International Journal of Applied Mathematical Sciences,
ISSN 0973-0176 Vol.2 No.1(2005), pp. 121-129

http://www.gbspublisher.com/ijams.htm

An Application of Higher Order Tangent Cones to Flow-Invariance

Elena Constantin
Department of Mathematics
University of Pittsburgh at Johnstown
Johnstown, PA 15904, USA
Email: constane@pitt.edu

Abstract

The goal of this paper is to give some necessary and sufficient conditions for the flow-invariance of a subset of a Banach space with respect to a high order autonomous differential equation.

AMS Subject Classification: 34G20.

Keywords and Phrases: Fréchet differentiability, locally Lipschitz functions, tangent vectors, flow-invariant sets.

1. Introduction

In this paper it is provided a characterization of the sets $S = G^{-1}(0) = \{x \in X;\, G(x) = 0\}$, that are flow-invariant with respect to the n-th order autonomous differential equation

$$u^{(n)}(t) = F(u(t)), t \geq 0, \tag{1}$$

where $G : U \to \mathbb{R}^m$, $m \geq 1$ is a n-times Fréchet differentiable mapping on an open subset U of a Banach space X, $n \geq 3$, and $F : U \to X$ is a locally Lipschitz mapping.

The invariant sets for the first order differential equations were studied by H. Brézis [1], M.G. Crandall [2], R.H. Martin, Jr. [4], N.H. Pavel and F. Iacob [11] and many other authors.

N.H. Pavel and C. Ursescu [13] treated the problem of flow-invariance of a set with respect to the second order differential equation $u''(t) = F(u(t))$, $t \geq 0$, using the theory of 'tangent sets'.

The purpose of this paper is to generalize their result Theorem 2.6, [13].

The flow-invariance with respect to equation (1) will be discussed in terms of n-th order tangent vectors defined by N.H. Pavel [10].

Definition 1. Let S be a nonempty subset of the normed space X and let $x \in S$. An element $y_n \in X$ is called a n-th order tangent vector to S at x if there exist some vectors $y_1, y_2, \ldots, y_{n-1}$ in X such that

$$\lim_{t\downarrow 0} \frac{1}{t^n} d(x + ty_1 + \frac{t^2}{2} y_2 + \frac{t^3}{3!} y_3 + \ldots + \frac{t^n}{n!} y_n; S) = 0. \tag{2}$$

The set of all n-th order tangent vectors to S at $x \in S$ is a cone denoted by $T_x^n S$.

One of our previous results will be used to prove the main theorems of this paper (Corollary 1.1, [3]). It gives a description of the n-th order tangent vectors to the set $S = G^{-1}(0) = \{z \in X; G(z) = 0\}$ at $x \in S$, where $G : X \to \mathbb{R}^m, m \geq 1$, has the properties specified below.

Theorem 1. Assume that $G : X \to \mathbb{R}^m$ is three times Fréchet differentiable at $x \in X$ with $G(x) = 0$, G is continuous near x, and $G'(x) : X \to \mathbb{R}^m$ is onto.

Then $y_3 \in T_x^3 S$ with associated vectors $y_i \in T_x^i S$, $i = 1, 2$ if and only if

$$G'(x)(y_1) = 0, G''(x)(y_1)(y_1) + G'(x)(y_2) = 0,$$

$$G'''(x)(y_1)(y_1)(y_1) + 3G''(x)(y_1)(y_2) + G'(x)(y_3) = 0.$$

Furthermore, if G is n times Fréchet differentiable at x, then $y_n \in T_x^n S$ with associated vectors $y_i \in T_x^i S$, $i = 1, \ldots, n-1$ if and only if

$$S_i^G(x; y_1, \ldots, y_i) = 0, \forall\, 1 \leq i \leq n,$$

where $S = G^{-1}(0), n \geq 3$.

Here, for any $i \geq 1$, $S_i^G(x, y_1, ..., y_i)$ denotes the expression

$$S_i^G(x, y_1, ..., y_i) = \sum_{k=1}^{i} \frac{i!}{k!} \Big[\sum_{\substack{i_1+\ldots+i_k=i \\ i_1,\ldots,i_k\in\{1,\ldots,i\}}} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(y_{i_1}) \ldots (y_{i_k}) \Big].$$

More precisely,

$$S_2^G(x, y_1, y_2) = G''(x)(y_1)(y_1) + G'(x)(y_2),$$

$$S_3^G(x, y_1, y_3, y_2) = G'''(x)(y_1)(y_1)(y_1) + 3G''(x)(y_1)(y_2) + G'(x)(y_3).$$

We recall the definition of a flow-invariant set with respect to the autonomous n-th order differential equation (1) (Definition 1.9, [5]).

