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Squeezed-State Eigenfunctions of the Schrödinger Equation and an Effective Hamiltonian

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Abstract

Quantum mechanical squeezing is shown to arise from confining particles to a finite region as described by the infinite square well and the bounded harmonic oscillator. The measure of squeezing is dependent on the width of the boundary. These confined states are expanded in terms of the ordinary harmonic oscillator number states. The resulting expansion coefficients are used to compute the squeezing parameter and variances in momentum and position. This information is used in the creation/annihilation representation of the system to develop an effective Hamiltonian for the bounded harmonic oscillator. This effective Hamiltonian contains boundary information. The ground-state energy of the effective Hamiltonian agrees well with the exact energy as computed from the solution of the Schrödinger equation for all boundary lengths.

Keywords: Eigenfunction, Schrödinger equation, quantum mechanics, squeezed states
1991 MSC: 34L40, 35Q40, 81R30, 81V80

1. Introduction

Quantum phenomena such as the Casimir effect, Aharonov-Bohm Effect, QCD (quark and gluon) confinement, and squeezing arise from constraints due to the geometry of the system. Such systems are highly dependent on the geometry of the boundaries and have varying properties depending on the boundary conditions.

Squeezed states arise in many areas of physics, such as quantum optics [21], solid-state physics [8], and cosmology [9]. Recent theoretical investigations examine applications for trapped ions [10] and multiphoton states [24]. Squeezed states have also been created experimentally through nonlinear optical processes such as four-wave mixing [18] and parametric down conversion [22]. The engineering of arbitrary squeezed states in cavities is currently under investigation [3].

Squeezed states are generated by many mechanisms, most of which involve quad-
ratic Hamiltonians. Ma, et. al. [11] have shown that squeezing can arise from a linear optical system involving a harmonic oscillator with time-dependent frequency. Blencowe [2] develops similar results in an investigation of the mechanical vibrations of a submicron cantilever.

Since many systems are modeled using harmonic oscillators, this paper focuses on squeezing in such systems. In all of its simplicity the harmonic oscillator (HO) is an idealized model of a single mode of an oscillating system. It successfully describes physical phenomena such as lattice vibrations (phonons) and electromagnetic field oscillations (photons). The bounded harmonic oscillator is equivalent to a harmonic oscillator in an infinite square well potential. This problem has been studied in the past as a model for nuclear binding [17] and as a more realistic model for solids [20]. While previous analyses [6, 19] have considered the convergence of the bounded harmonic oscillator system to the ordinary harmonic oscillator as the boundary width increases to infinity, *squeezing of position and momentum due to confinement* has not been considered. Further, an interesting property of the bounded harmonic oscillator is that the parity of the wave functions is dependent on the boundary. The bounded harmonic oscillator on the interval $(-l, l)$ has wavefunctions with even parity, while the wave functions for the interval $(0, l)$ have odd parity. Parity has been observed by others [6, 19], but its role in squeezing has not been previously recognized.

In this work we show that squeezed states arise from confinement. Confinement can be as simple as an atom confined by its neighboring atoms in a solid or as complicated as a vacuum where an infinite number of particle/anti-particle pairs are coming in and out of existence between two perfectly reflecting walls as in the Casimir effect. The wavefunctions defined on a finite interval have either even or odd parity. We expand these wavefunctions in terms of ordinary harmonic oscillator number states. The parity of the wavefunctions will determine the expansion coefficients. The noise properties of these states are investigated by looking at the variances in position and momentum. We investigate these properties as the width of the confinement approaches infinity and as the width decreases to zero. As examples the infinite square well (ISW) and bounded harmonic oscillator (BHO) are considered.

Although the present analysis is restricted to fixed boundaries we recognize that squeezing has also been shown to arise from time-dependent boundary conditions [1, 13, 15, 25]. Pereshogin, et. al. [13] has shown through a fiber bundle treatment of Schrödinger's equation that at each point in time the system has a distinct Hilbert space. The time derivatives associated with different points in time become connections, connecting the states in the Hilbert spaces at different points in time. The time-derivative is modified by the time-dependent boundary conditions in such a way that there is an effective Hamiltonian. Using this effective Hamiltonian formalism, Zou et. al. [25] have discussed squeezing for a particle in an infinite square well with a moving wall. It was shown that a particle initially in a coherent state evolves into a squeezed state.

There has also been interest in squeezing arising from the dynamic Casimir effect between two vibrating mirrors [4, 5, 23]. In these studies the properties of resulting photons are shown to exhibit squeezing for certain cases. Hence, squeezing due to confined boundaries is relevant in a variety of applications.

As an application of our work we create an effective Hamiltonian for the BHO using the results developed by our squeezing analysis. This Hamiltonian can be written in terms of creation/annihilation operators that create and destroy quanta in the infinite domain. With this, bounded regions can be studied in terms of operators in the infinite domain.

