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Nonresonance Impulsive Functional Dynamic Equations on Times Scales

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Abstract

In this paper, we investigate the existence of solutions of first order nonresonance impulsive functional dynamic equations on time scales. The proofs make use of the Leary-Schauder alternative.

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1. Introduction

In this paper, we are concerned with the existence of solutions of the nonresonance impulsive functional dynamic equation on a time scale,

$$y^{\Delta}(t) - \lambda y^{\sigma}(t) = f(t, y_t), \text{ a.e. } \ t \in J := [0, b], \ t \neq t_k, \quad k = 1, \dots, m, \tag{1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y(t) = \phi(t), \ \ t \in [-r, 0], \ \ y(0) = y(b), \tag{3}$$

where Δ denotes the delta derivative defined below, T is a time scale, $[0, b] \subset \mathrm{T}$, $\lambda \in \mathbb{R}_+$, $f : \mathrm{T} \times \mathbb{R} \to \mathbb{R}$, is a given function,

$$\mathcal{D} = \{\psi : [-r, 0] \to \mathbb{R} \mid \psi \ \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \psi(\bar{t}^-) \text{ and } \psi(\bar{t}^+) \text{ exist, and } \psi(\bar{t}^-) = \psi(\bar{t})\},$$

$\phi \in \mathcal{D}$, $0 < r < \infty$, $I_k \in C(\mathrm{I\!R}, \mathrm{I\!R})$, $t_k \in \mathrm{T}$, $0 = t_0 < t_1 < \dots t_m < b$, and

$$y_t(\theta) = y(t+\theta) \quad \text{for} \quad \theta \in [-r, 0] \quad \text{and} \quad t \in J - \{t_1, t_2, \dots, t_m\}.$$

Here, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k - h)$ represent the right and left hand limits of $y(t)$ at $t = t_k$ in the time scale sense. Moreover, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, whereas, if t_k is left scattered, then $y(t_k^-) = y(t_k)$. The function σ, known as the forward jump operator, will be defined below, and $y^\sigma(t) = y(\sigma(t))$.

Impulsive differential equations have become important in recent years as mathematical models of real processes; they arise in phenomena studied in physics, chemistry, population dynamics, biotechnology, and economics. There have been significant developments in impulse theory in recent years, especially in the area of impulsive differential equations with fixed moments; in this regard, see the monographs of Bainov and Simeonov [10], Lakshmikantham *et al.* [25], Samoilenko and Perestyuk [29], and the references contained therein.

Recently, dynamic equations on times scales have received much attention. We refer the reader to the books by Agarwal and O'Regan [5], Bohner and Peterson [16, 18], Lakshmikantham *et al.* [26], and the papers by Agarwal and Bohner [2], Agarwal *et al.* [3], Anderson [6, 9], Bohner and Guseinov [15], Bohner and Peterson [17], Bohner and Eloe [14], and Erbe and Peterson [20, 21]. The time scale calculus has tremendous potential for applications in mathematical models of real world processes in both the physical and social sciences, and we refer the reader to Leonov *et al.* [27] as well as [16, 18, 26] for discussions of some of these types of applications. The existence of solutions of boundary value problems on a measure chain (i.e., a time scale) was recently studied by Agarwal and O'Regan [4], Anderson [7, 8], Henderson [23], Henderson and Tisdell [24], and Sun [30], and impulsive dynamic equations on time scales were considered by Agarwal *et al.* [1], Benchohra *et al.* [12, 13], and Henderson [22]. The main theorems in this paper extend the results obtained on the problem (1)–(3) considered by Benchohra *et al.* [11] and Nieto [28]. Our approach here is based on a nonlinear alternative of Leray-Schauder type [19].

