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Impulsive Functional Differential Equations with Variable Times and Infinite Delay

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Abstract

In this paper we investigate the existence of solutions to some classes of impulsive functional and neutral differential equations with variable times and infinite delay, using the nonlinear alternative of Leray-Schauder type.

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1. Introduction

This paper is concerned with the existence of solutions to first order impulsive functional and neutral functional differential equations with variable times and infinite delay. In particular, in Section 3, we will consider the class of first order functional differential equations with impulsive effects,

$$y'(t) = f(t, y_t),\ a.e.\ t \in J = [0, b],\ t \neq \tau_k(y(t)),\ k = 1, \dots, m, \tag{1}$$

$$y(t^+) = I_k(y(t)),\quad t = \tau_k(y(t)),\ k = 1, \dots, m, \tag{2}$$

$$y(t)=\phi(t), \quad t\in(-\infty,0], \tag{3}$$

where $f: J\times B\to \mathbb{R}^n,\ \ \tau_k:\mathbb{R}^n\to\mathbb{R},\ I_k:\mathbb{R}^n\to\mathbb{R}^n,\ k=1,2,\dots,m$ are given functions satisfying some assumptions that will be specified later, $\phi\in B$ and B is called a *phase space* that will be defined later. Section 4 is devoted to impulsive neutral functional differential equations,

$$\frac{d}{dt}[y(t)-g(t,y_t)]=f(t,y_t),\ t\in J,\ t\neq\tau_k(y(t)),\ k=1,..,m, \tag{4}$$

$$y(t^+)=I_k(y(t)),\ \ t=\tau_k(y(t)),\ k=1,\dots,m, \tag{5}$$

$$y_0=\phi\in B, \tag{6}$$

where f, I_k, B are as in problem (1)-(3), and $g: J\times B\to\mathbb{R}^n$ is a given function.

The notion of the phase space B plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [15] (see also Kappel and Schappacher [18] and Schumacher [28]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [17]. For the case where the impulses are absent (i.e $I_k=0,\ k=1,\dots,m$), an extensive theory has been developed for the problem (1)-(3). We refer to Hale and Kato [15], Corduneanu and Lakshmikantham [9], Hino *et al* [17], Lakshmikantham *et al* [23] and Shin [29].

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [2], Lakshmikantham *et al* [22] and Samoilenko and Perestyuk [27] and the papers of Agur *et al* [1], Ballinger and Liu [4], Franco *et al* [11] and the references therein. The theory of impulsive differential equations with variable times is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting results have been done by Bajo and Liz [3], Frigon and O'Regan [12, 13, 14], Kaul *et al* [19], Kaul and Liu [20, 21], Lakshmikantham *et al* [24, 25] and Liu and Ballinger [26]. Very recently, impulsive functional differential equations with variable times was studied by Benchohra *et al* [7] and Benchohra and Ouahab [8]. The main theorems of the present paper extend the problem (1)-(3) considered by Benchohra *et al* [5, 7] when the delay is finite and the impulse times are constant or variable. Our approach here is based on the Leray-Schauder alternative [10].

2. Preliminaries

In this short section, we introduce notations and definitions which are used throughout this paper.

$C([0,b],\mathbb{R}^n)$ is the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : 0 \le t \le b\}.$$

$L^1([0,b],\mathbb{R}^n)$ denotes the Banach space of measurable functions $y : [0,b] \longrightarrow \mathbb{R}^n$ which are Lebesgue integrable and normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt \quad \text{for all} \quad y \in L^1([0,b],\mathbb{R}^n).$$

Definition 2..1 *The map $f : [0,b] \times B \longrightarrow \mathbb{R}^n$ is said to be L^1-Carathéodory if*

(i) $t \longmapsto f(t,x)$ *is measurable for each $x \in B$;*

(ii) $x \longmapsto f(t,x)$ *is continuous for almost all $t \in [0,b]$;*

(iii) For each $q > 0$, there exists $h_q \in L^1([0,b],\mathbb{R}_+)$ such that

$$\|f(t,x)\| \le h_q(t) \quad \textit{for all} \quad \|x\|_B \le q \quad \textit{and for almost all} \quad t \in [0,b].$$