2. Conditions for Squeezing on an Infinite Domain

In this section the features of squeezing for a harmonic oscillator on an infinite domain (HO) are reviewed [7, 12]. These features will serve as an indicator for squeezing on a finite domain. Squeezing is a very general effect that in the simplest case arises from Hamiltonians quadratic in position, x, and momentum, p, or equivalently a and $a^\dagger$, the creation/annihilation operators. However, such Hamiltonians do not guarantee squeezing. Squeezing depends on the value of the parameters in the Hamiltonian and can also depend on the initial state of the system. Squeezing arises from the terms containing two-particle creation/annihilation operators. The fluctuations in these two particles can interfere constructively and destructively so that fluctuations increase or decrease.

A generalized time-independent, (symmetric) quadratic Hamiltonian is given by,

$$H = \kappa p^2 + \chi(xp + px) + \eta x^2, \tag{1}$$

where x and p satisfy $[x, p] = i\hbar$. The HO Hamiltonian is retrieved for $\kappa = 1/(2m)$, $\chi = 0$ and $\eta = m\omega^2/2$. The ISW Hamiltonian is retrieved for $\kappa = 1/(2m)$ and $\chi = \eta = 0$. The BHO Hamiltonian is retrieved for $\kappa = 1/(2m)$, $\chi = 0$ and $\eta = m\omega^2/2$. In terms of creation/annihilation operators satisfying $[a, a^\dagger] = 1$ and $[a, a] = [a^\dagger, a^\dagger] = 0$, the generalized time-independent, (symmetric) quadratic Hamiltonian is

$$H = \frac{1}{2}\hbar\tilde{\omega}(a^\dagger a + aa^\dagger) + \gamma^* a^2 + \gamma(a^\dagger)^2. \tag{2}$$

The coefficients in Eq. (1) are related to those in Eq. (2) by $x/x_0 = (a^\dagger + a)/\sqrt{2}$ and $x_0 p/\hbar = i(a^\dagger - a)/\sqrt{2}$, where $x_0 = \sqrt{\hbar/(m\omega)}$ is the oscillation amplitude of the vacuum state setting the characteristic length scale of the system. Further $\hbar\tilde{\omega} = x_0^2\eta + 2\hbar x_0^{-2}\kappa$ and $\gamma = \frac{1}{2}x_0^2\eta - \frac{1}{2}\hbar^2 x_0^{-2}\kappa + i\hbar\chi$, where $x_0^2 = \hbar/(m\omega)$ is defined for the mass m and frequency ω of the infinite domain or ordinary harmonic oscillator. Using these relations we determine the Hamiltonians for the ISW and BHO in terms of the ordinary harmonic oscillator creation/annihilation operators. These are

$$H_{\text{ISW}} = \frac{p^2}{2m} \equiv \frac{\hbar^2}{2mx_0^2}(a^\dagger a + \frac{1}{2}) - \frac{\hbar}{4mx_0^2}\left(a^2 + (a^\dagger)^2\right) \tag{3}$$

$$\begin{aligned} H_{\text{BHO}} &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \\ &\equiv \left(\frac{1}{2}m\omega^2 x_0^2 + \frac{\hbar^2}{2mx_0^2}\right)(a^\dagger a + \frac{1}{2}) \\ &+ \frac{1}{2}\left(\frac{1}{2}m\omega^2 x_0^2 - \frac{\hbar^2}{2mx_0^2}\right)\left(a^2 + (a^\dagger)^2\right). \end{aligned} \tag{4}$$

These Hamiltonians are similar to the Hamiltonian in Eq. (2). All of these Hamiltonians have the characteristic two-particle terms, which can give rise to squeezing. Since the ISW and the HO model many systems, these simple examples demonstrate the prominence of squeezing.

The condition in which the Hamiltonian given by Eq. (2) generates squeezed states comes from the transformation that diagonalizes it. It is well known that such Hamiltonians can be diagonalized by the Bogoliubov transformation,

$$S^\dagger(z)aS(z) = \mu a - \nu a^\dagger \tag{5}$$

$$S^\dagger(z)a^\dagger S(z) = \mu a^\dagger - \nu^* a, \tag{6}$$

where μ and ν are coefficients that satisfy $|\mu|^2 - |\nu|^2 = 1$ in order to preserve the canonical commutation relations. The unitary operator that generates the Bogoliubov transformation is, $S(z) = \exp\left[\frac{1}{2}\left(z^* a^2 - z(a^\dagger)^2\right)\right]$, where $z = re^{i\theta}$ and $*$ denotes the conjugate. Here r is called the squeezing parameter, and θ is the squeezing angle. Using this notation and Eq. (5) and (6), one can show that

$$\mu = \cosh(r) \quad \text{and} \quad \nu = e^{i\theta}\sinh(r). \tag{7}$$

Applying the unitary operator to Eq. (2) we obtain the transformed Hamiltonian, $H \rightarrow H'$,

