

A Class of Oscillatory Singular Integrals

SHANZHEN LU ¹

*Department of Mathematics, Beijing Normal University
Beijing 100875, P.R.China
E-mail: lusz@bnu.edu.cn*

Abstract

This is a survey on a class of oscillatory singular integrals related closely to the Radon transform, which has important applications in the CT technology, based on a series of research works in the past twenty years. It includes their boundedness, the criterion of L^p -boundedness and some applications of the criterion to multilinear operators.

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0. Background

Many operators in harmonis analysis or partial differential equation are related to some versions of oscillatory integrals. Let us now consider the pseudo-differential operator

$$f \rightarrow \int_{R^n} e^{ix \cdot y} \sigma(x, y) \widehat{f}(y) dy,$$

where $\sigma \in S_{\rho, \delta}^m$. That means

$$|\partial_x^\beta \partial_y^\alpha \sigma(x, y)| \leq C_{\alpha, \beta} (1 + |y|)^{m - \rho|\alpha| + \delta|\beta|}.$$

If we take $g = \widehat{f}$, $m \leq 0$, $\rho \geq 0$ and $\delta = 0$, then the above pseudo-differential operator will become an oscillatory singular integral operator as follows

$$g \rightarrow \int_{R^n} e^{ix \cdot y} \sigma(x, y) g(y) dy,$$

where σ satisfies the condition of Calderón-Zygmund kernel. Let us now consider a convolution operator $(S * f)(\zeta, t)$ on the Heisenberg group $\mathbb{H}^n = C^n \times R$, where

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$\zeta \in C^n, t \in R$. If we apply the Fourier transform in t to $(S * f)(\zeta, t)$, then it will be the form of

$$\int_{R^n} e^{iB(x,y)} K(x, y) f(y) dy,$$

where $B(x, y)$ is a bilinear form and K is a Calderón-Zygmund kernel. It is easy to check that by a linear transform for x and y the above integral will be

$$\int_{R^n} e^{ix \cdot y} K(x, y) f(y) dy.$$

Moreover, it is well-known that the Radon transform, being an important role in the CT technology of medical sciences, is also closely related to this form of oscillatory singular integrals([34])

We will see this relation between the Radon transform and the above oscillatory singular integrals in Section 1. For oscillatory singular integrals and the Radon transform, we may refer to Stein's books [40], [41].

More general, let us now consider the oscillatory singular integral

$$\int_{R^n} e^{iP(x,y)} K(x, y) f(y) dy,$$

where P is a real valued polynomial and K is a Calderon-Zygmund kernel. This is just the object in this paper.

1. Boundedness

For simplicity, we denote the Calderón-Zygmund kernel by CZK. Let us first begin with the homogeneous CZK. K is said to be a homogeneous CZK if K satisfies (b):

(b) $K(x) = \Omega(x')/|x|^n$ with Ω homogeneous of degree 0 on S^{n-1} .

Since T is not a Calderón-Zygmund operator, we can not directly use the Calderón-Zygmund theory to study its L^p -boundedness. Also since T is not a convolution operator, we can not use Fourier transform method either. In fact, it went through several stages in the study of the boundedness of T .

(1) $P(x, y) = (Bx, y)$ is a bilinear form.

The first work was done by Phong and Stein [35] in this case.

Theorem 1.1 Let

$$Tf(x) = p.v. \int_{R^n} e^{iB(x,y)} K(x - y) f(y) dy,$$

where K satisfies (b),

$$(a) \quad K \in C^1(\mathbb{R}^n \setminus \{0\})$$

and

$$(c) \quad \int_{S^{n-1}} \Omega(x') d\sigma(x') = 0.$$

Then T is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$.