2. Preliminaries

We will briefly recall some basic definitions and facts from the time scale calculus that will be needed here. A *time scale* T is an nonempty closed subset of $\mathrm{I\!R}$. The forward and backward jump operators $\sigma, \rho : \mathrm{T} \to \mathrm{T}$ given by

$$\sigma(t) = \inf\{s \in \mathrm{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathrm{T} : s < t\},$$

respectively, (supplemented by $\inf \emptyset := \sup \mathrm{T}$ and $\sup \emptyset := \inf \mathrm{T}$) are well defined. The point $t \in \mathrm{T}$ is left-dense, left-scattered, right-dense, or right-scattered if

$\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, we define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[a, b], [a, b)$, etc. will denote time scales intervals

$$[a, b] = \{t \in \mathbb{T} : a \le t \le b\},$$

where $a, b \in \mathbb{T}$ with $a < \rho(b)$.

Definition 2.1. *Let X be a Banach space. The function $f : \mathbb{T} \to X$ is said to be rd−continuous provided it is continuous at each right-dense point and has a left-sided limit at each point; in this case, we will write $f \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, X)$.*

Definition 2.2. *For $t \in \mathbb{T}^k$, the Δ derivative of f at t, denoted by $f^{\Delta}(t)$, is the number (provided it exists) such that for every $\varepsilon > 0$ there exists a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$.

Remark 2.1. *It is known that if f is continuous, then it is rd−continuous. Moreover, if f is delta differentiable at t, then it is continuous there.*

Definition 2.3. *The function F is an antiderivative of $f : \mathbb{T} \to X$ provided*

$$F^{\Delta}(t) = f(t) \ \ \textit{for each } t \in \mathbb{T}^k.$$

The function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T},$$

where $\mu(t) = \sigma(t) - t$ is called the *graininess function.* We assume throughout this paper that

$$1 - \mu(t)\lambda \neq 0$$

for all $t \in \mathbb{T}$.

The *generalized exponential function* e_p is defined as the unique solution $y(t) = e_p(t, a)$ of the initial value problem

$$y^{\Delta} = p(t)y, \quad y(a) = 1,$$

where p is a regressive function. An explicit formula for $e_p(t, a)$ is given by

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \text{ with } \xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + \mathrm{hz})}{h}, & \text{if } h \neq 0, \\ z, & \text{if } \ h = 0. \end{cases}$$

For an additional discussion of the generalized exponential function, we refer the reader to the monograph by Bohner and Peterson [16]. Clearly, $e_p(t,s)$ never vanishes. We next give some fundamental properties of the exponential function.

Let $p, q : \mathbb{T} \to \mathbb{R}$ two regressive functions. We define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad \text{and} \quad p \ominus q := p \oplus (\ominus p).$$

Theorem 2.1 ([16, Theorem 2.36], [17, Theorem 3.1]). *Assume that $p,\ q : \mathbb{T} \to \mathbb{R}$ are regressive and rd−continuous functions; then the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s)\dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

We let $C([0,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}$, where $[0,b] \subset \mathbb{T}$, with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in [0,b]\}.$$

Let $[a,b]$ be an interval in $\mathbb{T}$ and let $L^1([a,b],\mathbb{R})$ denote the set of functions that are Lebesgue integrable in the time scale sense and let

$$AC^i((a,b),\mathbb{R}) = \{y \in C([a,b],\mathbb{R}) : \int y^\Delta(t)\Delta t = y(t)\}$$

be the space of i-times delta differentiable functions $y : [0,b] \to \mathbb{R}$ whose i^{th} delta derivative, $y^{\Delta^{(i)}}$, is absolutely continuous.

Definition 2.4. *A map $f : J \times \mathcal{D} \to \mathbb{R}$ is said to be L^1-Carathéodory if*

(i) $t \longmapsto f(t,u)$ *is measurable for each* $u \in \mathcal{D}$,

(ii) $u \longmapsto f(t,u)$ *is continuous for almost all* $t \in [0,b]$, *and*

(iii) *for each* $q > 0$ *there exists* $h_q \in L^1(J,\mathbb{R}_+)$ *such that*

$$|f(t,u)| \le h_q(t)$$

for all $\|u\|_{\mathcal{D}} \le q$ *and for almost all* $t \in [0,b]$.

