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Oscillations of Solutions of ODEs and Nevanlinna Theory

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Abstract

This paper has two novel components: we 1)investigate oscillation problems for solutions $(x(t), y(t))$, $t \in (t_1, t_2)$ of the autonomous system of equations $y' = F_1(x, y)$, $x' = F_2(x, y)$ and 2) transfer the spirit of classical Nevanlinna value distribution theory (complex analysis) into the theory of ordinary differential equations (ODE).

By analogy with the theory of oscillation for one ODE we consider the number $n(t_1, t_2, 0)$ of zeros τ_i in (t_1, t_2) of the solutions, that is, the number of those points τ_i, where $x(\tau_i) = 0$ and $y(\tau_i) = 0$. It turns out that upper bounds for $n(t_1, t_2, 0)$ can be given in terms of F_1, F_2, t_1 and t_2. By analogy with the concept of a-points in complex analysis we consider values $a := (a', a'')$ in the (x, y)-plane and define a-points of the solutions as those points τ_i, where $x(\tau_i) = a'$ and $y(\tau_i) = a''$. Denoting by $n(t_1, t_2, a)$ the number of a-points in (t_1, t_2) of the solutions, we give upper bounds for the sum $\sum_{\nu=1}^{q} n(t_1, t_2, a_\nu)$, where $a_1, a_2, ..., a_q$ is a given totality of pairwise different points. Thus we obtain for the solutions of the above equation an analogue of the second fundamental theorem in Nevanlinna theory; we consider also a similar sum for the number of a-points of meromorphic functions.

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1. Introduction, Definitions and Results

There are numerous investigations into oscillations (the distribution of zeros) for solutions of a given second-order ordinary differential equation. one equation). The reason for this is clear: in many applied problems the zeros of the solutions are of primarily interest. However the majority of these investigations relate to the standard, linear Schrödinger-type equations $y'' + A(t)y = 0$ or their immediate generalisations; they can be found in many textbooks and surveys, see for instance Ince [4], Sansone [6] or Hartman [3]. As for investigations of oscillations of the first-order nonlinear differential equations $y' + P(x, y) = 0$ we have quite a recent paper [2]; here we use methods of gamma-lines [1]. We are not aware of other investigations related to first-order equations.

Also we are not aware of investigations of zeros of solutions of systems of equations, even for an autonomous system of equations

$$\begin{cases} y' = F_1(x, y) \\ x' = F_2(x, y) \end{cases} . \tag{1.1}$$

In this paper we investigate oscillations of solutions of (1.1) assuming only that:
(a) the solutions $(x(t), y(t))$ belong to a given domain $D \subset \mathbb{R}^2$ when $t \in (t_1, t_2)$;
(b) functions $F_1(x, y)$, $F_2(x, y)$ and their derivatives $(F_1(x, y))'_x$, $(F_1(x, y))'_y$, $(F_2(x, y))'_x$, $(F_2(x, y))'_y$ are smooth functions in the closure $\bar{D}$ of D.

By analogy with the theory of oscillations for a single ODE we consider the number $n(t_1, t_2, 0)$ of zeros τ_i in (t_1, t_2) of the solutions, that is, the number of those points τ_i, where $x(\tau_i) = 0$ and $y(\tau_i) = 0$. Also by analogy with the concept of a-points in complex analysis we consider values $a := (a', a'')$ in the (x, y)-plane and define a-points of the solutions as those points τ_i, where $x(\tau_i) = a'$ and $y(\tau_i) = a''$. Respectively we denote by $n(t_1, t_2, a)$ the number of a-points in (t_1, t_2) of the solutions.

Below we give upper bounds respectively for the magnitude $n(t_1, t_2, a)$ with a given a and for the sum $\sum_{\nu=1}^{q} n(t_1, t_2, a_\nu)$, where $a_1, a_2, ..., a_q$ is a given totality of pairwise different points. Thus we obtain for the solutions of (1.1) an analogue of the second fundamental theorem in Nevanlinna value distribution theory; we consider also a similar sum for the number of a-points of meromorphic functions.

Clearly, under assumptions (a) and (b) there are some bounded constants $K(D)$ and $C(D)$ such that

$$|F_1(x, y)|, \ |F_2(x, y)| \leq K(D), \tag{1.2}$$

$$\left|(F_1(x, y))'_x\right|, \left|(F_1(x, y))'_y\right|, \left|(F_2(x, y))'_x\right|, \left|(F_2(x, y))'_y\right| \leq C(D) \tag{1.3}$$

when $(x, y) \in D$.

