = H^z(f)$.

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