$$\begin{aligned} H' &= S^\dagger(z) H S(z) = \left[\frac{1}{2}\hbar\tilde{\omega}\left(|\mu|^2 + |\nu|^2\right) - 2\mu\Re(\nu\gamma^*)\right](a^\dagger a + a a^\dagger). \\ &+ [\mu^2\gamma^* + (\nu^*)^2\gamma - \hbar\tilde{\omega}\mu\nu^*]a^2 + [\mu^2\gamma + \nu^2\gamma^* - \hbar\tilde{\omega}\mu\nu](a^\dagger)^2 \end{aligned} \tag{8}$$

which is diagonal for,

$$\mu^2\gamma^* + (\nu^*)^2\gamma - \hbar\tilde{\omega}\mu\nu^* \quad \text{and} \quad \mu^2\gamma + \nu^2\gamma^* - \hbar\tilde{\omega}\mu\nu = 0. \tag{9}$$

The corresponding transformed, energy eigenvalues are then,

$$E'_n = \left[\hbar\tilde{\omega}\left(|\mu|^2 + |\nu|^2\right) - 4\mu\Re(\nu\gamma^*)\right] n + \frac{1}{2}\hbar\tilde{\omega}\left(|\mu|^2 + |\nu|^2\right) - 2\mu\Re(\nu\gamma^*). \tag{10}$$

Solving for μ and ν in Eq. (9) and using Eq. (7), one finds

$$\theta = \pm i \ln\left(\text{sgn }(\gamma)\right) \tag{11}$$

$$r = \frac{1}{2}\coth^{-1}\left(\frac{2|\gamma|}{\hbar\omega}\right) \tag{12}$$

$$z = \frac{1}{2}\text{sgn}(\gamma)\coth^{-1}\left(\frac{2|\gamma|}{\hbar\omega}\right). \tag{13}$$

Here, $\text{sgn}(x) = x/|x|$, is the sign function. By definition, no squeezing corresponds to $r = 0$. Hence $\gamma = 0$ and the two-particle terms in Eq. (2) vanish. In this case the HO Hamiltonian is recovered.

The fluctuation in two conjugate variables such as momentum and position is an intrinsic property of quantum systems and is bounded by the uncertainty principle. The uncertainty product determines the amount of fluctuation. However, this gives no limit on the amount of fluctuation in the individual variables. Squeezing is a measure of the reduction of fluctuations in one conjugate variable at the expense of the other. In order to determine the degree of squeezing in position and momentum, non-dimensional position and momentum operators are defined by, $X = x/x_0$ and $P = x_0 p/\hbar$, where $x_0 = \sqrt{\hbar/(m\omega)}$ is the oscillation amplitude of the vacuum state. The vacuum is the state of lowest energy

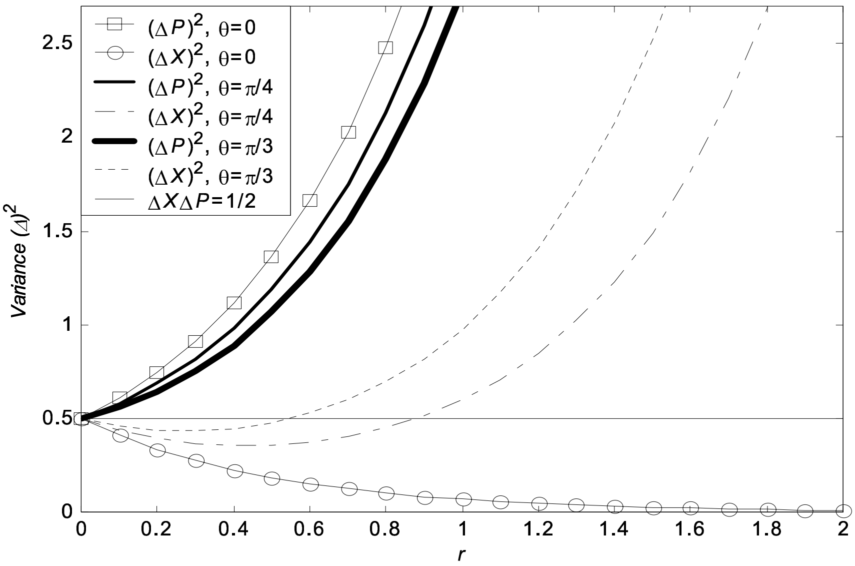

Figure 1: Variance in position and momentum for a squeezed vacuum vs. the squeezing parameter r . Different graphs correspond to different phases θ.