Let us now describe the outline of the proof in Theorem 1.1. Since one does not know whether T maps H^1 into L^1 , we can not use the standard theory of Calderón-Zygmund to deal with T . In fact, the authors in [35] introduced a variant of $H^1(\mathbb{R}^n)$, so called $H_E^1(\mathbb{R}^n)$:

$$H_E^1(\mathbb{R}^n) = \{f : f = \sum_k \lambda_k a_k, \text{ each } a_k \text{ is a } H_E^1 \text{ atom, } \sum_k |\lambda_k| < \infty\}.$$

A function $a(x)$ is called a H_E^1 atom if it satisfies

$$\text{supp } a \subset Q, \|a\|_\infty \leq 1/|Q|$$

and

$$\int a(y) e^{iB(x,y)} dy = 0.$$

Denote the dual space of $H_E^1(\mathbb{R}^n)$ by $BMO_E(\mathbb{R}^n)$. Then the authors in [35] proved that T maps $H_E^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and maps $L^\infty(\mathbb{R}^n)$ into $BMO_E(\mathbb{R}^n)$. Thus it follows from a interpolation that the conclusion of Theorem 1.1 holds.

It is easy to see that if T maps H^1 into L^1 then the proof of Theorem 1.1 will become very simple by a standard argument. Therefore an interesting problem will be formulated as follows. **Question 1.** $T: H^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$?

(2) General case for $P(x, y)$

When $P(x, y)$ is a polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, Ricci and Stein [36] obtained a celebrated result.

Theorem 1.2 Let

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y) dy,$$

where K satisfies (a),(b) and (c). Then T is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of the polynomial $P(x, y)$.

It should be pointed out that the proof in [36] is based on a series of Von der corput lemmas and the following estimate on polynomials. **Proposition 1.1** Let $P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ be a polynomial of degree d and let $\epsilon < 1/d$. Then

$$\int_{|x| < 1} |P(x)|^{-\epsilon} dx \leq C_\epsilon \left(\sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\epsilon},$$

where C_ϵ is independent of the coefficients of P .

(3) The case of $p = 1$.

In this case, Chanillo and Christ [3] gave a supplement.

Theorem 1.3 Let T be as in Theorem 1.2. Then

$$\|Tf\|_{WL^1} \leq C \|f\|_{L^1},$$

where $WL^1(R^n)$ is the weak $L^1(R^n)$ space and C is independent of coefficients of $P(x, y)$.

In addition, Sato [38] gave the weighted result of Theorem 1.3.

Note that we need the smooth condition (a) in the above theorems. It naturally raise the following problem.

Question 2. Is the smooth condition (a) in Theorem 1.2 and 1.3 necessary ?

(4) Part answer for Question 1 when $P(x, y) = P(x - y)$. In this case, T is a convolution operator. Hu-Pan [13] gave the following result by using Fourier transforms.

Theorem 1.4 Let

$$Tf(x) = p.v. \int_{R^n} e^{iP(x-y)} K(x-y) f(y) dy,$$

where K satisfies (a), (b) and (c), and $P(u)$ is a polynomial defined on R^n . Then $T : H^1(R^n) \rightarrow H^1(R^n)$ with the norm of T independent of coefficients of P .

(5) Answer for Question 2 with $1 < p < \infty$. We will prove that the smooth condition (a) on kernel K in Theorem 1.2 can be replaced by certain non-smooth condition. K is called a rough kernel if K satisfies (b), (c) and (d) $\Omega \in \bigcup_{q > 1} L^q(S^{n-1})$.

In this case, Lu and Zhang [28] obtained

Theorem 1.5 Let

$$Tf(x) = p.v. \int_{R^n} e^{iP(x,y)} K(x-y)f(y) dy,$$

where $P(x, y)$ is a polynomial on $R^n \times R^n$, K satisfies (b),(c) and (d). Then $T : L^p(R^n) \rightarrow L^p(R^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of $P(x, y)$.

It should be pointed out that the proof of Theorem 1.5 is based on a criterion on L^p -boundedness of T stated in Section 2.

The condition on $\Omega \in L^q(S^{n-1})$ in Theorem 1.5 was weakened into $\Omega \in L \log^+ L(S^{n-1})$ by Jiang and Lu [16] in 1995.

Theorem 1.6 Let T be as in Theorem 1.5, where K satisfies (b),(c) and

$$(e) \quad \Omega \in L \log^+ L(S^{n-1}).$$

Then $T : L^p(R^n) \rightarrow L^p(R^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of $P(x, y)$.