Theorem 1.1. For any smooth solution $(x(t), y(t))$, $t \in (t_1, t_2)$, of Equation (1.1) satisfying (a) and (b) with $D = \mathbb{R}^2$ and any point a, we have

$$n(t_1, t_2, a) \leq \frac{3C(\mathbb{R}^2)}{\pi} |t_2 - t_1| + 1. \tag{1.4}$$

Note that in this result we suppose that (1.3) holds for any $(x, y) \in \mathbb{R}^2$ (since $D = \mathbb{R}^2$). Now we suppose that these restrictions take place only in a "small" neighbourhood $D(a)$ of $a = (a', a'')$. We will deal with an arbitrary $D(a) \ni a$ with minimal distance ρ between a and boundary $\partial D(a)$. Then we have the following result.

Theorem 1.2. For any $a \in \mathbb{R}^2$ and any smooth solution $(x(t), y(t))$, $t \in (t_1, t_2)$, of Equation (1.1) satisfying (a) and (b) with $D := D(a)$ we have

$$n(t_1, t_2, a) \leq \left\{ \frac{3C(D(a))}{\pi} + \frac{\sqrt{2}K(D(a))}{2\rho} \right\} |t_2 - t_1| + 1. \tag{1.5}$$

Now let $a_1, a_2, ..., a_q$, $q \geq 2$, be a totality of pairwise different points and $D(a_1), D(a_2), ..., D(a_q)$, $q \geq 2$, corresponding neighbourhoods of values $a_1, a_2, ...,$ a_q. We assume that $D(a_i) \cap D(a_j) = \emptyset$, $i \neq j$, and denote $\rho = \min_{1 \leq \nu \leq q} \rho(a_\nu)\ \rho(a_\nu)$, where $\rho(a_\nu)$ is the minimal distance between a_ν and boundary $\partial D(a_\nu)$.

The next result gives upper bounds for the sums $\sum_{\nu=1}^{q} n(t_1, t_2, a_\nu)$ under some very general restrictions on the behaviour of $a(t)\, F_1$ and $b(t)\, F_2$ in the "small" neighbourhoods of these points, that is, in $d := d(a_1, ..., a_q) := \cup_{\nu=1}^{q} D(a_\nu)$.

Theorem 1.3. For any totality of pairwise different points $a_1, a_2, ..., a_q$ and for any smooth solution $(x(t), y(t))$, $t \in (t_1, t_2)$, of Equation (1.1) satisfying (a) and (b) in $D = d$, we have

$$\sum_{\nu=1}^{q} n(t_1, t_2, a_\nu) \leq \left\{ \frac{3C(d)}{\pi} + \frac{\sqrt{2}K(d)}{\rho} \right\} |t_2 - t_1| + 1. \tag{1.6}$$

The reader can easily observe that a dependance on $C(d)$ and $K(d)$ arises necessarily: to obtain a conclusion about the behaviour γ in neighbourhoods of $a_1, a_2, ...,$ a_q we should have information about the behaviour of F_1 and F_2 in these neighbourhoods.

Interplay between Theorem 1.3 and Nevanlinna value distribution theory [5]. The second fundamental theorem in Nevanlinna theory studies lower bounds for a similar sum $\sum_{\nu=1}^{q} N(r, a_\nu, w)$, where the magnitude $N(r, a_\nu, w)$ is the number of a-points in the disks $\{z|\ |z| < r\}$ of a meromorphic function w in the complex plane. A very important relationship in Nevanlinna theory (that leads to the known deficiency relation) is that the bounds in the Nevanlinna theorem "do not depend strongly" on the number q of values a_ν. A similar circumstance can also be observed in Theorem 1.3: in the right-hand side of (1.8) only the magnitude ρ depends on the totality of points a_ν.

2. Proofs.

The proof of Theorem 2.1. We utilise a geometric idea that has been used in a number of investigations related to geometric aspects of value distribution theory and gamma-lines (see [1]). Consider the part $\gamma_i(a)$ of the curve $\gamma := (x(t), y(t))$, $t \in (t_1, t_2)$, lying between two successive a-points of τ_i and τ_{i+1}. Since this part is a smooth curve and is closed so that $x(\tau_i) = x(\tau_{i+1}) = a'$ and $y(\tau_i) = y(\tau_{i+1}) = a''$ we conclude that the variation of the angle $\alpha(t)$ between the tangent to the curve and x-axis is greater than or equal to π so that we have

$$\mathrm{Var}_{\tau_i \le t \le \tau_{i+1}}\, \alpha(t) := \int_{\tau_i}^{\tau_{i+1}} \left| \frac{d}{dt} \arctan \frac{y'(t)}{x'(t)} \right| dt \ge \pi.$$

