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On the Global Character of $x_{n+1} = \dfrac{\beta x_n + \delta x_{n-k}}{A + x_{n-k}}$

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Abstract

We investigate the global character of solutions of the difference equation in the title with nonnegative parameters, nonnegative initial conditions, and where k is a positive integer.

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1. Introduction and Preliminaries

Consider the difference equation

$$x_{n+1} = \frac{\beta x_n + \delta x_{n-k}}{A + x_{n-k}}, \qquad n = 0, 1, \dots, \tag{1}$$

where k is a positive integer, the parameters $\beta,\ \delta$, and A are non-negative real numbers, and the initial conditions $x_{-k},\ x_{-k+1}, \dots,\ x_0$ are arbitrary non-negative real numbers such that the denominator of Eq.(1) is never zero.

The case $k = 1$ was investigated in [5]. See also [4]. In this case, when $A \geq \beta + \delta$ all solutions converge to the zero equilibrium and when $A < \beta + \delta$ all positive solutions converge to the positive equilibrium $\bar{x} = \beta + \delta - A$.

When $\beta = 0$, Eq.(1) is of the Riccati type, every solution of which converges to the zero equilibrium when $A \geq \delta$, and all solutions with positive initial conditions converge to the positive equilibrium $\bar{x} = \delta - A$ when $0 < A < \delta$.

When $\delta = 0$, Eq.(1) reduces to Pielou's discrete delay logistic model. This equation was investigated in [6] where it was shown that the positive equilibrium $\bar{x} = \beta - A$ is globally asymptotically stable when $(\beta - A)(k-1) \leq A$. It is also easy to see that the zero equilibrium is globally asymptotically stable when $\beta \leq A$.

When $A = 0$, the change of variables $x_n = \beta y_n$ reduces Eq.(1) to the equation

$$y_{n+1} = a + \frac{y_n}{y_{n-k}}$$

where $a = \delta/\beta$. This equation has been investigated in [3] where it was shown that the positive equilibrium $\bar{y} = 1 + a$ is globally asymptotically stable for the case where $k = 1$. The case $k \in \{2, 3, \ldots\}$ was investigated in [1], where it was shown that the positive equilibrium $\bar{y} = 1 + a$ is globally asymptotically stable when $a > 1$.

Thus, we may assume that the parameters β, δ, and A are all positive hereafter. The change of variables

$$x_n = \delta y_n$$

reduces Eq.(1) to the difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{q + y_{n-k}}, \qquad n = 0, 1, \ldots, \tag{2}$$

with $k \in \{2, 3, \ldots\}$ and with positive parameters and non-negative initial conditions.

Our goal is to investigate the global stability of the solutions of Eq.(2).

We now present some known results which will be useful in our investigation of Eq.(2).

The following theorem, which was given by C. Clark in [2], provides a sufficient condition for local asymptotic stability of the equilibria of Eq.(2).

Theorem A (Clark's Theorem)

Consider the difference equation

$$z_{n+1} + a_k z_n + \ldots + a_0 z_{n-k} = 0, \quad n = 0, 1, \ldots, \tag{3}$$

where $k \in \{1, 2, \ldots\}$ and a_i real numbers for all i. Then

$$\sum_{i=0}^{k} |a_i| < 1$$

is a sufficient condition for the asymptotic stability of Eq.(3).
The next theorem will be useful in establishing the global stability of the equilibria. See [4].
Theorem B
Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots \tag{4}$$

where $k \in \{1, 2, \ldots\}$. Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(u, v)$ *is non-decreasing in* u *and* v.

(b) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$M = f(M, M) \qquad \text{and} \qquad m = f(m, m)$$

then $M = m$.

Then Eq.(4) has a unique equilibrium $\bar{y} \in [a, b]$, and every solution of Eq.(4) converges to $\bar{y}$.
The next result about the global attractivity of the positive equilibrium has been established in [6] and [7].
Theorem C
Consider the difference equation

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-k}}, \quad n = 0, 1, \ldots, \tag{5}$$

where

$$a, b \in [0, \infty) \quad \text{with} \quad a + b > 0,\ A \in (0, \infty), \quad \text{and} \quad k \in \{1, 2, \ldots\}.$$

Then the positive equilibrium $\bar{x}$ of Eq.(5) is a global attractor of all nontrivial solutions of Eq.(5) in the following cases:

(i) $a > 0$ *and* $A > b > 0$;

(ii) $b > 0, \quad k \geq 2, \quad a \leq Ab$, *and* $\bar{x}(k - 1) \leq A$.