3. Functional Differential Equations

In order to define the phase space and the solution of (1)–(3) we shall consider the space

$$\begin{aligned} B_b \ &= \Big\{y : (-\infty,b] \to \mathbb{R}^n, \text{ and there exist } 0 < t_0 < t_1 < \ldots < t_m < b \\ &\quad \text{such that, } t_k = \tau_k(y(t_k)),\ y(t_k^-), y(t_k^+), \text{ exist with } \ y(t_k) = y(t_k^-), \\ &\quad y(t) = \phi(t), t \le 0,\ y_k \in C(J_k,\mathbb{R}^n)\Big\}, \end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0,\ldots,m$. Let $\|\cdot\|_b$ be the seminorm in B_b defined by

$$\|y\|_b = \|y_0\|_B + \sup\{|y(s)| : 0 \le s \le b\},\ y \in B_b.$$

We will assume that B satisfies the following axioms:

(A) If $y : (-\infty,b] \to \mathbb{R}^n, b > 0$ and $y_0 \in B$, then for every t in $[0,b)$ the following conditions hold:

(i) y_t is in B;

(ii) $\|y_t\|_B \leq K(t)\sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_B$,

where $H \geq 0$ is a constant, $K : [0,\infty) \to [0,\infty)$ is continuous, $M : [0,\infty) \to [0,\infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a B-valued continuous function on $[0, b)$.

(A-2) The space B is complete.

Let us start by defining what we mean by a solution of problem (1)–(3).

Definition 3..1 *A function $y \in B_b$, is said to be a solution of (1)–(3) if y satisfies (1)–(3).*

Theorem 3..1 *Let $f : J \times B \to \mathrm{I\!R}^n$ be an L^1-Carathéodory function. Assume that:*

(H1) The functions $\tau_k \in C^1(\mathrm{I\!R}^n, \mathrm{I\!R})$ for $k = 1, \ldots, m$. Moreover,

$$0 < \tau_1(x) < \ldots < \tau_m(x) < b \quad \text{for all } x \in \mathrm{I\!R}^n;$$

(H2) There exist a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1([0,b], \mathrm{I\!R}_+)$ such that

$$\|f(t,x)\| \leq p(t)\psi(\|x\|_B) \quad \text{for a.e. } t \in [0,b] \text{ and each } x \in B,$$

with

$$K_b \int_0^b p(s)ds < \int_c^\infty \frac{dx}{\psi(x)};$$

where $K_b = \sup\{|K(t)| : t \in [0,b]\}, M_b = \sup\{|M(t)| : t \in [0,b]\}$ and $c = M_b\|\phi\|_B + K_b\|\phi(0)\|$.

(H3) For all $(t,x) \in [0,b] \times \mathrm{I\!R}^n$ and for all $y_t \in B$ we have

$$\langle \tau_k'(x), f(t, y_t)\rangle \neq 1 \text{ for } k = 1, \ldots, m,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathrm{I\!R}^n$;

(H4) For all $x \in \mathrm{I\!R}^n$

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)) \text{ for } k = 1, \ldots, m.$$

Then the initial value problem (1)-(3) has at least one solution.

Proof. The proof will be given in several steps.

Step 1: Consider the problem,

$$y'(t)) = f(t, y_t), \; a.e. \; t \in [0, b], \tag{7}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{8}$$

Transform the problem (7)–(8) into a fixed point problem. Consider the operator $N : C((-\infty, b], \mathrm{I\!R}^n) \longrightarrow C((-\infty, b], \mathrm{I\!R}^n)$ defined by,

$$N(y)(t) = \begin{cases} \phi(t), & t \leq 0, \\ \phi(0) + \int_0^t f(s, y_s) ds, & t \in [0, b]. \end{cases}$$

Let $x(\cdot) : (-\infty, b) \to \mathrm{I\!R}^n$ be the function defined by

$$x(t) = \begin{cases} \phi(0), & \text{if } \; t \in [0, b], \\ \phi(t), & \text{if } \; t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in C([0, b], \mathrm{I\!R}^n)$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0, b], \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation,

$$y(t) = \phi(0) + \int_0^t f(s, y_s) ds,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \leq t \leq b$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t \leq b$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t f(s, \bar{z}_s + x_s) ds. \tag{9}$$

Set

$$C_0 = \{z \in C([0, b], \mathrm{I\!R}^n) : z(0) = 0\}.$$

Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = \int_0^t f(s, \bar{z}_s + x_s) ds, \quad t \in [0, b].$$

Obviously, that the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. We shall use the Leray-Schauder alternative to prove that P has fixed point.

