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Sharp Estimates on Approximation of Exponential Type Operators

Xiao-Ming Zeng
Department of Mathematics, Xiamen University,
Xiamen 361005, People's Republic of China
E-mail: xmzeng@jingxian.xmu.edu.cn
and
Fuhua (Frank) Cheng
Department of Computer Science, University of Kentucky,
Lexington, KY 40506-0046, U.S.A.
E-mail: cheng@cs.engr.uky.edu

Abstract

New estimates of Bernstein basis functions $p_{n,[nx]}(x),\ p_{n,[nx]+1}(x)$ and first order absolute moment of Szász operators $S_n(|t-x|,x)$ are presented. The new estimates of the first order absolute moment of the Szász operators $S_n(|t-x|,x)$ optimize the result of Theorem 2 in [7]. The estimates of the basis functions $p_{n,[nx]}(x),\ p_{n,[nx]+1}(x)$, together with a quantitative central limit theorem, are then used to derive a sharper asymptotic estimate on pointwise approximation of Bernstein operators for bounded functions.

1. Introduction

For a function f defined on $[0,1]$ the *Bernstein operator* B_n is defined by

$$B_n(f,x) = \sum_{k=0}^{n} f(k/n)p_{nk}(x), \qquad p_{nk}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k}. \tag{1}$$

For a function f defined on $[0,\infty)$ the Szász operator S_n is defined by

$$S_n(f,x) = \sum_{k=0}^{\infty} f(k/n)q_{nk}(x), \qquad q_{nk}(x) = e^{-nx}\frac{(nx)^k}{k!}. \tag{2}$$

For the rate of convergence of these operators for various function types, refer to [1,3,4,5,7] for a survey. The following is a most recent result on pointwise approximation of Bernstein operator for bounded functions [7]:

Theorem 1. *Let f be a bounded function on* $[0,\ 1]$, $f(x+)$ *and* $f(x-)$ *exist at a fixed point* $x \in (0,1)$. *Define* $g_x(t)$ *as*

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \le 1; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \le t < x. \end{cases}$$

Then for n sufficiently large we have

$$|B_n(f,x) - \frac{f(x+)+f(x-)}{2} - \frac{\mu(f,n,x)}{\sqrt{2\pi x(1-x)n}}| \le \frac{2}{nx(1-x)} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}) + o(n^{-1/2}), \tag{3}$$

where

$$\Omega(x, g_x, \sqrt{k}) = \sup\nolimits_{t\in[x-x/\sqrt{k},\, x+(1-x)/\sqrt{k}]} |g_x(t) - g_x(x)|.$$

and

$$\mu(f,n,x) = (f(x+)-f(x-))(nx-[nx]+(x-2)/3) + (f(x+)-f(x-))(1-\operatorname{sgn}(nx-[nx])),$$

As far as Szász operator is concerned, Zeng and Cheng [7] proved that

$$\left| S_n(|t-x|,x) - \sqrt{\frac{2x}{\pi n}} \right| \le \frac{2}{n\sqrt{nx}}. \tag{4}$$

This result is then used to estimate the rate of convergence of Szász operator for functions having derivatives of bounded variation in every finite subinterval of $[0,\infty)$. The above estimate improves the result obtained by Bojanic and Khan [1].

The aim of this paper is to improve both (3) and (4). The improvement of (3) is achieved by first getting better estimates of the Bernstein basis functions $p_{n,[nx]}(x)$ and $p_{n,[nx]+1}(x)$, and then using the new estimates, together with a quantitative central limit theorem in which the fourth moments of random variables exist, to derive an asymptotic estimate sharper than the one given in (3). This work is shown in Section 2. The improvement of (4) is achieved by providing sharper upper and lower bounds of the first order absolute moment of Szász operator $S_n(|t-x|,x)$. The new estimates optimize the result of Theorem 2 in [7]. This work is shown in Section 3.

2. Rate of Convergence of Bernstein operators

We first estimate Bernstein basis functions $p_{n,[nx]}(x)$ and $p_{n,[nx]+1}(x)$.

Lemma 1. *For* $x \in (0,1)$ *there holds*

$$p_{n,[nx]}(x) = \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}} + O(n^{-3/2}), \tag{5}$$

and

$$p_{n,[nx]+1}(x) = \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}} + O(n^{-3/2}). \tag{6}$$

Proof. By direct calculation and Stirling's formula $n! = (n/e)^n\sqrt{2\pi n}e^{\theta_n/12n}$, $0 < \theta_n < 1$, we have

$$p_{n,[nx]}(x) - \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}}$$

$$= \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}}\left(e^{c(n,x)}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}\left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2} - 1\right), \tag{7}$$

where

$$-\frac{1}{12[nx]} - \frac{1}{12(n-[nx])} < c(x,n) < \frac{1}{12n}. \tag{8}$$

In the following we prove that the terms inside the parentheses on the right side of (7) are of magnitude $O(n^{-1})$,

$$\left|e^{c(n,x)}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}\left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2} - 1\right| = O(n^{-1}).$$