Note that, for initial conditions that are not all zero we have an eventually positive solution of Eq.(2). Without loss of generality, in the sequel we need to consider only the positive solutions of Eq.(2).

2. Attracting Invariant Intervals

In this section we show that when $q > p$, the interval $[0, 1]$ is a globally attracting invariant interval for all solutions of Eq.(2). We also show that when $q < p$, the interval $(1, \infty)$ is a globally attracting invariant interval for all solutions of Eq.(2).
The equilibrium points of Eq.(2) are the non-negative solutions of the equation

$$\bar{y} = \frac{p\bar{y} + \bar{y}}{q + \bar{y}}.$$

Clearly, zero is always an equilibrium point of Eq.(2). However, when $q < p+1$, Eq.(2) also has the unique positive equilibrium point $\bar{y} = p+1-q$. The following lemma, the proof of which is straightforward and will be omitted, exhibits two identities that will be useful in the study of Eq.(2).

Lemma 0.1 *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2). Then for $n \geq 0$ the following two identities hold:*

$$y_{n+1} - 1 = \frac{p(y_n - \frac{q}{p})}{q + y_{n-k}} \tag{6}$$

$$y_{n+1} - y_n = \frac{(p-q)y_n + (1-y_n)y_{n-k}}{q + y_{n-k}} \tag{7}$$

The following lemma establishes the existence of invariant intervals for solutions of Eq.(2).

Lemma 0.2 *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2). Then the following statements are true.*

(a) Suppose that $q > p$ and that there exists $N \geq 0$ such that $y_N \leq 1$. Then $y_n < 1$ for all $n > N$.

(b) Suppose that $q = p$.

(i) If $y_0 < 1$, then $y_n < 1$ for all $n \geq 0$.

(ii) If $y_0 = 1$, then $y_n = 1$ for all $n \geq 0$.

(iii) If $y_0 > 1$, then $y_n > 1$ for all $n \geq 0$.

(c) Suppose that $q < p$ and that there exists $N \geq 0$ such that $y_N \geq 1$. Then $y_n > 1$ for all $n > N$.

Proof: We will prove (a). The proofs of (b) and (c) are similar and will be omitted. From Eq.(6) we see that $y_N \leq 1 < \frac{q}{p}$ and so $y_{N+1} < 1$. The result now follows by induction.

The next result gives conditions for the existence of globally attracting intervals for the solutions of Eq.(2).

Theorem 0.3 *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2). Then the following statements are true.*

(a) Suppose that $q > p$. Then there exists $N > 0$ such that $y_n \leq 1$ for all $n \geq N$.

(b) Suppose that $q < p$. Then there exists $N > 0$ such that $y_n > 1$ for all $n \geq N$.

Proof: We will prove (a). The proof of (b) is similar and will be omitted. Assume for the sake of contradiction that (a) is false. By Lemma (0.2), it must be true that $y_n > 1$ for all $n \geq 0$. Since $q > p$ it follows from Eq.(7) that the solution is decreasing. Hence $\lim_{n\to\infty} y_n$ exists and lies in the interval $[1, \infty)$. Thus,

$$\lim_{n\to\infty} y_n = \bar{y} = p + 1 - q \geq 1.$$

From here it follows that $q \leq p$ which is a contradiction. The proof is complete.

3. The Stability Character of the Zero Equilibrium

In this section we investigate the stability of the zero equilibrium of Eq.(2). The linearized equation of Eq.(2) with respect to the zero equilibrium is

$$z_{n+1} - \frac{p}{q} z_n - \frac{1}{q} z_{n-k} = 0, \qquad n = 0, 1, \ldots,$$

with associated characteristic equation

$$\lambda^{k+1} - \frac{p}{q}\lambda^k - \frac{1}{q} = 0.$$

Lemma 0.4 *When $q > p + 1$, the zero equilibrium of Eq.(2) is locally asymptotically stable, while if $q < p + 1$ it is unstable. In the case $q = p + 1$ the zero equilibrium is stable.*

Proof: It follows by Theorem A that the zero equilibrium is locally asymptotically stable when

$$q > p + 1.$$

Set

$$f(\lambda) = \lambda^{k+1} - \frac{p}{q}\lambda^k - \frac{1}{q}.$$

Note that $f(1) = \frac{q-(p+1)}{q} < 0$ when $q < p + 1$, and so f has a root with absolute value greater than 1. Thus, the zero equilibrium is unstable when $q < p + 1$.