Claim 1: *P is continuous.*

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in C_0. Then

$$\|(Pz_n)(t) - (Pz)(t)\| \;\leq\; \int_0^b \|f(s, \bar{z}_{n_s} + x_s) - f(s, \bar{z}_s + x_s)\| ds.$$

Since f is L^1-Carathéodory, then we have

$$\|P(z_n) - P(z)\|_\infty \;\leq\; \|f(\cdot, \bar{z}_{n_{(\cdot)}} + x_{(\cdot)}) - f(\cdot, \bar{z}_{(\cdot)} + x_{(\cdot)})\|_{L^1} \to 0 \text{ as } n \to \infty.$$

Claim 2: *P maps bounded sets into bounded sets in C_0.*

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that for each $z \in \mathcal{B}_q = \{z \in C_0 : \|z\|_\infty \leq q\}$ one has $\|P(z)\|_\infty \leq \ell$. Let $z \in \mathcal{B}_q$. Since f is an L^1-Carathéodory function, we have for each $t \in [0, b]$

$$\|(Pz)(t)\| \;\leq\; \int_0^b h_{q_*}(s) ds := \ell,$$

where

$$\|\bar{z}_s + x_s\|_B \leq \|\bar{z}_s\|_B + \|x_s\|_B \leq K_b q + K_b\|\phi(0)\| + M_b\|\phi\|_B := q_*.$$

Claim 3: *P maps bounded sets into equicontinuous sets of $C([0, b], \mathrm{IR}^n)$.*

Let $l_1, l_2 \in [0, b]$, $l_1 < l_2$ and $\mathcal{B}_q$ be a bounded set of C_0 as in Claim 2. Let $z \in \mathcal{B}_q$. Then for each $t \in [0, b]$ we have

$$\begin{aligned} \|(Pz)(l_2) - (Pz)(l_1)\| &\leq \int_{l_1}^{l_2} \|f(s, \bar{z}_s + x_s)\| ds \\ &\leq \int_{l_1}^{l_2} h_{q_*}(s) ds. \end{aligned}$$

We see that $\|(Pz)(l_2) - (Pz)(l_1)\|$ tend to zero independently of $z \in \mathcal{B}_q$ as $l_2 - l_1 \to 0$. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem we can conclude that $P : C_0 \longrightarrow C_0$ is continuous and completely continuous.

Claim 4: *A priori bounds on solutions.*

Let z be a possible solution of the equation $z = \lambda P(z)$ and $z_0 = \lambda\phi$ for some $\lambda \in (0, 1)$. Then

$$\|z(t)\| \leq \int_0^t \|f(s, \bar{z}_s + x_s)\| ds \leq \int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B) ds. \tag{10}$$

But

$$\begin{aligned}\|\bar{z}_s + x_s\|_B &\leq \|\bar{z}_s\|_B + \|x_s\|_B \\ &\leq K(t)\sup\{\|z(s)\| : 0 \leq s \leq t\} + M(t)\|z_0\|_B \\ &\quad + K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t)\|x_0\|_B \\ &\leq K_b \sup\{\|z(s)\| : 0 \leq s \leq t\} + M_b\|\phi\|_B + K_b M\|\phi(0)\|.\end{aligned} \tag{11}$$

If we name $w(t)$ the right hand side of (11), then we have

$$\|\bar{z}_s + x_s\|_B \leq w(t),$$

and therefore (10) becomes

$$\|z(t)\| \leq \int_0^t p(s)\psi(w(s))ds, \quad t \in [0,b]. \tag{12}$$

Using (12) in the definition of w, we have that

$$w(t) \leq K_b \int_0^t p(s)\psi(w(s))ds + M_b\|\phi\|_B + K_b\|\phi(0)\|, \quad t \in [0,b].$$