Note that the condition (e) is sharp in the following sense: (e) can not be replaced by any $\Omega \in L(\log^+ L)^\epsilon(S^{n-1})$ with $0 < \epsilon < 1$. In addition, Ojanen [30] established the weighted result of Theorem 1.6.

It should be pointed out that for T being as in Theorem 1.4 where $P(x, y) = P(x - y)$, the conclusion of Theorem 1.6 was also obtained by Al-Salman and Pan [1] in a different way.

Recently, (e) was weakened into $\Omega \in H^1(S^{n-1})$ by Fan and Pan ([11]) when $P(x, y) = P(x - y)$. For general $P(x, y)$, it is still an open problem.

Theorem 1.7 Let T be as in Theorem 1.4, where $P(x, y) = P(x - y)$, K satisfies (b),(c) and

$$(f) \quad \Omega \in H^1(S^{n-1}).$$

Then $T : L^p(R^n) \rightarrow L^p(R^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of P . Let us turn to the idea of the proof in Theorem 1.7. Let

$$Tf(x) = p.v. \int_{R^n} e^{iP(x-y)} K(x-y)f(y)dy,$$

where K satisfies (b), (c) and (f). In fact, the proof of Theorem 1.7 does not begin with dealing with T , but begin with a singular integral operator with kernel supported on a subvariety defined by

$$T_{\mathcal{P}}f(x) = p.v. \int_{R^n} K(y)f(x - \mathcal{P}(y))dy,$$

where $x \in R^d$, $\mathcal{P} = (P_1, P_2, \dots, P_d)$ is a mapping from R^d into R^n with P_j being polynomials in $y \in R^n$. The authors in [11] first proved that under the conditions on K in Theorem 1.7 $T_{\mathcal{P}}$ satisfies

$$\|T_{\mathcal{P}}f\|_{L^p(R^d)} \leq C\|\Omega\|_{H^1(S^{n-1})}\|f\|_{L^p(R^d)},$$

where C is independent of the coefficients of P_j . Then let $d = n + 1$, $x = (x_1, x_2, \dots, x_n)$ and $\mathcal{P}(x) = (x, P(x))$. For $f \in L^p(R^n)$ and $N > 0$, let the function F_N on R^{n+1} be

$$F_N(x, x_{n+1}) = f(x)e^{-ix_{n+1}}\chi_{[-N, N]}(x_{n+1}).$$

Applying the above inequality with F_N instead of f , we then get

$$\left\| \int_{R^n} K(y)F_N(x - y, x_{n+1} - P(y))dy \right\|_{L^p(R^{n+1})} \leq C\|F_N\|_{L^p(R^{n+1})}.$$

Finally the conclusion of Theorem 1.7 will be obtained by the last inequality as $N \rightarrow \infty$. This also shows the close relation between the Radon transform and the corresponding oscillatory singular integral.

Of course, the following problem is still open up to now.

Question 3. Does T map L^p into itself for general $P(x, y)$ when K satisfies (b),(c) and (f) ?

Let us now turn to nonhomogeneous CZK. $K(x, y)$ is said to be a standard CZK if it satisfies

$$(i) \quad |K(x, y)| \leq \frac{C}{|x - y|^n}, x \neq y;$$

and

$$(ii) \quad |\nabla_x K(x, y)| + |\nabla_y K(x, y)| \leq \frac{C}{|x - y|^{n+1}}.$$

Let

$$Tf(x) = p.v. \int_{R^n} e^{iP(x, y)} K(x, y)f(y)dy$$

and

$$\tilde{T}f(x) = p.v. \int_{R^n} K(x, y)f(y)dy.$$

Ricci and Stein [36] use L^2 -boundedness of \tilde{T} to get L^p -boundedness of T .

Theorem 1.8. If \tilde{T} is bounded on $L^2(R^n)$, then T is bounded on $L^p(R^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of $P(x, y)$. Using the interpolation theorem of operators with change of measures due to Stein and Weiss [42], Lu and Zhang [27] obtained the weighted result of Theorem 1.8.