and will therefore be the state with the smallest uncertainty. The degree of fluctuation in a measurement of an observable A is characterized by the variance $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$. The HO squeezed vacuum is found by applying the squeezing operator to the lowest harmonic oscillator number state, $|0\rangle_S = S^\dagger(z)\,|0\,\rangle$. The variances in X and P for the HO in a squeezed vacuum are,

$$(\Delta X)^2 = \frac{1}{2}\left(\cosh(2r) + \cos(\theta)\sinh(2r)\right), \tag{14}$$

$$(\Delta P)^2 = \frac{1}{2}\left(\cosh(2r) - \cos(\theta)\sinh(2r)\right). \tag{15}$$

The plots of Eqs. (14) and (15) in Fig. 1 show how the variances in position and momentum vary with r for different values of θ. The variances in position can be less than, or greater than 1/2. The uncertainty principle is not violated since the variance in position is decreased at the expense of the momentum. Clearly the standard quantum limit determined by the uncertainty product, $\Delta X \Delta P \geq 1/2$ is satisfied. If either of these values is less than 1/2, then there is squeezing. Reduction of the fluctuations in one of the conjugate variables results in the increase of fluctuations in the other.

3. Squeezing from Confinement via Boundary Conditions

The wave functions for the ISW and the BHO can be found by solving Schrödinger's equation with the appropriate boundary conditions. In both cases the parity of the wavefunction depends on the interval. Only two types of boundaries, $(0,l)$ and $(-l,l)$, need to be considered, since any arbitrary boundary (a,b) can be transformed into one or the other.

3..1 Infinite Square Well

In order to show that an ISW generates squeezed states we solve Schrödinger's equation directly. Schrödinger's equation in one-dimension for a particle in an ISW on an interval (a,b) is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x), \quad \psi(a) = \psi(b) = 0, \tag{16}$$

where the interval (a,b) is either $(0,l)$ or $(-l,l)$. The wave functions and energy eigenvalues for $x \in (0,l)$ and $x \in (-l,l)$ are given by,

$$\psi_m^{(0,l)}(x) = \sqrt{\frac{2}{l}}\sin\left(\frac{m\pi}{l}x\right), \quad \psi_m^{(-l,l)}(x) = \sqrt{\frac{1}{l}}\cos\left(\frac{(2m+1)\pi}{2l}x\right) \tag{17}$$

$$E_m^{(0,l)} = \frac{\pi^2\hbar^2 m^2}{2ml^2}, \quad E_m^{(-l,l)} = \frac{\pi^2\hbar^2 m^2}{8ml^2}. \tag{18}$$

3..2 Bounded Harmonic Oscillator

In order to show that a harmonic oscillator confined to a finite region generates squeezed states, we solve Schrödinger's equation directly. Schrödinger's equation in one-dimension for a particle in a harmonic-oscillator potential is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x), \quad \psi(a) = \psi(b) = 0. \tag{19}$$

Before solving the above consider recasting this spatial problem into a time-dependent problem by replacing x with time, t. The oscillator equation that results from this BHO model is interpreted as having a time modulation of the spring constant. This equation can be used to model the fundamental modes of vibration for the free oscillations of a cantilever, such as those in an atomic force microscope (AFM) tip. Blencowe [3] investigates this concept to examine the role of squeezing in such quantum electromechanical systems at the sub-micron level. In this context the analysis presented next may serve as the leading order solution in an asymptotic analysis of squeezing the electromechanical cantilever system under the limits of small nonlinearities in the spring constant, small damping, and small forcing.

Scaling the position coordinate x by l, we find that Eq. (19) becomes

$$\left\{\frac{d^2}{dz^2}+\epsilon-\alpha^2z^2\right\}\psi(z)=0,\quad \psi(a/l)=\psi(b/l)=0, \tag{20}$$

where $\epsilon=(l/x_0)^2\,E/(\frac{1}{2}\hbar\omega)$ is the non-dimensional energy and $\alpha=(l/x_0)^2$. To find a general solution Eq. (20) is converted to the confluent hypergeometric equation. Writing the wave function as a product $\psi(z)=e^{-\frac{1}{2}\alpha z^2}\phi(z)$ and changing the independent variable, $\xi=\alpha z^2$, we obtain the confluent hypergeometric equation

$$\left\{\xi\frac{d^2}{d\xi^2}+\left(\frac{1}{2}-\xi\right)\frac{d}{d\xi}-\mu\right\}\phi(\xi)=0, \tag{21}$$

where $\mu=\frac{1}{4}(1-\epsilon/\alpha)$. The general solution of the confluent hypergeometric equation, given in terms of confluent hypergeometric functions $M(a;b;z)$ [16], is

$$\phi_m(z)=AM(\mu_m;\frac{1}{2};\alpha z^2)+B\sqrt{\alpha}zM(\mu_m+\frac{1}{2};\frac{3}{2};\alpha z^2). \tag{22}$$

Hence, the general solution of Eq. (19) is,

$$\psi_m(z)=\left[AM(\mu_m;\frac{1}{2};\alpha z^2)+B\sqrt{\alpha}zM(\mu_m+\frac{1}{2};\frac{3}{2};\alpha z^2)\right]e^{-\frac{1}{2}\alpha z^2}. \tag{23}$$