Denoting by $\beta(t)$ the right hand side of the last inequality we have

$$w(t) \leq \beta(t), \quad t \in [0,b],$$

$$\beta(0) = M_b\|\phi\|_B + K_b\|\phi(0)\|,$$

and

$$\begin{aligned}\beta'(t) &= K_b p(t)\psi(w(t)) \\ &\leq K_b p(t)\psi(\beta(t)), \quad t \in [0,b].\end{aligned}$$

This implies that for each $t \in [0,b]$

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq K_b \int_0^b p(s)ds < \int_c^\infty \frac{ds}{\psi(s)}.$$

Thus from (H2) there exists a constant K_* such that $\beta(t) \leq K_*,\ t \in [0,b]$, and hence $\|\bar{z}_t + x_t\|_B \leq w(t) \leq K_*,\ t \in [0,b]$. From equation (12) we have that

$$\|z(t)\| \leq \int_0^b p(s)\psi(K_*)ds := K_1.$$

Set

$$U = \{z \in C_0 : \sup\{\|z(t)\| : 0 \leq t \leq b\} < K_1 + 1\}.$$

$P : \overline{U}_0 \to C_0$ is continuous and completely continuous. From the choice of U_0, there is no $z \in \partial U_0$ such that $z = \lambda P(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [10], we deduce that P has a fixed point z in U_0. Hence N has a fixed point y which is a solution to problem (7)–(8). Denote this solution by y_0.

Define the function

$$r_{k,1}(t) = \tau_k(y_0(t)) - t \quad \text{for} \quad t \geq 0.$$

(H1) implies that

$$r_{k,1}(0) \neq 0 \ \text{ for } \ k = 1, \ldots, m.$$

If

$$r_{k,1}(t) \neq 0 \ \text{ on } \ [0,b], \ \text{ for } \ k = 1, \ldots, m,$$

i.e.,

$$t \neq \tau_k(y_0(t)) \text{ on } \ [0,b] \ \text{ and for } \ k = 1, \ldots, m,$$

then y_0 is a solution of the problem (1)-(3).

It remains to consider the case when

$$r_{1,1}(t) = 0 \ \text{ for some } \ t \in [0,b].$$

Now since

$$r_{1,1}(0) \neq 0,$$

and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0, \ \text{ and } \ r_{1,1}(t) \neq 0 \ \text{ for all } \ t \in [0, t_1).$$

Thus by $(H1)$ we have

$$r_{k,1}(t) \neq 0 \ \text{ for all } \ t \in [0, t_1), \ \text{ and } \ k = 1, \ldots, m.$$

Step 2: Consider now the following problem

$$y'(t) = f(t, y_t), \ a.e. \ t \in [t_1, b], \tag{13}$$

$$y(t_1^+) = I_1(y_0(t_1)), \quad y(t) = y_0(t), \, t \in (-\infty, t_1]. \tag{14}$$

Let

$$C_1 = \{y \in C((t_1, b], \mathbb{R}^n) : y(t_1^+) \text{ exist}\}.$$

Set $C_* = C((-\infty, t_1], \mathbb{R}^n) \cap C_1$. Consider the operator $N_1 : C_* \to C_*$ defined by:

$$N_1(y)(t) = \begin{cases} y_0(t), & (-\infty, t_1], \\ \\ I_1(y_0(t_1)) + \displaystyle\int_{t_1}^{t} f(s, y_s)ds, & t \in (t_1, b]. \end{cases}$$

Let $x(\cdot) : (-\infty, b) \to \mathrm{I\!R}^n$ be the function defined by

$$x(t) = \begin{cases} I_1(y_0(t_1)), & \text{if } \ t \in (t_1, b], \\ y_0(t), & \text{if } \ t \in (-\infty, t_1]. \end{cases}$$

Then $x_{t_1} = y_0$. For each $z \in C((-\infty, t_1], \mathrm{I\!R}^n) \cap C((t_1, b], \mathrm{I\!R}^n)$ with $z(t_1) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [t_1, b], \\ 0, & \text{if } t \in (-\infty, t_1]. \end{cases}$$