Now assume $q = p + 1$. Let $\epsilon > 0$, and let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of Eq.(2). Letting $\delta = \epsilon$, if the initial conditions are such that

$$y_{-k},\ y_{-k+1}, \ldots,\ y_{-1},\ y_0 < \delta,$$

we have

$$0 \le y_1 = \frac{py_0 + y_{-k}}{p + 1 + y_{-k}} \le \frac{py_0 + y_{-k}}{p+1} < \frac{p\delta + \delta}{p+1} = \delta = \epsilon.$$

The proof follows by induction.

The next result gives the global stability character of the zero equilibrium of Eq.(2).

Theorem 0.5 *Assume that $q \ge p + 1$. Then the zero equilibrium of Eq.(2) is globally asymptotically stable.*

Proof: It suffices to show that the zero equilibrium is a global attractor of all solutions of Eq.(2). Since $q > p$, Theorem (0.3) (a) implies that the interval $(0, 1)$ attracts all solutions. Thus we may assume without loss of generality that $y_n < 1$ for all $n \ge -k$. Set

$$f(u, v) = \frac{pu + v}{q + v}.$$

Then it is easy to see that f satisfies the hypothesis of Theorem B on the interval $[0, 1]$. The proof is complete.

4. Stability of the Positive Equilibrium

Recall that the unique positive equilibrium $\bar{y} = p+1-q$ of Eq.(2) exists when $q < p + 1$. We first establish the local stability character of the positive equilibrium. The linearized equation of Eq.(2) with respect to the positive equilibrium is

$$z_{n+1} - \frac{p}{p+1} z_n + \frac{p-q}{p+1} z_{n-k} = 0, \qquad n = 0, 1, \ldots,$$

with associated characteristic equation

$$\lambda^{k+1} - \frac{p}{p+1}\lambda^k + \frac{p-q}{p+1} = 0.$$

The following lemma is a consequence of Theorem A and the conditions given in [8].

Lemma 0.6 *Suppose that $q < p + 1$. Then in each of the following cases, the positive equilibrium of Eq.(2) is locally asymptotically stable:*

(a) $q > p - 1$;

(b) $q \ge \frac{(k-1)p-1}{k}$.

We now discuss the global stability of the positive equilibrium of Eq.(2).

Theorem 0.7 *Assume that $q < p + 1$. Then the positive equilibrium of Eq.(2) is globally asymptotically stable in the following cases:*

(a) $q > p - 1$;

(b) $q \geq \frac{(k-1)p-1}{k}$.

Proof: It suffices to show that the positive equilibrium is a global attractor of all solutions of Eq.(2).
Case(i): Suppose that $p < q < p + 1$. Since $q > p$, Theorem (0.3) (a) implies that the interval $(0, 1)$ attracts all solutions. So we may assume without loss of generality that $y_n < 1$ for all $n \geq -k$. Let

$$L = \min\{y_{-k}, y_{-k+1}, \ldots, y_0, \bar{y} = p + 1 - q\}$$

and note that $0 < L \leq y_n < 1$ for all $n \geq -k$. Set

$$f(u, v) = \frac{pu + v}{q + v}.$$

Then it is easy to see that f satisfies the hypothesis of Theorem B on the interval $[L, 1]$.

Case(ii): Suppose that $q = p$. Note that, in this case $\bar{y} = 1$. From Lemma (0.2)(b) it is obvious that if $y_0 = 1$, then $\lim_{n\to\infty} y_n = 1$. In the case $y_0 \neq 1$, Eq.(7) shows that the solutions $\{y_n\}_{n=-k}^{\infty}$ are monotonic. Therefore, from Lemma (0.2)(b) we obtain $\lim_{n\to\infty} y_n = 1$.

Case(iii): Suppose that $q < p$. In this case $\bar{y} > 1$ and from Theorem (0.3)(b) we see that $(1, \infty)$ is an invariant and attracting interval for solutions of Eq.(2). Without loss of generality, we can assume $y_n > 1$ for all $n \geq -k$.
Introducing the substitution

$$y_n = 1 + w_n,$$

in Eq.(2), we obtain the difference equation

$$w_{n+1} = \frac{(p - q) + pw_n}{(q + 1) + w_{n-k}}, \qquad n = 0, 1, \ldots.$$

The result is a consequence of Theorem C. The proof is complete.

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