3. Main Results

In order to define a solution of the problem (1)–(3), we consider the space

$$\Omega = \{y : [0,b] \to \mathrm{I\!R} \mid y_k \in C(J_k, \mathrm{I\!R}),\ k = 0, \dots, m,\ y(t_k^-) \text{ and } y(t_k^+) \text{ exist},\ k = 1, \dots, m, \text{ and } y(t_k^-) = y(t_k)\},$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k},\ k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}] \subset [0,b]$, $k = 0, \dots, m$. Set

$$PC = \{y : [-r,b] \to \mathrm{I\!R} : y \in D \cap \Omega\}.$$

Then PC is a Banach space with the norm

$$\|y\|_{PC} = \sup\{|y(t)| : t \in [-r,b]\}.$$

Definition 3.1. *A function $y \in PC \cap AC((t_k, t_{k+1}), \mathrm{I\!R})$, $k = 0, \dots, m$, is said to be a solution of (1)–(3) if y satisfies the dynamic equation*

$$y^\Delta(t) - \lambda y^\sigma(t) = f(t, y_t) \ \text{ a.e. on } J \backslash \{t_k\},\ k = 1, \dots, m,$$

and for each $k = 1, \dots, m$, the function y satisfies $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$, $y(t) = \phi(t)$ for $t \in [-r, 0]$, and $y(0) = y(b)$.

We will need the following auxiliary result in order to prove our main existence theorems.

Lemma 3.1. *Let $f : \mathrm{T} \to \mathrm{I\!R}$ be an rd−continuous function. Then y is the unique solution of the problem*

$$y^\Delta(t) - \lambda y^\sigma(t) = f(t), \quad t \in J := [0,b],\ t \neq t_k, \quad k = 1, \dots, m, \tag{4}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad k = 1, \dots, m, \tag{5}$$

$$y(0) = y(b), \tag{6}$$

if and only if

$$y(t) = \int_0^b H_{-\lambda}(t,s) f(s) \Delta s + \sum_{k=1}^m H_{-\lambda}(t, t_k) I_k(y(t_k)), \tag{7}$$

where

$$H_{-\lambda}(t,s) = (1 - e_{-\lambda}(0,b))^{-1} \begin{cases} e_{-\lambda}(s,t), & \text{if } 0 \le s < t \le b, \\ e_{-\lambda}(0,b) e_{-\lambda}(s,t), & \text{if } 0 \le t \le s \le b, \end{cases}$$

and $e_{-\lambda}(\cdot, 0)$ is the unique solution of the initial value problem

$$x^\Delta(t) = -\lambda x(t), \quad t \in [0,b], \ x(0) = 1.$$

Proof. Let y be a solution of the problem (4)–(6). Then

$$y^{\Delta}(t) - \lambda y^{\sigma}(t) = f(t) \quad \text{for} \quad t \in [0, b] \subset \mathbf{T}.$$

Thus,

$$e_{-\lambda}(t, 0)y^{\Delta}(t) - \lambda e_{-\lambda}(t, 0)y(\sigma(t)) = e_{-\lambda}(t, 0)f(t)$$

or

$$[e_{-\lambda}(t, 0)y(t)]^{\Delta} = e_{-\lambda}(t, 0)f(t).$$

Assume that $t_k < t \leq t_{k+1},\ k = 1, \dots, m$. Integrations yield

$$\begin{aligned}
e_{-\lambda}(t_1, 0)y(t_1) - y(0) &= \int_0^{t_1} e_{-\lambda}(s, 0)f(s)\Delta s,\\
e_{-\lambda}(t_2, 0)y(t_2) - e_{-\lambda}(t_1, 0)y(t_1^+) &= \int_{t_1}^{t_2} e_{-\lambda}(s, 0)f(s)\Delta s,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
e_{-\lambda}(t_k, 0)y(t_k) - e_{-\lambda}(t_{k-1}, 0)y(t_{k-1}^+) &= \int_{t_{k-1}}^{t_k} e_{-\lambda}(s, 0)f(s)\Delta s,\\
e_{-\lambda}(t, 0)y(t) - e_{-\lambda}(t_k, 0)y(t_k^+) &= \int_{t_k}^{t} e_{-\lambda}(s, 0)f(s)\Delta s.
\end{aligned}$$