Theorem 1.9 If $\omega \in A_p, 1 < p < \infty$ and \tilde{T} is bounded on $L^2(R^n)$, then T is also bounded on $L^p_\omega(R^n)$, where A_p is the Muckenhoupt class and the norm of T is independent of coefficients of $P(x, y)$. **Remark 1.1** For real Hardy spaces $H^p(R^n)$ with $0 < p < 1$, Pan [32] gave a counter example to show that there is a non-trivial polynomial $P(x, y)$ such that the corresponding oscillatory singular integral operator T is unbounded on $H^p(R^n)$. For the case $p = 1$, as we mentioned above, we do not know whether T maps H^1 into L^1 . As a substitution of positive results in the case $p = 1$, Lu and Yang [26] showed that T maps Herz type Hardy spaces into Herz spaces. Moreover, as a development of Lu-Yang's result above to the case $0 < p < 1$, Li [18] gave a systematic work.

Remark 1.2 The oscillatory singular integral operator T in Theorem 1.5 and 1.6 can be replaced by a more general form of

$$Tf(x) = p.v. \int_{R^n} e^{iP(x,y)} K(x-y)h(|x-y|)f(y)dy,$$

where $h \in BV(R_+)$ the class of bounded variation functions on $[0, \infty)$. The corresponding truncated operator \tilde{T}_0 will be replaced by

$$\tilde{T}_0f(x) = p.v. \int_{|x-y|<1} K(x-y)h(|x-y|)f(y)dy.$$

2. Criterion on L^p -boundedness

Let

$$Tf(x) = p.v. \int_{R^n} e^{iP(x,y)} K(x-y)f(y) dy$$

and

$$\tilde{T}f(x) = p.v. \int_{R^n} K(x-y)f(y) dy.$$

A quite interesting problem is what about the relation between the boundedness of T and one of \tilde{T} .

Note that when $P(x, y) = P_0(x) + P_1(y)$,

$$T : L^p(R^n) \rightarrow L^p(R^n) \iff \tilde{T} : L^p(R^n) \rightarrow L^p(R^n) \quad (1 < p < \infty).$$

However, for nontrivial polynomial $P(x, y)$, that means, $P(x, y)$ does not take the form of $P_0(x) + P_1(y)$, the above conclusion is false.

Lu and Zhang in [28] get a convenient criterion to judge whether T is bounded on L^p .

Theorem 2.1 Let

$$Tf(x) = p.v. \int_{R^n} e^{iP(x,y)} K(x-y)f(y) dy$$

and

$$\widetilde{T}_0 f(x) = p.v. \int_{|x-y|<1} K(x-y)f(y) dy,$$

where $P(x, y)$ is a nontrivial polynomial, K satisfies (b) and (d). Then for any p with $1 < p < \infty$, T is bounded on $L^p(\mathbb{R}^n)$ if and only if \widetilde{T}_0 is bounded on $L^p(\mathbb{R}^n)$.

It should be pointed out that we use the rotation method due to Calderón and Zygmund [2] in the proof of Theorem 2.1.

Also we point out that the proof of Theorem 1.6 is based on the following criterion.

Theorem 2.2 Let T and \widetilde{T}_0 be as in Theorem 2.1, where K satisfies (b) and (e). Then the conclusion in Theorem 2.1 holds (see [16]). It will be worth mentioning that we do not need the cancellation condition (c) in Theorem 2.1 and Theorem 2.2. Note that the idea that the boundedness of oscillatory singular integral operator with polynomial phase is reduced to that of the corresponding truncated operator as in Theorem 2.1 and Theorem 2.2 is also applied to studying the boundedness of some oscillatory singular integral operators with non-polynomial phase. Let

$$Tf(x) = p.v. \int_{\mathbb{R}} e^{i|x|^\alpha|y|^\beta} K(x, y)f(y) dy$$

and

$$\widetilde{T}_0 f(x) = p.v. \int_{|x-y|<1} K(x, y)f(y) dy,$$

where $K(x, y)$ is a standard Calderón-Zygmund kernel and $\alpha, \beta \geq 1$. Pan and Sampson [33] proved that if \widetilde{T}_0 is bounded on $L^2(\mathbb{R})$, then T is a bounded operator from $H_E^1(\mathbb{R})$ to $L^1(\mathbb{R})$ (see the definition of $H_E^1(\mathbb{R})$ in Section 1). In addition, Sampson, Xu and Yang [37] also use the L^2 boundedness of \widetilde{T}_0 to show that T is a bounded operator from certain Herz space to Herz-type Hardy space.