Definition 2. The nonempty set $S \subset U$ is said to be (right-hand) flow-invariant with respect to the n-th order differential equation (1) if the solution $u : [0, T) \to X$ to the Cauchy problem (1) determined by the initial conditions

$$u(0) = x, u'(0) = y_1, \ldots, u^{(n-1)}(0) = y_{n-1}, \tag{3}$$

with $x \in S$, $y_1 \in T_x S, \ldots, y_{n-1} \in T_x^{n-1} S$, $F(x) \in T_x^n S$ having correspondent vectors $y_1, \ldots, y_{n-1}$, satisfies

$$u(t) \in S, \forall\, t \geq 0, t \in \mathrm{dom}\, u. \tag{4}$$

The constraints imposed to $(x, y_1, \ldots, y_{n-1})$ are necessary conditions to have the invariance property (4).

In [13], N.H. Pavel and C. Ursescu introduced the set

$$M_S^{(n)} = \{(x, y_1, \ldots, y_{n-1}) \in S \times X^{n-1} : y_i \in T_x^i S, i = 1, \ldots, n-1,$$

$$F(x) \in T_x^n S \text{ whose associated vectors are } y_1, \ldots, y_{n-1}\}, n \geq 2, \tag{5}$$

and they expressed the choice in (3) for the initial conditions by means of (5) as follows

$$(u(0), u'(0), \ldots, u^{(n-1)}(0)) = (x, y_1, \ldots, y_{n-1}) \in M_S^{(n)}.$$

This was justified by the following result.

Proposition 1. If $u : [0, T) \to X$ is a solution of (1) which satisfies the invariance property (4), then one has

$$(u(t), u'(t), \ldots, u^{(n-1)}(t)) \in M_S^{(n)}, \forall\, t \in [0, T). \tag{6}$$

Also, N.H. Pavel and C. Ursescu reduced the problem of invariant sets for (1) to a similar problem for a first order differential equation, fact that allowed them to utilize a theorem proved by M. Nagumo [6] and, independently, by H. Brézis [1], in order to obtain a characterization of flow-invariant sets $S \subset U$ with respect to the n-th order differential equation (1), which we will use for deriving the main results of this paper (see [5] and [13]) .

Theorem 2. Assume that $M_S^{(n)}$ is a nonempty closed subset of $U \times X^{n-1}$, for a closed subset S of U, $n \geq 2$. Then $S \subset U$ is a flow-invariant set with respect with the n-th order differential equation (1) if and only if $(y_1, \ldots, y_{n-1}, F(x)) \in X^n$ is a tangent vector to $M_S^{(n)}$ at $(x, y_1, \ldots, y_{n-1})$, for any $(x, y_1, \ldots, y_{n-1}) \in M_S^n$, i.e.,

$$\lim_{t \downarrow 0} t^{-1} d((x, y_1, \ldots, y_{n-1}) + t(y_1, \ldots, y_{n-1}, F(x)); M_S^{(n)}) = 0.$$

2. Main Results

In this section we determine $M_S^{(n)}$, $n \geq 3$, when $S = G^{-1}(0)$ for a mapping $G : U \to \mathbb{R}^m, m \geq 1$ and we provide an explicit description for those sets of this form that are flow-invariant with respect to (1).

First we analyze the case $n = 3$. We mention that the case $n = 2$ has been considered in [13].

Theorem 3. Assume that $G : U \to \mathbb{R}^m$, $m \geq 1$ is three times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^m$ is onto for each $x \in S = G^{-1}(0)$. Then $M_S^{(3)}$ is given by

$$M_S^{(3)} = \{(x, y_1, y_2) \in U \times X \times X : G(x) = 0, G'(x)(y_1) = 0,$$
$$G''(x)(y_1)(y_1) + G'(x)(y_2) = 0,$$
$$G'''(x)(y_1)(y_1)(y_1) + 3G''(x)(y_1)(y_2) + G'(x)(F(x)) = 0\}. \tag{7}$$

Suppose further that G is four times Fréchet differentiable on U, the function $h : U \to \mathbb{R}^m$ given by

$$h(x) = G'(x)(F(x)), \forall\, x \in U, \tag{8}$$

is Fréchet differentiable, $M_S^{(3)}$ is nonempty and the mapping $(G'(x)(\cdot), G''(x)(y_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m$ is onto for every $(x, y_1, y_2) \in M_S^{(3)}$.