If $y(\cdot)$ satisfies the following integral equation

$$y(t) = I_1(y_0(t_1)) + \int_{t_1}^{t} f(s, y_s)ds,$$

we can decompose it as $y(t) = \bar{z}(t) + x(t), t_1 \le t \le b$, which implies $y_t = \bar{z}_t + x_t$, for every $t_1 \le t \le b$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_{t_1}^{t} f(s, \bar{z}_s + x_s)ds. \tag{15}$$

Set

$$C_{t_1} = \{z \in C_* : z(t_1) = 0\}.$$

Let the operator $P : C_{t_1} \to C_{t_1}$ defined by:

$$(P_1 z)(t) = \int_{t_1}^{t} f(s, \bar{z}_s + x_s)ds, \quad t \in [t_1, b].$$

As in Step 1 we can show that P_1 is continuous and completely continuous, and if z is a possible solution of the equations $z = \lambda P_1(z)$ and $z_0 = \lambda y_0$ for some $\lambda \in (0, 1)$, there exists $K_{*_1} > 0$ such that

$$\|z\|_\infty \le K_{*_1}.$$

Set

$$U_1 = \{z \in C_{t_1} : \sup\{\|z(t)\| : t_1 \le t \le b\} \le K_{*_1} + 1\}.$$

As a consequence of the nonlinear alternative of Leray-Schauder type [10], we deduce that P_1 has a fixed point z in U_1. Thus N_1 has a fixed point y which is an solution to problem (13)–(14). Denote this solution by y_1. Define

$$r_{k,2}(t) = \tau_k(y_1(t)) - t \quad \text{for } t \ge t_1.$$

If

$$r_{k,2}(t) \ne 0 \ \text{ on } \ (t_1, b] \ \text{ and for all } \ k = 1, \ldots, m$$

then

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_1(t), & \text{if } t \in (t_1, b], \end{cases}$$

is a solution of the problem (1)–(3).

It remains to consider the case when

$$r_{2,2}(t) = 0, \quad \text{for some} \quad t \in (t_1, b].$$

By $(H4)$ we have

$$\begin{array}{rcl} r_{2,2}(t_1^+) & = & \tau_2(y_1(t_1^+)) - t_1 \\ & = & \tau_2(I_1(y_0(t_1)) - t_1 \\ & > & \tau_1(y_0(t_1)) - t_1 \\ & = & r_{1,1}(t_1) = 0. \end{array}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{2,2}(t_2) = 0,$$

and

$$r_{2,2}(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2).$$

It is clear by (H1) that

$$r_{k,2}(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2), \quad k = 2, \ldots, m.$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0.$$

From (H4) it follows that

$$\begin{array}{rcl} r_{1,2}(t_1^+) & = & \tau_1(y_1(t_1^+)) - t_1 \\ & = & \tau_1(I_1(y_0(t_1)) - t_1 \\ & \leq & \tau_1(y_0(t_1) - t_1 \\ & = & r_{1,1}(t_1) = 0. \end{array}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, b]$. Since

$$y_1'(t) = f(t, y_{1_t}),$$

then

$$r_{1,2}'(s_1) = \tau_1'(y_1(s_1))y_1'(s) - 1 = 0.$$

Therefore

$$\langle \tau_1'(y_1(s_1)), f(s_1, y_{1_{s_1}}) \rangle = 1,$$

which is a contradiction by $(H3)$.

Step 3: We continue this process and taking into account that $y_m := y\Big|_{[t_m,b]}$ is a solution to the problem

$$y'(t) = f(t, y_t), \;\; a.e. \; t \in (t_m, b], \tag{16}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)), \;\; y(t) = y_{m-1}(t), \; t \in (-\infty, t_{m-1}]. \tag{17}$$

The solution y of the problem (1)-(3) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

4. Neutral Functional Differential Equations

This section is concerned with the existence of solutions for initial value problems for first order neutral functional differential equations with impulsive effects at variable times (4)-(6).

Neutral functional differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the book by Hale and Verduyn Lunel [16].

In the last few years the impulsive neutral functional differential equations with bounded delay where the impulses are constant or variable were studied by Benchohra *et al* [6], Benchohra and Ouahab [8], and the goal of this section is to extend those results.