Define ϵ, $\epsilon_{1,n}$ and $\epsilon_{2,n}$ as follows:

$$\epsilon = nx - [nx],$$

$$\epsilon_{1,n} = \left(\tfrac{nx}{[nx]}\right)^{[nx]+1/2}\left(\tfrac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2},$$

$$\epsilon_{2,n} = \left(\tfrac{nx}{[nx]}\right)^{[nx]+1/2}.$$

It is obvious that $0 \leq \varepsilon < 1$, $\varepsilon_{1,n} \leq e^2$, and $\varepsilon_{2,n} \leq e^2$. Hence we have

$$\left|e^{c(n,x)}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}\left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2} - 1\right|$$

$$\leq \varepsilon_{1,n}|e^{c(n,x)}-1| + \varepsilon_{2,n}\left|e^{-\varepsilon} - \left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}\right| + \left|e^{-\varepsilon}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2} - 1\right|.$$

From (6), it is not difficult to show that

$$\varepsilon_{1,n}|e^{c(x,n)}-1| = O(n^{-1}).$$

Straightforward calculation shows that

$$nx\left|e^{-\varepsilon}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2} - 1\right| = \frac{nx}{[nx]}[nx]\left(e^{-\varepsilon}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2} - 1\right) \leq \frac{nx}{[nx]}\varepsilon \leq 2.$$

Hence (5) is proved if we can show that

$$\left|e^{-\varepsilon} - \left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}\right| = O(n^{-1}).$$

Note that

$$\left(\frac{n-nx}{n-[nx]}\right)^{2} < e^{\varepsilon}\left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}.$$

Hence we have

$$\begin{aligned}
&(1-x)n\left|e^{-\varepsilon} - \left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}\right| \\
&= (1-x)n\left(e^{-\varepsilon} - \left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}\right) \\
&= e^{-\varepsilon}\frac{n-nx}{n-[nx]}(n-[nx])\left(1 - e^{\varepsilon}\left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}\right) \\
&\leq (n-[nx])\left(1 - e^{\varepsilon}\left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2}\right) \\
&\leq 2\varepsilon - \frac{\varepsilon^2}{n-[nx]} < 2,
\end{aligned}$$

and formula (5) is proved. On the other hand, simple calculation shows that

$$p_{n,[nx]+1}(x) - p_{n,[nx]}(x) = \frac{nx-[nx]+x-1}{([nx]+1)(1-x)}p_{n,[nx]}(x) = O(n^{-3/2}).$$

Hence one gets (5) directly from (4).

Lemma 2. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of independent and identically distributed random variables with the expectation* $E(\xi_1) = a_1$, *the variance* $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E(\xi_1 - a_1)^4 < \infty$, *and let* F_n *stand for the distribution function of* $\sum_{k=1}^{n} (\xi_k - a_1)/\sigma\sqrt{n}$. *If* F_n *is a lattice distribution and* F_n^* *is a polygonal approximant of* F_n, *then the following equation holds for all* $t \in (-\infty, +\infty)$

$$F_n^*(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du - \frac{E(\xi_1 - a_1)^3}{6\sigma^3\sqrt{n}}(1 - t^2)\frac{1}{\sqrt{2\pi}} = e^{-t^2/2} = O(n^{-1}). \quad (9)$$

The proof of Lemma 2 can be found in [2, p. 540-542].

For two-point distribution $P(\xi_1 = j) = x^j(1 - x)^{1-j}$, $j = 0, 1$, direct calculation gives $E\xi_1 = x$, $E(\xi_1 - E\xi_1)^2 = x(1 - x)$, and $E(\xi_1 - E\xi_1)^4 = x(1 - x)(1 - 3x + 3x^2) < \infty$. By using Lemmas 1, 2 and proceeding along the same line as in [1], we get the following result:

Theorem 2. *Let* f *be a bounded function on* $[0, 1]$, $f(x+)$ *and* $f(x-)$ *exist at a fixed point* $x \in (0, 1)$. *Then for* n *sufficiently large we have*

$$\left| B_n(f, x) - \frac{f(x+) + f(x-)}{2} - \frac{\mu(f, n, x)}{\sqrt{2\pi x(1 - x)n}} \right|$$

$$\leq \frac{2}{nx(1 - x)} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}) + O(n^{-1}), \quad (10)$$

where $g_x(t)$, $\mu(f, n, x)$, $\Omega(x, g_x, \sqrt{k})$ *are defined in Theorem* 1.

Estimate (8) improves the result of Theorem 1 in that it replaces the term $o(n^{-1/2})$ on the right hand side of inequality (3) with $O(n^{-1})$.

3. Rate of Convergence of Szász operators

The following is a sharp estimate of $|S_n(|t - x|, x)|$.