Thus for any $i = 1, 2, ..., n(t_1, t_2, a) - 1$

$$1 \le \frac{1}{\pi} \int_{\tau_i}^{\tau_{i+1}} \left| \frac{d}{dt} \arctan \frac{y'(t)}{x'(t)} \right| dt \tag{2.1}$$

and summation over i provides

$$n(t_1, t_2, a) \le \frac{1}{\pi} \int_{t_1}^{t_2} \left| \frac{d}{dt} \arctan \frac{y'(t)}{x'(t)} \right| dt + 1. \tag{2.2}$$

Further we have

$$\left| \frac{d}{dt} \arctan \frac{y'(t)}{x'(t)} \right| := \left| \frac{y''x' - x''y'}{(x')^2 + (y')^2} \right|.$$

Since

$$\begin{aligned} y''(t) &= (F_1(x,y))'_x\, x' + (F_1(x,y))'_y\, y' \\ &= (F_1(x,y))'_x\, F_2(x,y) + (F_1(x,y))'_y\, F_1(x,y) \end{aligned}$$

and

$$\begin{aligned} x''(t) &= (F_2(x,y))'_x\, x' + (F_2(x,y))'_y\, y' \\ &= (F_2(x,y))'_x\, F_2(x,y) + (F_2(x,y))'_y\, F_1(x,y), \end{aligned}$$

we have

$$\begin{aligned} \frac{y''x' - x''y'}{(x')^2 + (y')^2} &= \frac{\left[(F_1(x,y))'_x\, F_2(x,y) + (F_1(x,y))'_y\, F_1(x,y)\right] F_2(x,y)}{F_2^2(x,y) + F_1^2(x,y)} \\ &\quad - \frac{\left[(F_2(x,y))'_x\, F_2(x,y) + (F_2(x,y))'_y\, F_1(x,y)\right] F_1(x,y)}{F_2^2(x,y) + F_1^2(x,y)} \\ &= \frac{(F_1(x,y))'_x\, F_2^2(x,y) - (F_2(x,y))'_y\, F_1^2(x,y)}{F_2^2(x,y) + F_1^2(x,y)} \\ &\quad + \frac{\left[(F_1(x,y))'_y - (F_2(x,y))'_x\right] F_1(x,y)\, F_2(x,y)}{F_2^2(x,y) + F_1^2(x,y)}, \end{aligned}$$

so that

$$\left|\frac{y''x'-x''y'}{(x')^2+(y')^2}\right| \leq$$

$$\left|(F_1(x,y))'_x\right| + \frac{\left|(F_1(x,y))'_y\right| + \left|(F_2(x,y))'_x\right|}{2} + \left|(F_2(x,y))'_y\right|. \qquad (2.3)$$

Consequently due to (2.2) we have

$$n(t_1,t_2,a) \leq \frac{1}{\pi}\int_{t_1}^{t_2}\left\{\left|(F_1(x,y))'_x\right| + \frac{\left|(F_1(x,y))'_y\right| + \left|(F_2(x,y))'_x\right|}{2} + \left|(F_2(x,y))'_y\right|\right\} +1$$

and taking into account (1.2) we obtain inequality (1.4) of Theorem 1.1.

The proof of Theorem 2.2. Under the conditions of this theorem we have restrictions (1.2) and (1.3) only when $(x,y) \in D(a)$. We consider in the proof two types of indices:
indices i^* such that $\gamma_{i^*}(a)$ lie completely in $D(a)$ and
indices i^{**} such that $\gamma_{i^{**}}(a)$ does not lie completely in $D(a)$.

We denote by $n^*(t_1,t_2,a)$ and $n^{**}(t_1,t_2,a)$ respectively the numbers of corresponding indices. Clearly we have

$$n(t_1,t_2,a) \leq n^*(t_1,t_2,a) + n^{**}(t_1,t_2,a) + 1. \qquad (2.4)$$

On taking into account that (1.3) holds now for $(x,y) \in D(a)$, we obtain from (2.1) and (2.3) that

$$n^*(t_1,t_2,a) \leq \frac{3C(D(a))}{\pi}\sum_{(\gamma_{i^*}(a))}\left|T(\gamma_{i^*}(a))\right|, \qquad (2.5)$$

where $T(\gamma_{i^*}(a))$ are intervals in (t_1,t_2) corresponding to $\gamma_{i^*}(a)$ and $|T(\gamma_{i^*}(a))|$ is the length of $T(\gamma_{i^*}(a))$. Since $\sum_{(\gamma_{i^*}(a))}|T(\gamma_{i^*}(a))| \leq |t_2-t_1|$ the last inequality yields