Then $S = G^{-1}(0)$ is flow-invariant with respect to the differential equation $u'''(t) = F(u(t)), t \geq 0$ if and only if

$$G^{(4)}(x)(y_1)(y_1)(y_1)(y_1) + 6G'''(x)(y_1)(y_1)(y_2) +$$
$$+3G''(x)(y_2)(y_2) + 3G''(x)(y_1)(F(x)) + h'(x)(y_1) = 0. \tag{9}$$

Proof. Formula (7) follows directly from Theorem 1.

To prove the second part, we notice that, due to (7), the set $M_S^{(3)}$ can be rewritten as

$$M_S^{(3)} = g^{-1}(0),$$

where $g : U \times X \times X \to \mathbb{R}^{4m}$ is defined by

$$g(x, y_1, y_2) = (G(x), G'(x)(y_1), G''(x)(y_1)(y_1) + G'(x)(y_2),$$
$$G'''(x)(y_1)(y_1)(y_1) + 3G''(x)(y_1)(y_2) + G'(x)(F(x))),$$

for all $(x, y_1, y_2) \in U \times X \times X$.

It can be easily seen that under the hypothesis of the theorem, g is Fréchet differentiable and its Fréchet derivative is determined by the relation

$$g'(x, y_1, y_2)(u, v_1, v_2) = (G'(x)(u), G''(x)(u)(y_1) + G'(x)(v_1),$$
$$G'''(x)(u)(y_1)(y_1) + 2G''(x)(y_1)(v_1) + G''(x)(u)(y_2) + G'(x)(v_2),$$
$$G^{(4)}(x)(u)(y_1)(y_1)(y_1) + 3G'''(x)(y_1)(y_1)(v_1) + 3G'''(x)(u)(y_1)(y_2) +$$
$$+3G''(x)(v_1)(y_2) + 3G''(x)(y_1)(v_2) + h'(x)(u)).$$

We now show that $g'(x, y_1, y_2) : X \times X \times X \to \mathbb{R}^{4m}$ is onto for each $(x, y_1, y_2) \in M_S^{(3)}$, i.e., the equation $g'(x, y_1, y_2)(u, v_1, v_2) = (z_1, z_2, z_3, z_4)$ has a solution $(u, v_1, v_2) \in X \times X \times X$ for any $(z_1, z_2, z_3, z_4) \in \mathbb{R}^{4m}$. Since $G'(x)$ is onto, there are $u \in X$ and $v_1 \in X$ such that $G'(x)(u) = z_1$ and $G''(x)(u)(y_1) + G'(x)(v_1) = z_2$. Then the element $v_2 \in X$ can be obtained using the fact that the mapping $(G'(x), G''(x)(y_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m$ is onto.

Thus, $T_{(x,y_1,y_2)}M_S^{(3)} = g'(x, y_1, y_2)^{-1}(0)$.

Finally, Theorem 2 completes the proof.

Theorem 3 has a geometrical interpretation: suppose a mass particle is launched from the initial point x in a set $S = G^{-1}(0)$ with the initial velocity y_1 and the initial acceleration y_2, and its trajectory satisfies the equation $u'''(t) = F(u(t)), t \geq 0$. If our sufficient condition for flow-invariance (9) is verified, then the trajectory of the particle remains in the set $S = G^{-1}(0)$.

The following result represents a generalization of the previous theorem for the case of equation (1).

Theorem 4. Assume that $G : U \to \mathbb{R}^m$ is n times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^n$ is onto for each $x \in S = G^{-1}(0)$. Then $M_S^{(n)}$ is given by

$$M_S^{(n)} = \Big\{(x, y_1, ..., y_{n-1}) \in U \times X^{n-1} : G(x) = 0,$$

$$S_j^G(x, y_1, ..., y_j) = 0, 1 \leq j \leq n-1,$$

$$G'(x)(F(x)) + \sum_{k=2}^{n} \frac{n!}{k!}\Big[\sum_{\substack{i_1+...+i_k=n \\ i_1,...,i_k \in \{1,...,n-1\}}} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(y_{i_1})\dots(y_{i_k})\Big] = 0\Big\}. \quad (10)$$

Suppose further that G is $n+1$ times Fréchet differentiable on U, the function $h : U \to \mathbb{R}^m$ given by

$$h(x) = G'(x)(F(x)),\ \forall\, x \in U,$$

is Fréchet differentiable, $M_S^{(n)}$ is nonempty and the mapping
$(G'(x)(\cdot), G''(x)(y_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m$ is onto for any $(x, y_1, \dots y_{n-1}) \in M_S^{(n)}$.