Some recent results on oscillatory singular integrals will involve kernels related to certain block spaces. The spaces generated by blocks was first introduced by Taibleson-Weiss [43] in one dimension and by Lu-Taibleson-Weiss [22] in high dimension in the study of almost everywhere convergence of Fourier series and the Bochner-Riesz means at critical index respectively. Later, block spaces were applied to many subjects in harmonic analysis (see [29], [19],[20], [39] or [23]).

A q -block on S^{n-1} is an $L^q(1 < q \leq \infty)$ function $b(x)$ that satisfies

$$(i) \quad \text{supp } b \subset Q; (ii) \quad \|b\|_q \leq |Q|^{1/q-1},$$

where $Q = S^{n-1} \cap \{y \in R^n : |y - \xi| < \rho, \text{ for some } \xi \in S^{n-1}, 0 < \rho \leq 1\}$. The following block space $B_q^{0,\nu}(S^{n-1})$ was introduced by Jiang-Lu [15] in the study of the boundedness of singular integral operators (see [23]):

$$B_q^{0,\nu}(S^{n-1}) = \left\{ \Omega \in L^1(S^{n-1}) : \Omega(y') = \sum_k c_k b_k(y'), M_q^{0,\nu} < \infty \right\},$$

where each c_k is a complex number, each b_k is a q -block supported on Q_k , and

$$M_q^{0,\nu} = \sum_k |c_k| \left\{ 1 + \left(\log^+ \frac{1}{|Q_k|} \right)^{\nu+1} \right\}.$$

In 1994, Keitoku and Sato [17] showed that for any $q > 1$, the inclusion $\cup_{r>1} L^r(S^{n-1}) \subset B_q^{0,0}(S^{n-1})$ is strict. In addition, the relationship between $B_q^{0,0}(S^{n-1})$ and $L \log^+ L(S^{n-1})$ remains open. And even the relationship between $B_q^{0,0}(S^{n-1})$ and $H^1(S^{n-1})$ also remains open.

Recently, Hussain and Pan [14] obtained **Theorem 2.3** Let T be as in Theorem 1.4 where $P(x, y) = P(x - y)$. If K satisfies (b), (c) and

$$(g) \quad \Omega \in B_q^{0,0}(S^{n-1}), q > 1,$$

then T is bounded on $L^p(R^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of P .

Note that T is a convolution operator in Theorem 2.3. Thus the authors in [14] can employ Fourier transform in the proof. However, for non-trivial polynomial $P(x, y)$, the method in [14] does not work. As in [28], the idea of Lu and Wu [24] is first to establish a criterion on the L^p - boundedness of T by employing the rotation method together with some delicate estimates on polynomials, and then the following result on boundedness will follows directly from the criterion.

Theorem 2.4 Let T and \tilde{T}_0 be as in Theorem 2.1 and K satisfy (b) and (g). Then we have (i) For any a non-trivial polynomial $P(x, y)$, T is bounded on $L^p(R^n)$ if and only if \tilde{T}_0 is bounded on $L^p(R^n)$; (ii) If K also satisfies (c), then T is bounded on $L^p(R^n)$, $1 < p < \infty$, with the norm of T independent of coefficients of $P(x, y)$.

Remark 2.1 As in Remark 1.2, the oscillatory singular integral operator T and the corresponding truncated operator \tilde{T}_0 in Theorem 2.1, Theorem 2.2 and Theorem 2.4 can be replaced by

$$Tf(x) = p.v. \int_{R^n} e^{iP(x,y)} K(x - y)h(|x - y|)f(y)dy$$

and

$$\tilde{T}_0 f(x) = p.v. \int_{|x-y|<1} K(x-y)h(|x-y|)f(y)dy$$

respectively, where $h \in \text{BV}(R_+)$.