Different intervals will correspond to different solutions. Here the intervals $(0,l)$ and $(-l,l)$ are considered. We pose $\psi(0)=A=0$ and $\psi(1)=M(\mu+\frac{1}{2};\frac{3}{2};\alpha)=0$ on the interval $(0,l)$ and, $\psi'(0)=B=0$ and $\psi(1)=M(\mu;\frac{1}{2};\alpha)=0$ on the interval $(-l,l)$. The condition that the derivative vanish at the origin results from the mapping of the independent variable $\xi\to\alpha z^2$, taking the interval $(-l,l)\to(0,l)$, and the symmetry of the eigenfunctions on the interval $(-l,l)$. Thus, the first boundary condition determines the parity of the wave function and the second condition yields the energy spectrum, which depends on l. The wave functions are given by,

$$\psi_m^{(0,l)}(x)=N_m^{(0,l)}(l)(x/x_0)M\left(\mu_m(l)+\frac{1}{2};\frac{3}{2};(x/x_0)^2\right)e^{-\frac{1}{2}(x/x_0)^2} \tag{24}$$

$$\psi_m^{(-l,l)}(x)=N_m^{(-l,l)}(l)M\left(\mu_m(l);\frac{1}{2};(x/x_0)^2\right)e^{-\frac{1}{2}(x/x_0)^2}. \tag{25}$$

Notice that for both the ISW and the BHO, the wavefunctions on the interval $(-l,l)$ have even parity and the wave functions on the interval $(0,l)$ have odd parity.

4. Expansion in Ordinary Harmonic Oscillator Number States

In order to show that these wave functions correspond to squeezed states, we expand the states associated with $\psi_m^{(a,b)}(x) = \left\langle x \left| \psi_m^{(a,b)} \right.\right\rangle$ in terms of ordinary harmonic oscillator number states, $\left|\psi_m^{(a,b)}\right\rangle = \sum_{n=0}^{\infty} A_{m,n}^{(a,b)} \left|n\right\rangle$, using standard eigenfunction expansions. Each wavefunction has the expansion $\left|\psi^{(0,l)}\right\rangle = \sum_{n=0}^{\infty} A_{m,2n+1}^{(0,l)} \left|2n+1\right\rangle$ and $\left|\psi^{(-l,l)}\right\rangle = \sum_{n=0}^{\infty} A_{m,2n}^{(-l,l)} \left|2n\right\rangle$. These are the characteristic expansions of squeezed states, since they lead to constructive and destructive interference that can raise and lower the amount of quantum noise in an observable. Due to the parity of the wavefunctions, the only non-zero coefficients are

$$\begin{aligned} A_{m,2n+1}^{(0,l)} &= \left\langle 2n+1 \right| \left. \psi_m^{(0,l)} \right\rangle \\ &= \frac{1}{2^n \sqrt{2x_0\sqrt{\pi}(2n+1)!}} \int_0^l dx \psi_m^{(0,l)}(x) H_{2n+1}(x) e^{-\frac{1}{2}(x/x_0)^2}, \end{aligned} \tag{26}$$

$$A_{m,2n}^{(-l,l)} = \left\langle 2n \right| \left. \psi_m^{(-l,l)} \right\rangle = \frac{1}{2^n \sqrt{x_0\sqrt{\pi}(2n)!}} \int_{-l}^l dx \psi_m^{(-l,l)}(x) H_{2n}(x) e^{-\frac{1}{2}(x/x_0)^2}, \tag{27}$$

where $H_n(x)$ are Hermite polynomials. Since $\psi_m^{(0,l)}(x)$ has odd parity, the states on the interval $(0, l)$ correspond to single particle excited states,

$$\left|1\right\rangle_S = \left|\psi_0^{(0,l)}\right\rangle = \sum_{n=0}^{\infty} A_{0,2n+1}^{(0,l)} \left|2n+1\right\rangle = a^\dagger \sum_{n=0}^{\infty} \frac{A_{0,2n+1}^{(0,l)}}{\sqrt{2n+1}} \left|2n\right\rangle. \tag{28}$$

For illustrative purposes, only the lowest energy state associated with $\psi_0^{(0,l)}(x)$ will be considered here. It was stated earlier that a squeezed vacuum is obtained by transforming the ordinary harmonic oscillator vacuum by $\left|0\right\rangle_S = S(z)\left|0\right\rangle$. The operator $S(z)$ acting on the vacuum state is the sum over all even powers of $a^\dagger$. The squeezed vacuum is then the sum over all even numbered states,

$$\left|0\right\rangle_S = \sum_{k=0}^{\infty} A_{2k}(\mu,\nu) \left|2k\right\rangle, \tag{29}$$

where

$$A_{2k}(\mu,\nu) = \left[1 - \left(\frac{|\nu|}{\mu}\right)^2\right]^{1/4} \frac{\sqrt{(2k)!}}{2^k k!} \left(\frac{\nu}{\mu}\right)^k. \tag{30}$$