Then $S = G^{-1}(0)$ is flow-invariant with respect to the differential equation $u^{(n)}(t) = F(u(t)), t \geq 0$ if and only if

$$h'(x)(y_1) + \sum_{k=3}^{n} \frac{n!}{k!}\Big\{\sum_{\substack{i_1+...+i_k=n \\ i_1,...,i_k \in \{1,...,n-2\}}} \frac{1}{i_1! \cdots i_k!}\Big[G^{(k+1)}(x)(y_1)(y_{i_1})\dots(y_{i_k}) +$$

$$+G^{(k)}(x)(y_{i_1+1})(y_{i_2})\dots(y_{i_k}) + \dots + G^{(k)}(x)(y_{i_1})\dots(y_{i_{k-1}})(y_{i_k+1})\Big]\Big\} +$$

$$+nG'''(x)(y_1)(y_1)(y_{n-1}) + nG''(x)(y_2)(y_{n-1}) + nG''(x)(y_1)(F(x)) = 0, \quad (11)$$

for any $(x, y_1, ..., y_{n-1}) \in M_S^{(n)}$ or, equivalently

$$h^{'}(x)(y_1) + \sum_{k=2}^{n} \frac{n!}{k!}\{ \sum_{\substack{i_1,...,i_k \in \{1,...,n-1\} \\ i_1+...+i_k=n}} \frac{1}{i_1! i_2! \dots i_k!} [G^{(k+1)}(x)(y_1)(y_{i_1}) \cdots (y_{i_k}) +$$

$$+ G^{(k)}(x)(y_{i_1+1})(y_{i_2}) \cdots (y_{i_k}) + \dots + G^{(k)}(x)(y_{i_1}) \cdots (y_{i_{k-1}})(y_{i_k+1})]\} = 0, \quad (11')$$

for any $(x, y_1, ..., y_{n-1}) \in M_S^{(n)}$, $y_n = F(x)$.

Here, $S_j^G(x, y_1, ..., y_j)$, $j \geq 1$ denotes the expression

$$S_j^G(x, y_1, ..., y_j) = \sum_{k=1}^{j} \frac{j!}{k!} \Big[\sum_{\substack{i_1+...+i_k=j \\ i_1,...,i_k \in \{1,...,j\}}} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(y_{i_1}) \dots (y_{i_k}) \Big].$$

Proof. This result can be proved analogously to the previous theorem. We have $M_S^{(n)} = g^{-1}(0)$, where $g : U \times X^{n-1} \to \mathbb{R}^{n+1}$ is equal to

$$g(x, y_1, ..., y_{n-1}) = \Big(G(x), S_1^G(x, y_1), \dots, S_{n-1}^G(x, y_1, ..., y_{n-1}),$$

$$G'(x)(F(x)) + \sum_{k=2}^{n} \frac{n!}{k!} \Big[\sum_{\substack{i_1+...+i_k=n \\ i_1,...,i_k \in \{1,...,n-1\}}} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(y_{i_1}) \dots (y_{i_k}) \Big] \Big),$$

it is Fréchet differentiable and its first Fréchet derivative is

$$g'(x, y_1, ..., y_{n-1})(v_1, \dots, v_n) = \Big(G'(x)(v_1), [S_1^G]'(x, y_1)(v_1, v_2), \dots,$$

$$[S_{n-1}^G]'(x, y_1, ..., y_{n-1})(v_1, \dots, v_n),$$

$$h'(x)(v_1) + \sum_{k=2}^{n} \frac{n!}{k!} \Big\{ \sum_{\substack{i_1+...+i_k=n \\ i_1,...,i_k \in \{1,...,n-1\}}} \frac{1}{i_1! \cdots i_k!} \Big[G^{(k+1)}(x)(v_1)(y_{i_1}) \dots (y_{i_k}) +$$

$$+ G^{(k)}(x)(v_{i_1+1})(y_{i_2}) \dots (y_{i_k}) + \dots + G^{(k)}(x)(y_{i_1}) \dots (y_{i_{k-1}})(v_{i_k+1}) \Big] \Big\} \Big),$$