Adding these expressions, we obtain

$$e_{-\lambda}(t, 0)y(t) = y(0) + \sum_{0<t_k<t} e_{-\lambda}(t_k, 0)I_k(y(t_k)) + \int_0^t e_{-\lambda}(s, 0)f(s)\Delta s. \tag{8}$$

In view of (8) and the fact that $y(0) = y(b)$, we have

$$e_{-\lambda}(b, 0)y(b) = y(0) + \sum_{k=1}^{m} e_{-\lambda}(t_k, 0)I_k(y(t_k)) + \int_0^b e_{-\lambda}(s, 0)f(s)\Delta s,$$

and so

$$y(0) = (e_{-\lambda}(b, 0) - 1)^{-1}\left[\sum_{k=1}^{m} e_{-\lambda}(t_k, 0)I_k(y(t_k)) + \int_0^b e_{-\lambda}(s, 0)f(s)\Delta s\right]. \tag{9}$$

Substituting (9) into (8) yields

$$\begin{aligned}
e_{-\lambda}(t, 0)y(t) = (e_{-\lambda}(b, 0) - 1)^{-1}&\left[\sum_{k=1}^{m} e_{-\lambda}(t_k, 0)I_k(y(t_k))\right]\\
&+ (e_{-\lambda}(b, 0) - 1)^{-1}\left[\int_0^b e_{-\lambda}(s, 0)f(s)\Delta s\right]\\
&\quad + \sum_{0<t_k<t} e_{-\lambda}(t_k, 0)I_k(y(t_k)) + \int_0^t e_{-\lambda}(s, 0)f(s)\Delta s.
\end{aligned} \tag{10}$$

From (10) and the fact that

$$\sum_{k=1}^{m} I_k(y(t_k)) = \sum_{0<t_k<t} I_k(y(t_k)) + \sum_{t\le t_k<b} I_k(y(t_k)),$$

we have

$$\begin{aligned}
e_{-\lambda}(t,0)y(t) = (e_{-\lambda}(b,0)-1)^{-1}&\left[\sum_{0<t_k<t} \frac{e_{-\lambda}(0,b)e_{-\lambda}(t_k,0)}{e_{-\lambda}(0,b)} I_k(y(t_k))\right]\\
&+ (e_{-\lambda}(b,0)-1)^{-1}\left[\sum_{t\le t_k<b} \frac{e_{-\lambda}(0,b)e_{-\lambda}(t_k,0)}{e_{-\lambda}(0,b)} I_k(y(t_k))\right]\\
&+ (e_{-\lambda}(b,0)-1)^{-1}\left[\int_0^t \frac{e_{-\lambda}(0,b)e_{-\lambda}(s,0)}{e_{-\lambda}(0,b)} f(s)\Delta s\right.\\
&+ \left.\int_t^b \frac{e_{-\lambda}(0,b)e_{-\lambda}(s,0)}{e_{-\lambda}(0,b)} f(s)\Delta s\right]\\
&+ \sum_{0<t_k<t} \frac{e_{-\lambda}(0,b)e_{-\lambda}(t_k,0)}{e_{-\lambda}(0,b)} I_k(y(t_k))\\
&+ \int_0^t \frac{e_{-\lambda}(0,b)e_{-\lambda}(s,0)}{e_{-\lambda}(0,b)} f(s)\Delta s\,.
\end{aligned}$$

Multiplying the above expression by $e_{-\lambda}(0,t)$ gives

$$\begin{aligned}
y(t) = (1-e_{-\lambda}(0,b))^{-1}&\left[\sum_{0<t_k<t} e_{-\lambda}(t_k,t) I_k(y(t_k))\right.\\
&\qquad\left.+ \sum_{t\le t_k<b} e_{-\lambda}(0,b)e_{-\lambda}(t_k,t) I_k(y(t_k))\right]\\
+ (1-e_{-\lambda}(0,b))^{-1}&\left[\int_0^t e_{-\lambda}(s,t)f(s)\Delta s + \int_t^b e_{-\lambda}(0,b)e_{-\lambda}(s,t)f(s)\Delta s\right].
\end{aligned}$$