Remark 2.2 It should be pointed out that the following precise estimate on polynomials will play a crucial role in proving Theorem 2.4. It can be also regarded as a preciseness of Proposition 1.1. **Proposition 2.1** Suppose $P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ is a polynomial of degree d and let $\epsilon < 1/2d$, then

$$\int_{|x| \leq 1} |P(x)|^{-\epsilon} dx \leq CA^\epsilon \left(\sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\epsilon},$$

where C, A depend only on the dimension n and the total degree of $P(x)$, but not on the coefficients of P .

2. Generalizations and Applications of the Criterion.

The proof of Theorem 1.5 will give us an inspiration. To study the boundedness of some oscillatory singular integral with polynomial phase, we prefer first study the relation between this oscillatory singular integral and its truncated singular integral. This idea was first applied to study a class of multilinear oscillatory singular integrals which is closely related to commutators. In fact, in the case of singular integrals, multilinear singular integrals are generalization of commutators of singular integrals (see [8]).

Let us consider the commutator $[T, A]$ generated by T and a BMO function A , where

$$Tf(x) = p.v. \int_{R^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Further, inspired by the idea of Meyer, let us now pose a class of multilinear oscillatory integral operators as follows:

$$T_A f(x) = p.v. \int_{R^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y) dy,$$

where $R_m(A; x, y)$ denotes the m -th ($m \geq 2$) remainder of Taylor series of A at x about y , more precisely,

$$R_m(A; x, y) = A(x) - \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha A(y) (x-y)^\alpha,$$

where $D^\alpha A \in BMO(R^n)$ for all multi-indices $|\alpha| = m - 1$. Obviously, if $m = 1$, then the multilinear oscillatory integral operator is just the commutator $T_A = [T, A]$.

More general, let us consider the following multilinear oscillatory integral

$$T_{A_1, A_2} f(x) = p.v. \int_{R^n} e^{iP(x, y)} \frac{\Omega(x - y)}{|x - y|^{n+M-1}} \prod_{j=1}^2 R_{m_j}(A_j; x, y) f(y) dy,$$

where $M = m_1 + m_2$, Ω is homogeneous of degree zero. Suppose that A_1 has derivatives of order $m_1 - 1$ in $BMO(R^n)$, A_2 has derivatives of order m_2 in $L^{r_0}(R^n)$, $1 < r_0 \leq \infty$. Then Chen-Hu-Lu [6] gave a necessary and sufficient condition for the rough multilinear oscillatory integral operator T_{A_1, A_2} to enjoy (L^p, L^r) boundedness or enjoy (L^p, L^p) boundedness respectively.

Theorem 3.1 Let Ω be homogeneous of degree zero and satisfies (d). If A_1 and A_2 are defined as above, then for $1/r = 1/p + 1/r_0$, $1 < p, r < \infty$, the following two facts are equivalent:

(i) If $P(x, y)$ is a nontrivial real-valued polynomial, then

$$T_{A_1, A_2} : L^p(R^n) \rightarrow L^r(R^n).$$

(ii) The truncated operator

$$S_{A_1, A_2} : L^p(R^n) \rightarrow L^r(R^n),$$

where

$$S_{A_1, A_2} f(x) = \int_{|x-y|<1} \frac{\Omega(x - y)}{|x - y|^{n+M-1}} \prod_{j=1}^2 R_{m_j}(A_j; x, y) f(y) dy.$$

It should be pointed out that Wu [44] obtained the weighted result of Theorem 3.1 and a direct application of the weighted result to the weighted boundedness of T_{A_1, A_2} under the additional conditions: $\Omega \in Lip_1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(\theta) \theta^\gamma d\theta = 0, \quad \text{for all } |\gamma| < M = m_1 + m_2.$$

The following result is also included in [6].