with $[S_j^G]'(x, y_1, ..., y_j)(v_1, \dots, v_{j+1})$, $1 \leq j \leq n-1$ as below

$$[S_j^G]'(x, y_1, ..., y_j)(v_1, \dots, v_{j+1}) =$$

$$\sum_{k=1}^{j} \frac{j!}{k!} \Big\{ \sum_{\substack{i_1+...+i_k=j \\ i_1,...,i_k \in \{1,...,j\}}} \frac{1}{i_1! \cdots i_k!} \Big[G^{(k+1)}(x)(v_1)(y_{i_1}) \dots (y_{i_k}) +$$

$$+G^{(k)}(x)(v_{i_1+1})(y_{i_2})\dots(y_{i_k})+\dots+G^{(k)}(x)(y_{i_1})\dots(y_{i_{k-1}})(v_{i_k+1})\Big]\Big\}.$$

Taking into account that the mapping $(G'(x)(\cdot), G''(x)(y_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m$ is onto for every $(x, y_1, \dots, y_n) \in M_S^{(n)}$, it can be seen that $g'(x, y_1, ..., y_{n-1})$ is onto as well, since the derivative of the last component of g in the direction $(v_1, \dots, v_n)$, can be written more explicitly as

$$h'(x)(v_1)+\sum_{k=3}^{n}\frac{n!}{k!}\Big\{\sum_{\substack{i_1+\dots+i_k=n\\ i_1,\dots,i_k\in\{1,\dots,n-2\}}}\frac{1}{i_1!\cdots i_k!}\Big[G^{(k+1)}(x)(v_1)(y_{i_1})\dots(y_{i_k})+$$

$$+G^{(k)}(x)(v_{i_1+1})(y_{i_2})\dots(y_{i_k})+\dots+G^{(k)}(x)(y_{i_1})\dots(y_{i_{k-1}})(v_{i_k+1})\Big]\Big\}+$$

$$+nG'''(x)(v_1)(y_1)(y_{n-1})+nG''(x)(v_2)(y_{n-1})+nG''(x)(y_1)(v_n),$$

and

$$[S_{n-1}^G]'(x, y_1, \dots, y_{n-1})(v_1, \dots, v_n) =$$

$$=\sum_{k=2}^{n-1}\frac{(n-1)!}{k!}\Big\{\sum_{\substack{i_1+\dots+i_k=n-1\\ i_1,\dots,i_k\in\{1,\dots,n-1\}}}\frac{1}{i_1!\cdots i_k!}\Big[G^{(k+1)}(x)(v_1)(y_{i_1})\dots(y_{i_k})+$$

$$+G^{(k)}(x)(v_{i_1+1})(y_{i_2})\dots(y_{i_k})+\dots+G^{(k)}(x)(y_{i_1})\dots(y_{i_{k-1}})(v_{i_k+1})\Big]\Big\}+$$

$$+G''(x)(v_1)(y_{n-1})+G'(x)(v_n).$$

We deduce $T_{(x,y_1,\dots,y_{n-1})}M_S^{(n)} = g'(x, y_1, \dots, y_{n-1})^{-1}(0)$ and by Theorem 2 we obtain (11).

Corollary 1. Let H be a real Hilbert space of inner product $<,>$ and norm $\|\cdot\|$.

Then, in the case of the sphere $S(r) = \{x \in H, \|x\| = r\}$, $r > 0$, the sets given by (7) and (10) become respectively

$$M_{S(r)}^{(3)} = \big\{(x, y_1, y_2) \in U \times H \times H, \|x\| = r, < x, y_1 >= 0,$$

$$\|y_1\|^2+ < x, y_2 >= 0, < x, F(x) > +3 < y_1, y_2 >= 0\big\}, \tag{12}$$

and

$$M_{S(r)}^{(n)} = \big\{(x, y_1, \dots, y_{n-1}) \in U \times H^{n-1}, \|x\| = r, < x, y_1 >= 0,$$

$$< x, y_j > +\frac{1}{2}\sum_{k=1}^{j-1}\binom{j}{k} < y_k, y_{j-k} >= 0, 2 \le j \le n-1,$$

$$< x, F(x) > +\frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k} < y_k, y_{n-k} >= 0\big\},\ n \ge 3. \tag{13}$$

Proof. In this case $S(r) = G^{-1}(0)$, with $G(x) = \frac{1}{2}(\|x\|^2 - r^2), \forall\, x \in H$. Then, from (7) and (10) we obtain (12) and (13), respectively, since

$$G'(x)(y) =< x, y >, \ \forall\, y \in H,$$

$$G''(x)(y)(v) =< y, v >, \ \forall\, y, v \in H,$$

and $G^{(k)}(x)$, $k \geq 3$ are identically zero.