In terms of the squeezing parameter r, Eq. (30) may be written as

$$A_{2k}(r,\theta) = \left[1-\tanh^2(r)\right]^{1/4} \frac{\sqrt{(2k)!}}{2^k k!} e^{ik\theta} \tanh^k(r). \tag{31}$$

Equating, $|0\rangle_S = \left|\psi_0^{(-l,l)}\right\rangle$, we define the ratio,

$$\begin{aligned} \frac{A_{2k+2}(r,\theta)}{A_{2k}(r,\theta)} &= \frac{\frac{\sqrt{(2k+2)!}}{2^{k+1}(k+1)!} e^{i(k+1)\theta} \coth^{k+1}(r)}{\frac{\sqrt{(2k)!}}{2^k k!} e^{ik\theta} \coth^k(r)} \\ &= \frac{1}{\sqrt{2k+2}} e^{i\theta} \coth(r) \\ &= \frac{\left\langle 2k+2 \middle| \psi_0^{(-l,l)} \right\rangle}{\left\langle 2k \middle| \psi_0^{(-l,l)} \right\rangle} \end{aligned} \tag{32}$$

or,

$$e^{i\theta} \coth(r) = \sqrt{2k+2} \frac{A_{2k+2}(r,\theta)}{A_{2k}(r,\theta)} = c_k. \tag{33}$$

Using this ratio and Eq. (11), (12), and (13), we find $\gamma_k = \frac{1}{2}\hbar\omega e^{i\theta_k} \coth(2r_k) = \frac{1}{2}\hbar\omega c_k$. This expression will be used in the next section to construct an effective Hamiltonian for the BHO. The squeezing parameter and angle are given in terms of the ratios of the expansion coefficients,

$$r_k = \coth^{-1}\left(|c_k|\right) \tag{34}$$

$$\theta_k = \pm i \ln\left(\text{sgn}(c_k)\right). \tag{35}$$

The parameter r is a measure of the amount of squeezing in position and momentum. The greater the value of r, the greater the squeezing. As seen in Fig. 2, smaller cavities will produce more squeezing. As the cavity size increases the squeezing parameter approaches zero for the BHO. However, there is always squeezing of a state in the ISW. In this case the measure of squeezing initially decreases as the cavity length increases. However, past a critical point near $l/x_0 \approx 2$, the squeezing increases. The system acts like a harmonic oscillator when the ground state oscillation amplitude, x_0 is less than the boundary length, $2l$, i.e. $x_0 < 2l$, and acts like an infinite square well when $x_0 > 2l$. As we shall discuss below this minimum in the squeezing parameter r corresponds to a transition in the momentum and position variance profiles.

In Fig. 3 we consider the variance in both the position and momentum. Clearly there is squeezing in position for small l since the variance is always

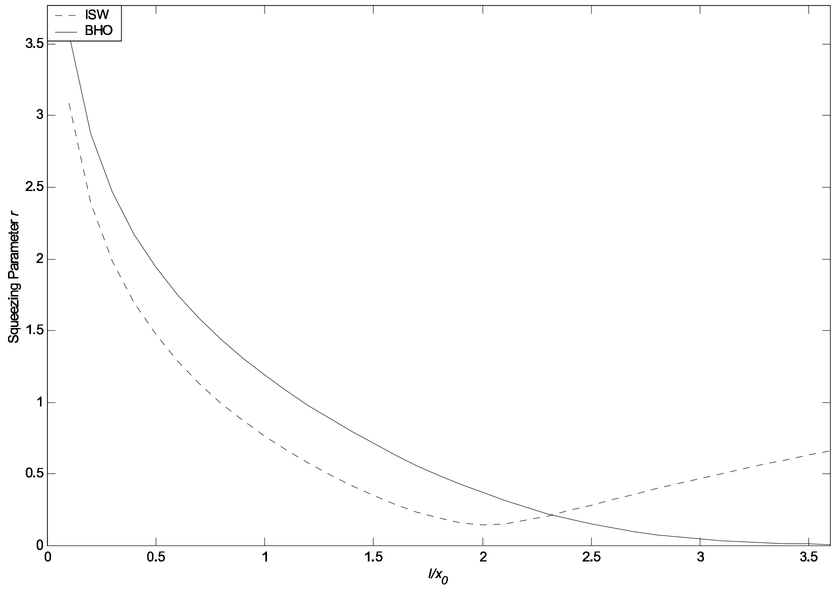

Figure 2: Squeezing parameter r vs. l/x_0 .