Theorem 4..1 *Let* $f : J \times B \to \mathbb{R}^n$ *be an* L^1*-Carathéodory function. Assume* $(H1)$ *and the conditions:*

(B1) The function g is continuous and completely continuous and for any bounded set $Q \subseteq C((-\infty, b], \mathbb{R}^n)$ *the set* $\{t \to g(t, x_t) : x \in Q\}$ *is equicontinuous in* $C([0, b], \mathbb{R}^n)$ *and there exist constants* $c_1, \; c_2 \geq 0$ *such that*

$$|g(t, u)| \leq c_1 \|u\|_B + c_2, \quad t \in [0, b], \; u \in B;$$

(B2) There exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ *and* $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$\|f(t, x)\| \leq p(t)\psi(\|x\|_B), \;\; \textit{for a.e. } t \in [0, b] \textit{ and each } y \in \mathbb{R}^n$$

with

$$\frac{K_b}{1-K_bc_1}\int_0^b p(s)ds < \int_{\bar{c}}^{\infty}\frac{dx}{\psi(x)},$$

where $c_1K_b < 1$, $\bar{c} = \dfrac{1}{1-c_1K_b}\left[K_b\|g(0,\phi)\| + c_2K_b + \alpha\right]$ *and* $\alpha = M_b\|\phi\|_B + K_b\|\phi(0)\| + M_b\|\phi\|_B$;

(B3) For all $t \in [0,b]$ *and for all* $y_t \in B$ *we have*

$$\langle \tau_k'(y(t) - g(t,y_t)), f(t,y_t)\rangle \neq 1 \textit{ for } k = 1,\dots,m,$$

where $\langle\cdot,\cdot\rangle$ *denotes the scalar product in* $\mathbb{R}^n$;

(B4) g *is a nonnegative function;*

(B5) τ_k *is a nonincreasing function and*

$$I_k(x) \leq x \quad \textit{for all} \quad x \in R^n, k = 1,\dots,m;$$

(B6) For all $x \in \mathbb{R}^n$

$$\tau_k(x) < \tau_{k+1}(I_k(x)) \textit{ for } k = 1,\dots,m,$$

are satisfied. Then the IVP (4)-(6) has at least one solution.

Proof. Consider the following problem

$$\frac{d}{dt}[y(t) - g(t,y_t)] = f(t,y_t),\ \textit{a.e. } t \in [0,b], \tag{18}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{19}$$

Transform the problem into a fixed point problem. Consider the operator $N : C((-\infty,b],\mathbb{R}^n) \to C((-\infty,b],\mathbb{R}^n)$ defined by,

$$N^*(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty,0], \\ \phi(0) - g(0,\phi(0)) + g(t,y_t) + \displaystyle\int_0^t f(s,y_s)ds, & \text{if } t \in [0,b]. \end{cases}$$

In analogy to Theorem 3..1, we consider the operator $P^* : C_0 \to C_0$ defined by

$$(P^*z)(t) = \ g(0,\phi) - g(t,\overline{z}_t + x_t) + \int_0^t f(s,\overline{z}_s + x_s)ds.$$

In order to use the Leray-Schauder alternative, we shall obtain a priori estimates for the solutions of the integral equation

$$z(t) = \lambda\left[g(0,\phi) - g(t,\overline{z}_t + x_t) + \int_0^t f(s,\overline{z}_s + x_s)ds\right],$$

where $z_0 = \lambda\phi$ for some $\lambda \in (0,1)$. Then

$$\begin{aligned}\|z(t)\| &\leq \|g(0,\phi(0))\| + \|g(t,\bar{z}_t + x_t)\| + \int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds\\ &\leq \|g(0,\phi)\| + c_1\|\bar{z}_t + x_t\|_B + c_2 + \int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds.\end{aligned}$$