Theorem 3.2 Let $1 < p < \infty$, Ω be the same as that in Theorem 3.1. Suppose A_i have derivatives of order $m_i - 1$ in $BMO(R^n)$ respectively, $i = 1, 2$. Then the following two facts are equivalent:

(i) If $P(x, y)$ is a nontrivial real-valued polynomial, then T_{A_1, A_2} is bounded on $L^p(\mathbb{R}^n)$ with the bound

$$C \left(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{BMO} \right) \left(\sum_{|\beta|=m_2-1} \|D^\beta A_2\|_{BMO} \right).$$

(ii) The truncated operator S_{A_1, A_2} is bounded on $L^p(\mathbb{R}^n)$ with the bound

$$C \left(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{BMO} \right) \left(\sum_{|\beta|=m_2-1} \|D^\beta A_2\|_{BMO} \right).$$

Remark 3.1 In fact, under the conditions of Theorem 3.1 and Theorem 3.2 together with (c), it follows from Theorem 3.1 and Theorem 3.2 that T_{A_1, A_2} enjoys (L^p, L^r) boundedness and (L^p, L^p) boundedness respectively. However, Chen-Hu-Lu previously obtained such results by different arguments (see [4], [5]).

It should be pointed out that the weighted result of Theorem 3.2 was also obtained by Ding-Lu-Yang [10].

Futhermore, Motivated by the idea of Paluszynski's paper [31], Chen and Lu [7] studied the relation between the boundedness of T_{A_1, A_2} and the smoothness of $A_i (i = 1, 2)$.

Theorem 3.3 Suppose that A_1 and A_2 are functions with derivatives of order $m_1 - 1$ and $m_2 - 1$ respectively in Besov-Lipschitz class $\dot{\Lambda}_\beta, 0 < \beta < 1$. For $1 < p < q < \infty, 1/q = 1/p - 2\beta/n$ and $0 < \beta < 1$, then

$$\|T_{A_1, A_2} f\|_q \leq C \prod_{j=1}^2 \|\nabla^{m_j-1} A_j\|_{\dot{\Lambda}_\beta} \|f\|_p.$$

Here the Besov-Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the k^{th} difference operator.

Let us point out that Lu and Wu [25] weaken the condition on Ω in Theorem 3.1 by using $\Omega \in B_q^{0,1}(S^{n-1})$ to replace (d) since Keitoku and Sato [17] have pointed out

$$\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,1}(S^{n-1}) \subset B_q^{0,0}(S^{n-1}).$$

Under other more weak condition on Ω than (d), we do not know whether the conclusion of Theorem 3.1 hold. Thus we have **Question 4**. Can the condition

(d) in Theorem 3.1 be improved into (e) ,(f) or (g) ?

Let us now turn to the standard Calderón-Zygmund kernel $K(x, y)$. Consider

$$T_{A_1, A_2} f(x) = p.v. \int_{R^n} e^{iP(x,y)} \frac{K(x, y)}{|x - y|^{M-1}} \prod_{j=1}^2 R_{m_j}(A_j; x, y) f(y) dy,$$

where $K(x, y)$ is a standard CZK. Lu [21] also established a necessary and sufficient condition for the multilinear oscillatory integral operator with standard CZK to enjoy (L^p, L^r) boundedness. **Theorem 3.4** Let A_1 and A_2 are the same

as in Theorem 3.1. Then for any $1 < p, r < \infty$ with $1/r = 1/p + 1/r_0$, the following two facts are equivalent:

(i) If $P(x, y)$ is a nontrivial polynomial, then

$$T_{A_1, A_2} : L^p(R^n) \rightarrow L^r(R^n).$$

(ii) The truncated operator

$$S_{A_1, A_2} : L^p(R^n) \rightarrow L^r(R^n),$$

where

$$S_{A_1, A_2} = p.v. \int_{|x-y|<1} \frac{K(x, y)}{|x - y|^{M-1}} \prod_{j=1}^2 R_{m_j}(A_j; x, y) f(y) dy.$$

Note that if we come to the case of T_A , where

$$T_A f(x) = p.v. \int_{R^n} e^{iP(x,y)} \frac{K(x, y)}{|x - y|^m} R_{m+1}(A; x, y) f(y) dy,$$

then we can get a criterion for T_A to enjoy L^p boundedness. Using this criterion, Yan [46] gave an easily verifiable sufficient condition for T_A to be bounded on $L^p(R^n)$.