Thus,

$$y(t) = \int_0^b H_{-\lambda}(t,s)f(s)\Delta s + \sum_{k=1}^{m} H_{-\lambda}(t,t_k)I_k(y(t_k)).$$

To prove that if y satisfies the integral equation (7), then y is solution of the problem (4)–(6), let $t \in [0,b]\backslash\{t_1,\dots,t_m\}$ and

$$y(t) = \int_0^b H_{-\lambda}(t,s)f(s)\Delta s + \sum_{k=1}^{m} H_{-\lambda}(t,t_k)I_k(y(t_k)).$$

Then,

$$\begin{aligned}
y(t) = (1 - e_{-\lambda}(0,b))^{-1} \Bigg[& \sum_{0<t_k<t} e_{-\lambda}(t_k,t) I_k(y(t_k)) \\
& + \sum_{t\leq t_k<b} e_{-\lambda}(0,b) e_{-\lambda}(t_k,t) I_k(y(t_k)) \Bigg] \\
& + (1 - e_{-\lambda}(0,b))^{-1} \left[\int_0^t e_{-\lambda}(s,t) f(s) \Delta s + \int_t^b e_{-\lambda}(0,b) e_{-\lambda}(s,t) f(s)\Delta s \right].
\end{aligned}$$

Now

$$\begin{aligned}
e_{-\lambda}(t,0)y(t) &= (1 - e_{-\lambda}(0,b))^{-1} \left[\sum_{0<t_k<t} e_{-\lambda}(t_k,0) I_k(y(t_k)) \right] \\
& + (1 - e_{-\lambda}(0,b))^{-1} \left[\sum_{t\leq t_k<b} e_{-\lambda}(0,b) e_{-\lambda}(t_k,0) I_k(y(t_k)) \right] \\
& + (1 - e_{-\lambda}(0,b))^{-1} \Bigg[\int_0^t e_{-\lambda}(s,0) f(s) \Delta s \\
& \qquad + \int_t^b e_{-\lambda}(0,b) e_{-\lambda}(s,0) f(s) \Delta s \Bigg],
\end{aligned}$$

so

$$\begin{aligned}
[e_{-\lambda}(t,0)y]^{\Delta}(t) &= (1 - e_{-\lambda}(b,0))^{-1} \left[\sum_{0<t_k<t} e_{-\lambda}(t_k,0) I_k(y(t_k)) \right]^{\Delta} \\
& + (1 - e_{-\lambda}(0,b))^{-1} \left[\sum_{t\leq t_k<b} e_{-\lambda}(0,b) e_{-\lambda}(t_k,0) I_k(y(t_k)) \right]^{\Delta} \\
& + (1 - e_{-\lambda}(0,b))^{-1} \Bigg[\int_0^t e_{-\lambda}(s,0) f(s) \Delta s \\
& \qquad + \int_t^b e_{-\lambda}(0,b) e_{-\lambda}(s,0) f(s) \Delta s \Bigg]^{\Delta} \\
&= (1 - e_{-\lambda}(0,b))^{-1} \left[\int_0^t e_{-\lambda}(s,0) f(s) \Delta s \right]^{\Delta} \\
& + (1 - e_{-\lambda}(0,b))^{-1} \left[\int_t^b e_{-\lambda}(0,b) e_{-\lambda}(s,0) f(s) \Delta s \right]^{\Delta}.
\end{aligned}$$

Then,

$$[e_{-\lambda}(t,0)y]^{\Delta}(t) = (1 - e_{-\lambda}(0,b))^{-1} [e_{-\lambda}(t,0) f(t) - e_{-\lambda}(0,b) e_{-\lambda}(t,0)] f(t),$$

or

$$e_{-\lambda}(t,0)y^{\Delta}(t) + e_{-\lambda}^{\Delta}(t,0)y(\sigma(t)) = e_{-\lambda}(t,0)f(t),$$

and so

$$e_{-\lambda}(t,0)y^{\Delta}(t) - \lambda e_{-\lambda}(t,0)y(\sigma(t)) = e_{-\lambda}(t,0)f(t).$$