Proposition 1. *For* $x \in [0, \infty)$ *there holds*

$$\left| S_n(|t - x|, x) - \sqrt{\frac{2x}{\pi n}} \right| \leq \frac{2}{n(1 + \sqrt{nx})}. \quad (11)$$

Proof. It is known from Lemma 4 of [7] that

$$S_n(|t - x|, x) = 2xe^{-nx}\frac{(nx)^{[nx]}}{[nx]!}.$$

If $n < 1/x$, then $[nx] = 0$. Straightforward calculation shows that

$$|2\sqrt{nx}e^{-nx} - \sqrt{2/\pi}| \leq 1.$$

Hence for $0 \leq x < 1/n$ we have

$$n(1+\sqrt{nx})\left|S_n(|t-x|,x) - \sqrt{\frac{2x}{\pi n}}\right| = (\sqrt{nx}+nx)|2\sqrt{nx}e^{-nx} - \sqrt{2/\pi}| \leq 2.$$

If $n \geq 1/x$, then $[nx] \geq 1$. Using Stirling's formula (cf. [6]):

$$n! = (n/e)^n\sqrt{2\pi n}e^{C_n}, \tag{12}$$

where

$$\frac{1}{12n+1} < C_n = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + \ldots < \frac{1}{12n},$$

we get

$$n(1+\sqrt{nx})\left(S_n(|t-x|,x) - \sqrt{\frac{2x}{\pi n}}\right)$$

$$= \sqrt{2/\pi}(\sqrt{nx}+nx)\left(e^{-nx+[nx]}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}e^c - 1\right)$$

$$= \sqrt{2/\pi}(\sqrt{nx}+nx)(e^c-1) + e^c\sqrt{2/\pi}(\sqrt{nx}+nx)\left(e^{-nx+[nx]}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2} - 1\right),$$

where

$$-\frac{1}{12[nx]} < c < -\frac{1}{12[nx]+1}.$$

Thus, from the expansion formula: $e^c = \sum\limits_{k=0}^{\infty} \frac{c^k}{k!}$, it is not difficult to show that

$$(\sqrt{nx}+nx)|e^c - 1| \leq \frac{1}{2}.$$

On the other hand, write $nx = [nx] + \varepsilon$ $(0 \leq \varepsilon < 1)$, then

$$e^c\sqrt{2/\pi}(\sqrt{nx}+nx)\left|e^{-nx+[nx]}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2} - 1\right|$$

$$= e^c\sqrt{2/\pi}(\sqrt{nx}+nx)\left(e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2} - 1\right)$$

$$\leq 2\sqrt{2/\pi}nx\left(e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2}-1\right).$$

Using inequalities

$$\left(1+\frac{\varepsilon}{[nx]}\right)^{1/2}<1+\frac{\varepsilon}{nx}, \qquad \left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]}<e^{\varepsilon},$$

we deduce that

$$nx\left(e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2}-1\right)\leq\varepsilon<1.$$

Consequently

$$n(1+\sqrt{nx})\left|S_n(|t-x|,x)-\sqrt{\frac{2x}{\pi n}}\right|\leq\frac{\sqrt{2}}{2\sqrt{\pi}}+\frac{2\sqrt{2}}{\sqrt{\pi}}\leq 2.$$

This completes the proof of Proposition 1.

Estimate (11) is stronger than the estimate of Bojanic and Khan [1, p. 167]:

$$S_n(|t-x|,x)=\sqrt{\frac{2x}{\pi n}}+O(n^{-1}),$$

and the estimate (4) [7, Lemma 4]. Furthermore, the following Lemma shows that the estimate (11) is the best possible in the asymptotic sense.

Lemma 3. For $x=1$ we have

$$\left|S_n(|t-1|,1)-\sqrt{\frac{2}{\pi n}}\right|\geq\frac{1}{15}\sqrt{\frac{2}{\pi}}n^{-3/2}.$$

Proof. Using Stirling's formula (12), we have

$$\left|S_n(|t-1|,1)-\sqrt{\frac{2}{\pi n}}\right|=\left|2e^{-n}\frac{n^n}{n!}-\sqrt{\frac{2}{\pi n}}\right|=\sqrt{\frac{2}{\pi n}}(1-e^{-C_n})$$

$$\geq\sqrt{\frac{2}{\pi n}}\frac{C_n}{e^{C_n}}\geq\sqrt{\frac{2}{\pi n}}\frac{e^{-1/12}}{12n+1}\geq\frac{1}{15}\sqrt{\frac{2}{\pi}}n^{-3/2}.$$

Using Lemma 3 we can show that the estimate (33) in Theorem 2 of [7] cannot be improved asympotically. For $x \in [0, \infty)$, by straightforward calculation we have

$$|t-x| - |0-x| = \int_0^t sgn(u-x)du, \qquad t \in [0, \infty).$$

By setting $h(t) = |t-x|$ in (33) of [7], we have $\tau = 2$, and $\varphi_x = 0$. Hence (33) of [1] becomes

$$\left| S_n(|t-x|, x) - \sqrt{\frac{2x}{\pi n}} \right| \leq \frac{2}{n^{3/2}x^{1/2}}. \tag{13}$$

For $x = 1$, from Lemma 3 and (10) we get

$$\frac{1}{15}\sqrt{\frac{2}{\pi}} n^{-3/2} \leq \left| S_n(|t-1|, 1) - \sqrt{\frac{2}{\pi n}} \right| \leq 2n^{-3/2}.$$

The above inequalities, combined with (42) of [7], show that the estimate of Theorem 2 of [7] cannot be improved asymptotically.

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