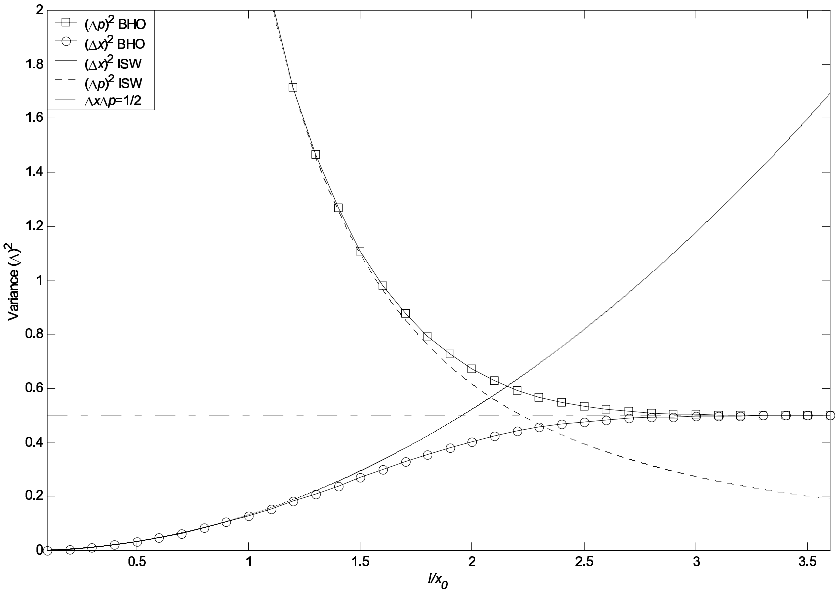

Figure 3: Variances of position and momentum for the BHO and ISW vs. l/x_0.

far from the value 1/2. As shown in Fig. 3, the BHO variances limit to that of the HO as l increases, since r approaches zero. For $l < x_0$ the BHO variances are almost identical to that of a particle in an infinite square well on the interval $(-l, l)$. The variances for the ISW can be calculated exactly,

$$(\Delta x)^2 = \frac{(\pi^2 - 6)}{3\pi^2} l^2, \tag{36}$$

$$(\Delta p)^2 = \frac{\pi^2 \hbar^2}{4l^2}. \tag{37}$$

The oscillation amplitude of the ground state bounded oscillator is x_0. Therefore the BHO behaves like a particle in an infinite square well potential when the oscillation amplitude of the vacuum state x_0 is larger than the boundary length l. Here the ISW is a good approximation for the BHO. Physically this makes sense since the oscillatory motion is restricted on small domains and the motion is essentially the same as a standing wave.

For the ISW we notice in Fig. 3 that for small cavities, $l/x_0 < 2$, the variance in position is small while the variance in momentum is large. The opposite is true for larger cavities. The explanation for this result is that it is easier to track the location of the particle in a thin cavity as compared to a wide cavity. Hence the variance in position increases as the width increases.

5. Effective Hamiltonian

We now consider an application of the results derived above to construct creation/annihilation operators for the BHO. In general, it is not straightforward to construct the BHO creation/annihilation operators. The operators on the finite interval could be constructed directly by the usual Hamiltonian factorization method [14], in which creation /annihilation operators are defined by,

$$A^{\pm} = \frac{1}{\sqrt{2}} \left(\mp \frac{d}{dx} + \frac{d}{dx} \ln (\psi_0(x)) \right), \tag{38}$$

where, $\psi_0(x)$ is the ground state wavefunction (defined by Eq. (24) and (25) for the BHO). This method is not useful in most cases, since the canonical commutation relations are not easily constructed and the creation/annihilation operators cannot be found analytically due to the dependence on the boundary conditions.

Rather than using Eq. (38) to calculate the creation/annihilation operators, our approach is to cast the bounded system in terms of operators in the infinite domain. We shall use the position (momentum) eigenfunctions and eigenvalues for the bounded oscillator that were found above. From these eigenstates, which depend on the boundary, an effective Hamiltonian will be constructed by computing the coefficients of the quadratic squeezing Hamiltonian, Eqs. (2) or (8), from these eigenstates. These coefficients will be piecewise defined in the sense that they vanish outside the boundaries. Using the effective Hamiltonian, one can extract the creation/annihilation operators for the BHO.