Thus,

$$y^{\Delta}(t) - \lambda y(\sigma(t)) = f(t).$$

Next, we show that $y(0) = y(b)$. Now

$$y(0) = \int_0^b H_{-\lambda}(0,s)f(s)\Delta s + \sum_{k=1}^{m} H_{-\lambda}(0,t_k)I_k(y(t_k)),$$

and from the definition of $H_{-\lambda}$, we obtain

$$\begin{aligned} y(0) &= (1 - e_{-\lambda}(0,b))^{-1}\left[\sum_{k=1}^{m} e_{-\lambda}(0,b)e_{-\lambda}(t_k,0)I_k(y(t_k)) \right. \\ &\qquad \left. + \int_0^b e_{-\lambda}(0,b)e_{-\lambda}(s,0)f(s)\Delta s\right] \\ &= (1 - e_{-\lambda}(0,b))^{-1}\left[\sum_{k=1}^{m} e_{-\lambda}(t_k,b)I_k(y(t_k)) + \int_0^b e_{-\lambda}(s,b)f(s)\Delta s\right] \\ &= \int_0^b H_{-\lambda}(b,s)f(s)\Delta s + \sum_{k=1}^{m} H_{-\lambda}(b,t_k)I_k(y(t_k)) = y(b). \end{aligned}$$

We can easily prove that $y(t_k^+) - y(t_k) = I_k(y(t_k))$, $k = 1,\dots,m$, and this completes the proof of the lemma. □

We now present an existence result for the problem (1)–(3).

Theorem 3.1. *Suppose that the following conditions hold.*

(H1) *The function* $f : [0,b] \times \mathcal{D} \to \mathbb{R}$ *is* L^1*-Carathéodory.*

(H2) *There exist functions* p, $\bar{q} \in L^1([0,b],\mathbb{R}_+)$, *and constants* $\alpha \in [0,1)$, c_k, $b_k \in \mathbb{R}_+$, *and* $\alpha_k \in [0,1)$, $k = 1,2,\dots,m$, *such that*

$$|f(t,u)| \le p(t)|u|^{\alpha} + \bar{q}(t) \text{ for each } (t,y) \in [0,b] \times \mathcal{D},$$

and

$$|I_k(y)| \le c_k + b_k|y|^{\alpha_k} \text{ for } y \in \mathbb{R}.$$

Then the problem (1)–(3) has at least one solution.

Proof. Consider the operator $G : PC \to PC$ defined by

$$G(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \displaystyle\int_0^b H_{-\lambda}(t,s) f(s, y_s)\Delta s + \sum_{k=1}^{m} H_{-\lambda}(t, t_k) I_k(y(t_k)), & \text{if } t \in [0, b], \end{cases}$$

where $H_{-\lambda}$ defined in Lemma 3.1. From Lemma 3.1, we can easily show that the fixed points of G are solutions to the problem (1)–(3).

We will show that G satisfies the assumptions of the nonlinear Leray-Schauder alternative. The proof will be given in several steps.

Step 1: *G is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC. Then

$$\begin{aligned} |G(y_n)(t) - G(y)(t)| \le \int_0^b & |H_{-\lambda}(t,s)||f(s,(y_n)_s) - f(s, y_s)|\Delta s \\ & + \sum_{k=1}^{m} |H_{-\lambda}(t, t_k)||I_k(y_n(t_k)) - I_k(y(t_k))|. \end{aligned}$$

Since f is L^1-Carathédory and the I_k are continuous functions, we have

$$\|G(y_n) - G(y)\|_{PC} \le H_* \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{L^1} + H_* \sum_{k=1}^{m} |I_k(y_n(t_k)) - I_k(y(t_k))|,$$

where

$$H_* = \sup\{|H_{-\lambda}(t,s)| : (t,s) \in [0,b] \times [0,b]\}.$$

Hence,

$$\|G(y_n) - G(y)\|_{PC} \to 0 \quad as\ n \to \infty,$$

and so G is continuous.