Let $\alpha = M_b\|\phi\|_B + K_b\|\phi(0)\| + M_b\|\phi\|_B$. We have

$$\|\bar{z}_t + x_t\|_B \leq K_b \sup_{s\in[0,t]} \|z(s)\| + \alpha := w(t)$$

and

$$\|z(t)\| \leq \|g(0,\phi)\| + c_1 w(t) + c_2 + \int_0^t p(s)\psi(w(s))ds.$$

But

$$w(t) \leq K_b\|g(0,\phi)\| + c_1K_b w(t) + c_2K_b + K_b\int_0^t p(s)\psi(w(s))ds + \alpha$$

or

$$w(t) \leq \frac{1}{1-c_1K_b}\left[K_b\|g(0,\phi)\| + c_2K_b + \alpha + K_b\int_0^t p(s)\psi(w(s))ds\right],\ t\in[0,b].$$

Taking the right hand side as $\beta(t)$ we have

$$w(t) \leq \beta(t), \quad t\in[0,b],$$

$$\beta(0) := \bar{c} = \frac{1}{1-c_1K_b}\left[K_b\|g(0,\phi)\| + c_2K_b + \alpha\right],$$

and

$$\beta'(t) = \frac{K_b}{1-c_1K_b}p(t)\psi(w(t)) \leq \frac{K_b}{1-c_1K_b}p(t)\psi(\beta(t)), \quad t\in[0,b],$$

and

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq \frac{K_b}{1-c_1K_b}\int_0^b p(s)ds < \int_{\bar{c}}^{\infty}\frac{ds}{\psi(s)}.$$

Therefore, there exists a constant K_* such that $\beta(t) \leq K_*,\ t\in[0,b]$, and hence $\|\bar{z}_t + x_t\|_B \leq w(t) \leq K_*,\ t\in[0,b]$, and

$$\|z(t)\| \leq \|g(0,\phi)\| + c_1K_* + c_2\int_0^b p(s)\psi(K_*)ds := K_2.$$

Set

$$U = \{z\in C_0 : \sup\{\|z(t)\| : 0\leq t\leq b\} < K_2 + 1\}.$$

From the choice of U_1, there is no $z \in \partial U_2$ such that $z = \lambda P^*(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [10], we deduce that P^* has a fixed point z in U_2. Then N^* has a fixed point y which is a solution to problem (18)–(19). Denote this solution by y_0. Define the function

$$r_{k,1}(t) = \tau_k(y_0(t)) - t \quad \text{for} \quad t \geq 0.$$

$(H1)$ implies that

$$r_{k,1}(0) \neq 0 \ \ \text{for} \ \ k = 1, \ldots, m.$$

If

$$r_{k,1}(t) \neq 0 \ \ \text{on} \ \ [0,b] \ \ \text{for} \ \ k = 1, \ldots, m$$

i.e.,

$$t \neq \tau_k(y_0(t)) \ \text{on} \ \ [0,b] \ \ \text{and for} \ \ k = 1, \ldots, m,$$

then y_1 is a solution of the problem (4)-(6). It remains to consider the case when

$$r_{1,1}(t) = 0 \ \ \text{for some} \ \ t \in [0,b].$$

Now since

$$r_{1,1}(0) \neq 0$$

and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0, \quad \text{and} \ \ r_{1,1}(t) \neq 0 \ \ \text{for all} \ \ t \in [0, t_1).$$

Thus by $(H1)$ we have

$$r_{k,1}(t) \neq 0 \ \ \text{for all} \ \ t \in [0, t_1), \quad \text{and} \ \ k = 1, \ldots, m.$$

Step 2: Consider now the problem,

$$\frac{d}{dt}[y(t) - g(t, y_t)] = f(t, y_t), \ \ a.e. \ t \in [t_1, b], \tag{20}$$

$$y(t_1^+) = I_1(y_0(t_1)), \quad y(t) = y_0(t), \quad t \in (-\infty, t_1]. \tag{21}$$

Let $N_1^* : C_* \to C_*$ be defined by

$$N_1^*(y)(t) = \begin{cases} y_0(t), & t \in (-\infty, t_1], \\ \\ I_1(y_0(t_1)) - g(t_1, y_{0_{t_1}}) \\ +g(t, y_t) + \displaystyle\int_{t_1}^t f(s, y_s)ds, & t \in (t_1, b]. \end{cases}$$