In terms of quantities calculated above, the state in Eq. (29) is an eigenstate of the quadratic Hamiltonian in Eq. (2). The Hamiltonian for the bounded states is the same as that given in Eq. (2), except the coefficient γ, is now dependent on the boundary as calculated for γ_k. The squeezing parameter and angle were found as functions of the boundary length l in Eq. (34) and Eq. (35). Using these results we can calculate μ and ν in Eq. (7). The energies corresponding to the effective Hamiltonian are given by the diagonalization in Eq. (10). However now the parameters, μ, ν , and γ all depend on the boundary length. The ground-state energy is,

$$E_0 = \frac{1}{2}\hbar\omega \left(\left|\mu^2\right|^2 + |\nu|^2 \right) - 2\mu\Re(\nu\gamma^*). \tag{39}$$

In order to check the accuracy of the effective Hamiltonian, the energy from Eq. (39) is compared to the exact energy as computed numerically using, $E_0 = \frac{1}{2}\hbar\omega(1 - 4\mu_0(l))$. In this expression, $\mu_0(l)$ is the first eigenvalue of the eigenfunction in Eq. (25). The results are shown in Fig. 4. The energy of the effective Hamiltonian is very close to the exact energy (labeled BHO) for all values of the boundary length, providing confidence in the effective Hamiltonian concept. Fig. 4 also shows the same critical point as in Fig. 3 such that the BHO energy matches the ISW energy for tight confinement, and matches the HO energy for widely spaced boundaries. In the latter case full numerical simulations [19] and the SWKB method [6] have also been used to demonstrate the convergence of the BHO energies to the HO spectrum as the boundary width increases. The simplicity and accuracy of the effective Hamiltonian concept illustrates its usefulness as an alternative means to study confined quantum systems.

6. Summary

We have examined solutions to the Schrödinger equation defined on a bounded domain using either ISW or BHO potentials. In all cases we have shown that squeezing can arise from confinement. Further systems with a definite parity, even or odd, are recognized as having the potential to exhibit squeezing. For $(0, l)$ domains the wavefunctions are odd, and this generates a single particle excited squeezed state for the lowest energy state. For $(-l, l)$ domains the wavefunctions are even and generate a squeezed vacuum state.

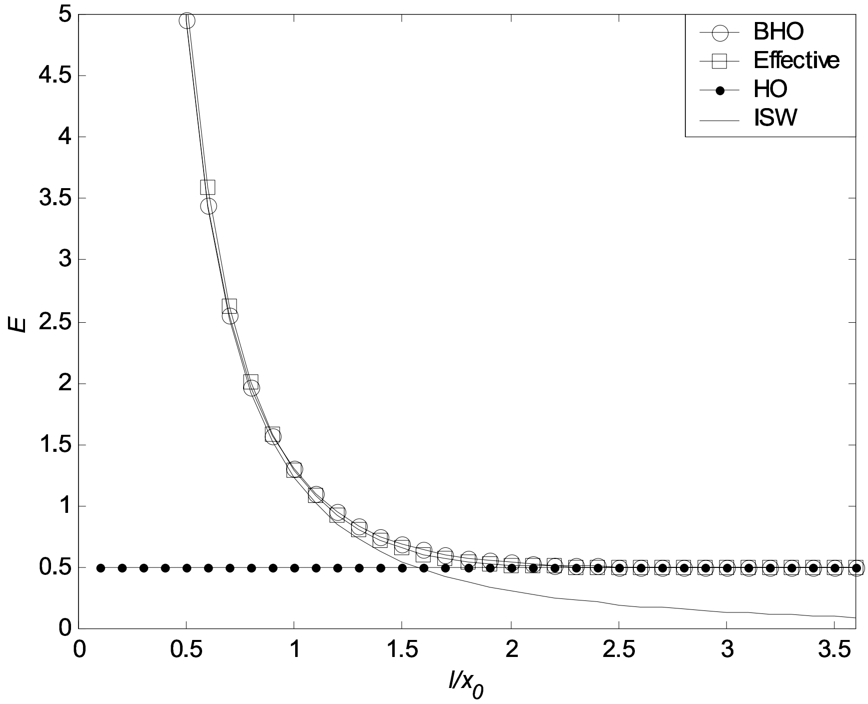

Figure 4: Non-dimensional energies ($E = E/(\hbar\omega)$) of the ISW, BHO and ordinary HO vs. l/x_0 .

For cavity lengths that are smaller than the oscillation amplitude of the vacuum state, we see that the ISW is a good approximation for the BHO. Hence, because of its mathematical simplicity the ISW model of the potential is preferred over the BHO for small domains. As the cavity length increases we find that the BHO approaches the HO as expected and squeezing of states vanishes. For the ISW system we find that there is always squeezing of a state. However the nature of the squeezing changes as the cavity length increases. For thin cavities the position variance is small while for larger cavities the momentum variance is small. The above results could be investigated through an experiment using a cavity of varying width.

We have also shown that the bounded HO system can be written in terms of the unbounded system. We accomplished this by creating an effective Hamiltonian that contains boundary information. The Hamiltonian for the BHO can be written in terms of creation/annihilation operators that create and destroy quanta in the infinite domain. With this, bounded regions can be studied in terms of operators in the infinite domain. We tested the accuracy of the effective Hamiltonian by comparing its ground-state energy due to the boundary to the exact energy as computed from the solution of the Schrödinger equation. These energies agreed for all boundary lengths. Hence, the effective Hamiltonian concept provides a useful alternative means to study confined quantum systems.

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