Step 2: G maps bounded sets into bounded sets in PC.

It suffices to show that there exists a positive constant ℓ such that $\|G(y)\|_{PC} \le \ell$ for each $y \in B_q = \{y \in PC : \|y\|_{PC} \le q\}$. Let $y \in B_q$; then, for each $t \in [0, b]$, we have

$$G(y)(t) = \int_0^b H_{-\lambda}(t,s) f(s, y_s)\Delta s + \sum_{k=1}^{m} H_{-\lambda}(t, t_k) I_k(y(t_k)).$$

By (H1), for each $t \in [0,b]$, we have

$$\begin{aligned}|G(y)(t)| \leq \int_0^b |H_{-\lambda}(t,s)||f(s,y_s)|\Delta s + \sum_{k=1}^{m} |H_{-\lambda}(t,t_k)||I_k(y(t_k))| \\ \leq H_* \int_0^b h_q(s)\Delta s + H_* \sum_{k=1}^{m}[c_k + q^{\alpha_k} b_k].\end{aligned}$$

Then, for each $y \in B_q$, we have

$$\|G\|_{PC} \leq H_*\|h_q\|_{L^1} + H_* \sum_{k=1}^{m}[c_k + q^{\alpha_k} b_k] := \ell.$$

Step 3: *G maps bounded sets into equicontinuous sets in PC.*

Let τ_1, $\tau_2 \in [0,b]$, $\tau_1 < \tau_2$, and let B_q be a bounded set in PC as in Step 2. Let $y \in B_q$ and $t \in [0,b]$; then

$$\begin{aligned}|G(y)(\tau_2) - G(y)(\tau_1)| \leq \int_0^b |H_{-\lambda}(\tau_2,s) - H_{-\lambda}(\tau_1,s)|h_q(s)\Delta s \\ + \sum_{k=1}^{m} |H_{-\lambda}(\tau_2,t_k) - H_{-\lambda}(\tau_1,t_k)|[c_k + qb_k].\end{aligned}$$

Now as $\tau_2 \longrightarrow \tau_1$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ are clear.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $G : PC \to PC$ is completely continuous.

Step 4: *A priori bounds on solutions.*

Let $y = \gamma G(y)$ for some $0 < \gamma < 1$. This implies, by (H2), that for each $t \in [0,b]$ we have

$$|y(t)| \leq H_* \left[\int_0^b p(s)\|y_s\|^{\alpha}\Delta s + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}[c_k + b_k|y(t_k)|^{\alpha_k}]\right]. \tag{11}$$

Consider the function ν defined by

$$\nu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r,t]$ be such that $\nu(t) = |y(t^*)|$. If $t^* \in [-r,0]$, we have $\nu(t) = \|\phi\|_{\mathcal{D}}$.

If $t^* \in [0, b]$, by (11), we have

$$\begin{aligned}
|\nu(t)| &\leq H_* \left[\int_0^b p(s)|\nu(s)|^\alpha \Delta s + \|\bar{q}\|_{L^1} + \sum_{k=1}^m [c_k + b_k|\nu(t_k)|^{\alpha_k}] \right] \\
&\leq H_* \left[\int_0^b p(s)|\nu(s)|^\beta \Delta s + \|\bar{q}\|_{L^1} + \sum_{k=1}^m [c_k + b^k|\nu(t_k)|^\beta] \right] \\
&\leq H_* \left[\|p\|_{L^1}\|y\|_{PC}^\beta + \|\bar{q}\|_{L^1} + \sum_{k=1}^m [c_k + b_k\|y\|_{PC}^\beta] \right]
\end{aligned}$$

for $t \in [0, b]$, where $\beta = \max(\alpha, \alpha_k)$, $k = 1, \dots, m$. Then,

$$\|y\|_{PC} \leq H_* \left[\|p\|_{L^1}\|y\|_{PC}^\beta + \|\bar{q}\|_{L^1} + \sum_{k=1}^m [c_k + b_k\|y\|_{PC}^\beta] \right].$$