Corollary 2. Let $U \subset H$ be an open subset of the Hilbert space H, with $S(r) \subset U$. Assume that $F : U \to H$ is locally Lipschitz and the mapping $h(x) =< x, F(x) >$ is Fréchet differentiable on U.

Then $S(r)$ is a flow-invariant set with respect to the equation $u'''(t) = F(u(t))$, $t \geq 0$, if and only if

$$3\|y_2\|^2 + 3 < y_1, F(x) > +h'(x)(y_1) = 0, \forall\, (x, y_1, y_2) \in M^{(3)}_{S(r)}, \tag{14}$$

and $S(r)$ is a flow-invariant set for (1) if and only if

$$n < y_2, y_{n-1} > +n < y_1, F(x) > +h'(x)(y_1) = 0, \forall\, (x, y_1, \ldots, y_{n-1}) \in M^{(n)}_{S(r)}. \tag{15}$$

Proof. These results follow straight forward from Theorems 3 and 4, taking into account the explicit form of the Fréchet derivatives of $G(x) = \frac{1}{2}(\|x\|^2 - r^2), \forall\, x \in H$, and the fact that $(G'(x)(\cdot), G''(x)(y_1)(\cdot)) : H \to \mathbb{R} \times \mathbb{R}$ is onto for every $(x, y_1, \ldots y_{n-1}) \in M^{(n)}_{S(r)}$. Indeed, for any $(v_1, v_2) \in \mathbb{R} \times \mathbb{R}$, we can find $u = \frac{xv_1}{r^2} + \frac{y_1 v_2}{\|y_1\|^2}$ such that $< x, u >= v_1$ and $< y_1, u >= v_2$, as $\|x\| = r$ and $< x, y_1 >= 0$.

References

[1] H. Brézis, On a characterization of flow-invariant sets, *Communs Pure Appl. Math.*, **23**, 1970, pp. 261-263.

[2] M.G. Crandall, A generalization of Peano's existence theorem and flow-invariance, *Proc. Am. Math. Soc.*, **36**, 1972, pp. 151-155.

[3] E. Constantin, Higher order necessary and sufficient conditions for optimality, *PanAmerican Math. J.*, **14**, 3, 2004, pp. 1-25.

[4] R. H. Martin, Jr., Differential equations on closed subsets of a Banach space, *Trans. Am. Math. Soc.*, **179**, 1973, pp. 399-414.

[5] D. Motreanu, N.H. Pavel, *Tangency, flow invariance for differential equations, and optimization problems*, Marcel Dekker, Inc., New York, 1999.

[6] M. Nagumo, Überdie Lage der Integralkurven gevöhnlicherö Differentialgleichungen, *Proc. Phys. Math. Soc. Japan*, **24**, 1942, pp. 551-559.

[7] N.H. Pavel, Invariant sets for a class of semilinear equations of evolution, *Nonlin. Anal. TMA,* **1**, 1977, 187-196.

[8] N.H. Pavel, Second order differential equation on closed subsets of a Banach space, *Boll. Un. Mat. Ital.*, **12**, 1975, pp. 348-353.

[9] N.H. Pavel, Differential Equations, flow-invariance and applications, *Pitman Res. Notes Math.*, **113**, Longman, London, 1984.

[10] N.H. Pavel, J.K. Huang, J.K. Kim, Higher order necessary conditions for optimization, *Libertas Math.*, **14**, 1994, pp. 41-50.

[11] N.H. Pavel, F. Iacob, Invariant sets for a class of perturbed differential equations of retarted type, *Israel J. Math.*, **28**, 1977, pp. 254-264.

[12] N.H. Pavel, C. Ursescu, *Flow-invariance for higher order differential equations*, Analele Ştiinţifice ale Universităţii 'Al.I. Cuza', Iaşi, **24**, 1978, pp. 91-100.

[13] N.H. Pavel, C. Ursescu, Flow-invariant sets for autonomous second order differential equations and applications in mechanics, *Nonlinear Anal.*, **6**, 1982, pp. 35-77.