In analogy to Theorem 3..1, we consider the operator $P_1^* : C_{t_1} \to C_{t_1}$ defined by

$$(P_1^* z)(t) = \ g(t_1, y_0) - g(t, \overline{z}_t + x_t) + \int_{t_1}^t f(s, \overline{z}_s + x_s)ds,$$

and there exists $\overline{M} > 0$ such that if z be a possible solutions of the integral equation

$$z(t) = \lambda \left[g(t_1, y_{0_{t_1}}) - g(t, \overline{z}_t + x_t) + \int_{t_1}^{t} f(s, \overline{z}_s + x_s) ds \right],$$

where $z_0 = \lambda y_0$ for some $\lambda \in (0,1)$, we have

$$\|z\|_\infty \leq \overline{M}.$$

Set

$$U_3 = \{z \in C_{t_1} : \sup\{\|z(t)\| : t_1 \leq t \leq b\} < \overline{M} + 1\}.$$

From the choice of U_3, there is no $z \in \partial U_3$ such that $z = \lambda P *_1 (z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [10], we deduce that P_1^* has a fixed point z in U_3. Then the problem (20)-(21) has at least one solution. Note this solution by y_1. Define

$$r_{k,2}(t) = \tau_k(y_1(t)) - t \quad \text{for } t \geq t_1.$$

If

$$r_{k,2}(t) \neq 0 \;\text{ on }\; (t_1, b] \;\text{ and for all }\; k = 1, \ldots, m$$

then

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_1(t), & \text{if } t \in (t_1, b], \end{cases}$$

is a solution of the problem (4)–(6). It remains to consider the case when

$$r_{2,2}(t) = 0, \;\text{ for some }\; t \in (t_1, b].$$

By $(B6)$ we have

$$\begin{array}{rcl} r_{2,2}(t_1^+) & = & \tau_2(y_1(t_1^+)) - t_1 \\ & = & \tau_2(I_1(y_0(t_1))) - t_1 \\ & > & \tau_1(y_0(t_1)) - t_1 \\ & = & r_{1,1}(t_1) = 0. \end{array}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{2,2}(t_2) = 0,$$

and

$$r_{2,2}(t) \neq 0 \;\text{ for all }\; t \in (t_1, t_2).$$

It is clear by $(H1)$ that

$$r_{k,2}(t) \neq 0 \;\text{ for all }\; t \in (t_1, t_2), \;\; k = 2, \ldots, m.$$

Suppose now that there is $\bar{s} \in (t_1, t_2]$ such that

$$r_{1,2}(\bar{s}) = 0.$$

Consider the function $L_1(t) = \tau_1(y_2(t) - g(t, y_{2_t})) - t$.

From $(B4) - (B6)$ it follows that

$$\begin{array}{rcl} L_1(\bar{s}) & = & \tau_1(y_1(\bar{s}) - g(\bar{s}, y_{1_{\bar{s}}})) - \bar{s} \\ & \geq & \tau_1(y_1(\bar{s})) - \bar{s} \\ & = & r_{1,2}(\bar{s}) = 0. \end{array}$$

Thus the function L_1 attains a nonnegative maximum at some point $s_1 \in (t_1, b]$. Since

$$\frac{d}{dt}[y_1(t) - g(t, y_{1_t})] = f(t, y_{1_t}),$$

then

$$L_1'(s_1) = \tau_1'(y_1(s_1) - g(s_1, y_{1_{s_1}}))\frac{d}{dt}[y_1(s_1) - g(s_1, y_{2_{s_1}})] - 1 = 0.$$

Therefore

$$\langle \tau_1'(y_1(s_1) - g(s_1, y_{1_{s_1}})), f(s_1, y_{1_{s_1}})\rangle = 1,$$

which contradicts $(B3)$.

Step 3: We continue this process and taking into account that $y_m := y\Big|_{[t_m,b]}$ is a solution to the problem

$$\frac{d}{dt}[y(t) - g(t, y_t)] = f(t, y_t), \ a.e.\ t \in (t_m, b], \tag{22}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m)), \ y(t) = y_{m-1}(t), \ \ t \in (-\infty, t_{m-1}]. \tag{23}$$

The solution y of the problem (4)-(6) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in (-\infty, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \dots & \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

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