If $\|y\|_{PC} > 1$, we have

$$\|y\|_{PC}^{1-\beta} \leq H_* \left[\|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^m [c_k + b_k] \right],$$

and so

$$\|y\|_{PC} \leq \left(H_*\|p\|_{L^1} + H_*\|\bar{q}\|_{L^1} + H_* \sum_{k=1}^m [c_k + b_k] \right)^{\dfrac{1}{1-\beta}} := \psi_*.$$

Hence,

$$\|y\|_{PC} \leq \max(1, \|\phi\|_{\mathcal{D}}, \psi_*) := M.$$

Set

$$U := \{y \in PC : \|y\|_{PC} < M + 1\},$$

and consider the operator $G : \overline{U} \to PC$. From the choice of U, there is no $y \in \partial U$ such that $y = \gamma G(y)$ for some $\gamma \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [19], we deduce that G has a fixed point y in U that is a solution of the problem (1)–(3). □

In our final result, we use the usual form of the Leray-Schauder alternative to obtain a slightly different existence result.

Theorem 3.2. *In addition to* $(H1)$, *assume that*

(H3) *there exist continuous non-decreasing functions* ψ_1, $\psi : [0,\infty) \to (0,\infty)$, *a function* $\bar{p} \in L^1([0,b],\mathbb{R}_+)$, *positive constants* d_k, $k = 1,\ldots,m$, *and a constant* $M_* > 0$ *such that*

$$|I_k(y)| \leq d_k\psi_1(|y|) \quad \textit{for all } y \in \mathbb{R},$$

$$|f(t,u)| \leq \bar{p}(t)\psi(\|u\|) \quad \textit{for each } (t,u) \in [0,b] \times \mathcal{D},$$

and

$$\frac{M_*}{H_*\left[\psi(M_*)\displaystyle\int_0^b \bar{p}(s)\Delta s + \psi_1(M_*)\sum_{k=0}^m d_k\right]} > 1.$$

Then the problem (1)–(3) has at least one solution.

Proof. Consider the operator G defined in the proof of Theorem 3.1. We will show that G satisfies the assumptions of the Leray-Schauder nonlinear alternative. Let y be a possible solution of the problem (1)–(3); then, similar to the proof of Theorem 3.1, we have

$$|y(t)| \leq H_*\left[\int_0^b \bar{p}(s)\psi(|y_s\|)\Delta s + \sum_{k=1}^m d_k\psi_1(|y(t_k)|)\right].$$

Consider the function ν defined by

$$\nu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r,t]$ be such that $\nu(t) = |y(t^*)|$. If $t^* \in [-r,0]$, we have $\nu(t) = \|\phi\|_{\mathcal{D}}$. If $t^* \in [0,b]$, the previous inequality yields

$$\nu(t) \leq H_*\left[\int_0^b \bar{p}(s)\psi(\nu(s))\Delta s + \sum_{k=1}^m d_k\psi_1(\nu(t_k))\right]$$

for $t \in [0,b]$. Consequently,

$$\frac{\|y\|_{PC}}{H_*\left[\displaystyle\int_0^b \bar{p}(s)\psi(\|y\|_{PC})\Delta s + \sum_{k=1}^m d_k\psi_1(\|y\|_{PC})\right]} \leq 1.$$

Then by $(H3)$, there exists M_* such that $\|y\|_{PC} \neq M_*$.

Set

$$U_1 = \{y \in PC : \|y\|_{PC} < M_*\}.$$

As in the proof of Theorem 3.1, the operator $G : \overline{U}_1 \to PC$ is continuous and completely continuous. From the choice of U_1, there is no $y \in \partial U_1$ such that $y = \lambda G(y)$ for some $\lambda \in (0,1)$. As a consequence of the Leray-Schauder nonlinear alternative [19], we conclude that G has a fixed point y in U_1 that is a solution of the problem (1)–(3). □

Remark 3.1. *Notice that the function ψ_1 must be non-decreasing but it does not necessarily need to be continuous.*

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