$$n^*(t_1,t_2,a) \leq \frac{3C(D(a))}{\pi}\left|t_2-t_1\right|. \qquad (2.6)$$

For the indices i^{**}, we note that the set $\gamma_{i^{**}}(a) \cap D(a)$ necessarily contains two curves such that each of them connects the point a with the boundary $\partial D(a)$ of $D(a)$. Let us denote by $\delta_{i^{**}}(a)$ the union of these two curves, by $T(\delta_{i^{**}}(a))$ corresponding intervals in (t_1,t_2) and by $|T(\delta_{i^{**}}(a))|$ the length of these intervals. Also we denote by $l(X)$ the total length of a totality of curves X. Since $l(\delta_{i^{**}}(a)) \geq 2\rho$, that is,

$$1 \leq \frac{l(\delta_{i^{**}}(a))}{2\rho},$$

we obtain

$$n^{**}(t_1,t_2,a) \le \sum_{(\gamma_{i^{**}}(a))} \frac{l(\delta_{i^{**}}(a))}{2\rho}.$$

Since

$$l(\delta_{i^{**}}(a)) := \int_{T(\delta_{i^{**}}(a))} \sqrt{(x'(t))^2 + (y'(t))^2}dt := \int_{T(\delta_{i^{**}}(a))} \sqrt{F_1^2(t)+F_2^2(t)}dt$$

we obtain on also taking into account (1.2) that

$$n^{**}(t_1,t_2,a) \le \frac{\sqrt{2}K(D(a))}{2\rho} \sum_{(\gamma_{i^{**}}(a))} |T(\delta_{i^{**}}(a))|\,. \tag{2.7}$$

Thus we derive from the obvious inequality $\sum_{(\gamma_{i^{**}}(a))} |T(\delta_{i^{**}}(a))| \le |t_2 - t_1|$ that

$$n^{**}(t_1,t_2,a) \le \frac{\sqrt{2}K(D(a))}{\rho} |t_2 - t_1|\,. \tag{2.8}$$

Theorem 2 follows from (2.4), (2.6) and (2.8).

The proof of Theorem 2.3. Denote $N = \sum n(t_1,t_2,a_\nu)$. By moving along the curve γ we can enumerate all those points τ_i, $t_1 \le \tau_1 < ... < \tau_N \le t_2$, where γ has a_ν-points, $\nu = 1,2,...,q$. Denote by γ_i the part of γ corresponding to the interval (τ_i,τ_{i+1}). We distinguish three types of indices for τ_i:
indices i^* such that endpoints of γ_{i^*} are the same a_ν-points and γ_{i^*} lie completely in $D(a_\nu)$;
indices i^{**} such that endpoints of $\gamma_{i^{**}}$ are the same a_ν-points and $\gamma_{i^{**}}$ does not lie completely in $D(a_\nu)$;
indexes i^{***} such that the $\gamma_{i^{***}}$ begins with a points a_ν but ends at another point a_j so that $a_\nu \ne a_j$, $\nu \ne j$, $\nu, j \in (1,2,...q)$.

The curves γ_{i^*}, $\gamma_{i^{**}}$, $\gamma_{i^{***}}$ which begin with points a_ν we will denote by $\gamma_{i^*}(a_\nu)$, $\gamma_{i^{**}}(a_\nu)$ and $\gamma_{i^{***}}(a_\nu)$. We denote by $n^*(t_1,t_2,a_\nu)$, $n^{**}(t_1,t_2,a_\nu)$ and $n^{***}(t_1,t_2,a_\nu)$ respectively the numbers of corresponding curves (or indices i^*, i^{**}, i^{***}). Clearly we have

$$\sum_{\nu=1}^{q} n(t_1,t_2,a_\nu) \le$$

$$\sum_{\nu=1}^{q} n^*(t_1,t_2,a_\nu) + \sum_{\nu=1}^{q} n^{**}(t_1,t_2,a_\nu) + \sum_{\nu=1}^{q} n^{***}(t_1,t_2,a_\nu) + 1. \tag{2.9}$$

Also denoting by $T(x)$ intervals in (t_1,t_2) corresponding to the part x and by $|T(x)|$ the length of $T(x)$, we note that

$$\sum_{\nu=1}^{q} \sum_{i^*=1}^{n^*(t_1,t_2,a_\nu)} |T(\gamma_{i^*}(a_\nu))| + \sum_{\nu=1}^{q} \sum_{i^{**}=1}^{n^{**}(t_1,t_2,a_\nu)} |T(\gamma_{i^{**}}(a_\nu))|$$

$$+\sum_{\nu=1}^{q}\sum_{i^{***}=1}^{n^{***}(t_1,t_2,a_\nu)}|T(\gamma_{i^{***}}(a_\nu))| \le |t_2-t_1|\,. \tag{2.10}$$