Let the singular integrals \widetilde{T}_A and K_α are defined by

$$\widetilde{T}_A f(x) = p.v. \int_{R^n} \frac{K(x, y)}{|x - y|^m} R_{m+1}(A; x, y) f(y) dy$$

and

$$K_\alpha f(x) = p.v. \int_{R^n} \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} f(y) dy, \quad |\alpha| = m,$$

respectively. **Theorem 3.5** Let A have derivative of order m in $BMO(R^n)$ and

$K(x, y)$ satisfy (i) and (ii). If \widetilde{T}_A and K_α are bounded on $L^2(R^n)$, then T_A is bounded on $L^p(R^n)$ with $1 < p < \infty$.

Note that Theorem 3.5 is somewhat similar to Theorem 1.8. Let us come back to a simple case of multilinear oscillatory singular integral operators with rough kernel. Let

$$T_A f(x) = p.v. \int_{R^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y) dy.$$

Then it is easy to get a criterion for T_A to enjoy L^p boundedness. Ding-Lu-Yang [10] extended the criterion on L^p boundedness of T_A into weighted case. Let us now only formulate this weighted result.

Theorem 3.6 Suppose that Ω satisfies (b) and (d), and A has derivatives of order $m-1$ ($m \geq 2$) in $BMO(R^n)$. If p, q and ω satisfy one of the following conditions:

- (a) $q' < p < \infty$, and $\omega \in A_{p/q'}$;
- (b) $1 < p < q$, and $\omega^{-1/(p-1)} \in A_{p'/q'}$;
- (c) $1 < p < \infty$, and $\omega^{q'} \in A_p$,

where A_p is the Muckenhoupt class. Then the following two statements are equivalent:

- (i) If $P(x, y)$ is a nontrivial real-valued polynomial, then

$$\|T_A f\|_{p,\omega} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{BMO} \|f\|_{p,\omega}.$$

- (ii) The truncated operator

$$S_A f(x) = p.v. \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y) dy$$

satisfies

$$\|S_A f\|_{p,\omega} \leq C \sum_{|\alpha|=m-1} \|D^\alpha A\|_{BMO} \|f\|_{p,\omega}.$$

Here the constant C depend only on n, p, q and the A_p constant of ω , the total degree of the polynomial $P(x, y)$. Inspired by Ding-Lu-Yang's paper [10], Wu [45] got a similar weighted result for T_A where $D^\gamma A \in L^r(R^n)$, $|\gamma| = m$ and $1 < r < \infty$.

For some non-standard oscillatory singular integrals, criterions on thier bound- edness have be also established. Precisely, Ding and Lu [9] even established a necessary and sufficient condition for higher order commutators of rough oscil- latory integral operators to enjoy weighted L^p -boundedness.

Theorem 3.7 Let $1 < p < \infty$, Ω satisfy (b),(c) and (e), $h(|x|) \in BV(R_+)$, $b(x) = b(|x|) \in BMO(R_+)$, and $\omega^2 = \omega(|x|)^2 \in A_p(R_+)$. Then the following two statements are equivalent:

(i) If $P(x, y)$ is a nontrivial polynomial, then the higher order commutator

$$T_b^m f(x) = p.v. \int e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

(ii) The truncated operator

$$S_{b,0}^m f(x) = p.v. \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

In addition, Ding and Lu [9] also established a result on the weighted L^p boundedness of T_b^m by using the weighted L^p boundedness of the singular integral operator \tilde{T} , which is defined by

$$\tilde{T}f(x) = p.v. \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy.$$

The above singular integral operator was first introduced by Fefferman [12].

Theorem 3.8 Let p, Ω, h, b, m and ω be as in Theorem 3.7. If \tilde{T} is bounded on $L^p(\omega)$, then T_b^m is also bounded on $L^p(\omega)$.

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