The situation with the indices i^* is identical with that we had in the proof of Theorem 1.2, so that applying (2.5) we have

$$n^*(t_1,t_2,a_\nu) \le \frac{3C(D(a_\nu))}{\pi}\sum_{i^*=1}^{n^*(t_1,t_2,a_\nu)}|T(\gamma_{i^*}(a_\nu))|$$

and taking into account also (2.10) and that $C(d) \ge C(D(a_\nu))$, $\nu = 1,2,...,q$, we obtain

$$\sum_{\nu=1}^{q} n^*(t_1,t_2,a_\nu) \le \frac{3C(d)}{\pi}\,|t_2-t_1|\,. \tag{2.11}$$

The situation with the indices i^{**} is also identical with that in the proof of Theorem 1.2, so that on applying (2.7) we have

$$n^{**}(t_1,t_2,a_\nu) \le \frac{\sqrt{2}K(D(a_\nu))}{2\rho}\sum_{i^{**}=1}^{n^{**}(t_1,t_2,a_\nu)}|T(\delta_{i^{**}}(a_\nu))|\,;$$

here $\delta_{i^{**}}(a))$ is defined similarly as in the proof of Theorem 1.2. Now taking into account that $|T(\delta_{i^{**}}(a_\nu))| \le |T(\gamma_{i^{**}}(a_\nu))|$ and that $K(d) \ge K(D(a_\nu))$, $\nu = 1,2,...,q$, we obtain

$$\sum_{\nu=1}^{q} n^{**}(t_1,t_2,a_\nu) \le \frac{\sqrt{2}K(d)}{2\rho}\sum_{\nu=1}^{q}\sum_{i^{**}=1}^{n^{**}(t_1,t_2,a_\nu)}|T(\gamma_{i^{**}}(a_\nu))|\,.$$

For the indices i^{***}, we may argue as for the indices i^{**}. We note that any $\gamma_{i^{***}}(a_\nu)$ involves a part $\delta_{i^{***}}(a_\nu)$ such that the length $l(\delta_{i^{***}}(a_\nu))$ is greater than or equal to ρ. Therefore repeating verbatim the corresponding reasoning in the proof of Theorem 1.2 and taking again into account that $K(d) \ge K(D(a_\nu))$, $\nu = 1,2,...,q$, we obtain

$$\begin{aligned}\sum_{\nu=1}^{q} n^{***}(t_1,t_2,a_\nu) &\le \frac{\sqrt{2}K(d)}{\rho}\sum_{\nu=1}^{q}\sum_{i^{**}=1}^{n^{***}(t_1,t_2,a_\nu)}|T(\delta_{i^{***}}(a_\nu))| \\ &\le \frac{\sqrt{2}K(d)}{\rho}\sum_{\nu=1}^{q}\sum_{i^{**}=1}^{n^{***}(t_1,t_2,a_\nu)}|T(\gamma_{i^{***}}(a_\nu))|\,.\end{aligned}$$

From the last two inequalities and (2.10) we have

$$\sum_{\nu=1}^{q} n^{**}(t_1,t_2,a_\nu) + \sum_{\nu=1}^{q} n^{***}(t_1,t_2,a_\nu) \le \frac{\sqrt{2}K(d)}{\rho}\,|t_2-t_1|$$

so that Theorem 1.3 follows immediately from (2.9), (2.11) and the last inequality.

References

[1] Barsegian G.A., *Gamma-lines: on the geometry of real and complex functions*, Taylor and Frances, London-New York, 2002.

[2] Barseghyan K. and Barsegian G., A method for studying oscillations of non linear differential equations. Applications to some equations in biology and economics, 121-146; *Topics in Analysis and its Applications*, Proceedings of the NATO Advanced Research Workshop, Yerevan, September 2002, Editors G. Barsegian, H. Begehr, Kluwer, Series: NATO Science Publications, 2004.

[3] Hartman Ph., *Ordinary Differential Equations*, John Wiley and Sons, New York-London-Sydney, 1964.

[4] Ince E. L., *Ordinary Differential Equations*, Dover, New York, 1956.

[5] Nevanlinna R., *Eindeutige Analytische Funktionen*, Springer, Berlin, 1936.

[6] Sansone G., *Equazioni Differentialy nel Campo Reale*, Seconda Edizione, Bologna, 1948.